Leaci and Tomarelli *Advances in Continuous and Discrete Models* (2023) 2023:14
https://doi.org/10.1186/s13662-023-03762-8

Advances in Continuous and Discrete Models

RESEARCH Open Access

Symmetrized fractional total variation for signal and image analysis

Antonio Leaci[1*] and Franco Tomarelli[2]

*Correspondence: antonio.leaci@unisalento.it
[1] Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce, Italy
Full list of author information is available at the end of the article

Abstract

We introduce and study a variational model for signal and image analysis based on Riemann–Liouville fractional derivatives. Both the one-dimensional and two-dimensional cases are studied. The model exploits a quadratic fitting data term together with both right and left Riemann–Liouville fractional derivatives as regularizing terms, with the aim of achieving an orientation-independent analysis.

MSC: 26A33; 26A45; 49J45

Keywords: Fractional derivatives; Distributional derivatives; Sobolev spaces; Bounded variation functions; Embeddings; Compactness; Calculus of variations; Abel equation; Signal analysis; Image analysis

1 Introduction

Variational methods in image processing are quite popular now and there is a plenty of papers on this topic both from the theoretical and computational point of view. The general (deterministic) variational model may be written as the minimization of a cost functional under some constraints that usually drive the underlying functional framework. The cost functional is the sum of two different terms. The first one is the *fitting data term* that measures the distance between the observed or measured object (the datum) and the computed one. The second term add priors on the object and it is known as the regularizing term; the role of this term is twofold: sometimes the minimization problem turns out to be ill-posed, thus such a term allows selection of the desired solution among objects whose properties are a priori known; in addition, this second term must be suitably tailored in order to make the whole cost functionals coercive and semicontinuous with respect to the underlying topology, in order to exploit the methods of Calculus of Variations.

Classical nonconvex (hence lacking the uniqueness of a minimizer) variational models are the Mumford–Shah and Blake–Zisserman ones ([1, 7, 16, 22]) that were extensively used in image segmentation and denoising.

Another classical model for image analysis is the Rudin–Osher–Fatemi model ([32]) also known as ROF or TV (total variation), which is based on a strictly convex minimization, and has several variations in the literature ([19, 30, 35]): in such a model, the regularizing term is a first-order term, namely the total variation that turns to be the ℓ^1 norm of the gradient in a discretized setting; to avoid artefacts such as staircasing that appears in the TV approach the total variation term has been updated to involve a second-order derivative, the most popular one being the so-called Total Generalized Variation (TGV) [8]. Similar models have been developed with different second-order terms for denoising and texture analysis in [3, 5, 24] or segmentation and inpainting as in [11–17].

All the aforementioned models provide good information of the contours and cartoons but few of them focus on texture analysis (from a deterministic point of view). A more appropriate tool for this task could be the fractional derivative (of order between 0 and 1) that has been extensively used in computer vision [19, 23, 27] for denoising and segmentation and textured images as well [30, 31, 34]; but these articles mostly deal with the finite-dimensional framework (after discretization) and very few propose an infinite-dimensional approach together with an appropriare mathematical analysis. However, we mention the work by Zhang and Chen [35] that proposes an anisotropic total fractional-order variation model for image restoration in a continuous setting and produces several numerical tests, showing better reconstruction of noisy images, with improved contrast and circumventing the staircasing effect.

In this paper, we introduce an isotropic FTV model, say a model based on Fractional Total Variation of restored images: a suitably weighted fidelity term matches with a regularizing term corresponding to the sum of total variations of both left and right fractional derivatives.

We investigate such a variational model in the framework of Riemann–Liouville fractional derivatives. We show that there exists a unique solution that belongs to the space of Fractional Bounded Variation space BV^s as defined in [26], see also [4, 21, 25]. Both 1D and 2D cases are considered, respectively, in Sects. 3 and 4.

A comparison with Zhang and Chen [35] (which is based on fractional derivative filtering too) is shown in Sect. 5.

We emphasize that, as long as fractional derivatives are involved, the isotropy is not for free but must be imposed, since both right and left derivatives have different nontrivial kernels: for instance, referring to the interval $(0,1)$ for simplicity, the Riemann–Liouville left derivative D^s_+ vanishes on every multiple of x^{s-1}, thus unbounded perturbations of the kind εx^{s-1} may affect any minimizing sequence with an arbitrary low cost. On the other hand, $D^s_-[x^{s-1}] \not\equiv 0$ since the kernel of the right derivative D^s_- consists in the multiples of $(1-x)^{s-1}$; moreover neither x^{s-1} nor $(1-x)^{s-1}$ belong to $BV^s(0,1)$. For this reason, we introduce a symmetrized approach to the fractional total variation in signal analysis, where the regularizing filtering is based on both left and right derivatives: see functional $\mathcal{F}$ defined in (3.3) and related optimization problem $\mathcal{P}_{\mathcal{F}}$ that fulfils isotropy, since it depends on both left and right fractional derivatives.

With regard to recent results concerning fractional Sobolev and fractional Bounded Variation spaces based on Riemann–Liouville fractional derivatives we mention also [6], [10] and [21]; about recent developments concerning fractional derivatives, by a different approach based on operator theory and tuned to the case of several variables, we mention [20]; about texture analysis models based on fractional derivatives and numerical implementation we mention [18], [29].

2 Preliminary tools for the 1D case

As we are dealing with Riemann–Liouville fractional calculus we recall the definitions and the main results useful in the following ([33]). For additional details concerning a bilateral approach we refer to [4, 25, 26].

The signal domain is a generic bounded interval (a,b) of $\mathbb{R}$ but, for the sake of simplicity, we choose $a=0$ and $b=1$ in the following. The notation d/dx stands for the classical pointwise derivative, D_x or, in short D, denotes the distributional derivative, $W^{1,1}(0,1)=\{u\in L^1(0,1)\mid Du\in L^1(0,1)\}$ denotes the usual Gagliardo–Sobolev space. Note that often (see, e.g., [33]) this space is referred to as the space of absolutely continuous functions $AC(0,1)$.

We recall the definition of the Riemann–Liouville fractional integrals and derivatives for L^1-functions, which, like their bilateral versions (Riesz potentials and related distributional derivatives) [25], can be represented by convolutions.

Definition 2.1 Let $u\in L^1(0,1)$. For every $0<s<1$, recall the classical left-side and right-side Riemann–Liouville fractional integrals, by setting, respectively,

$$I_+^s[u](x)=\frac{1}{\Gamma(s)}\int_0^x\frac{u(t)}{(x-t)^{1-s}}\,dt,\quad 0\le x\le 1, \tag{2.1}$$

$$I_-^s[u](x):=\frac{1}{\Gamma(s)}\int_x^1\frac{u(t)}{(t-x)^{1-s}}\,dt,\quad 0\le x\le 1, \tag{2.2}$$

$$I^s[u](x):=\frac{1}{2\Gamma(s)}\int_0^1\frac{u(t)}{|x-t|^{1-s}}\,dt,\quad 0\le x\le 1. \tag{2.3}$$

Here, and in the following, Γ stands for the classical Gamma function (see [28]).

Next, we define the Riemann–Liouville fractional derivative as

Definition 2.2 (Distributional Riemann–Liouville fractional derivative) Let $u\in L^1(0,1)$ and $0<s<1$. Then, referring to [25] and [26], the left, right, and bilateral Riemann–Liouville derivatives of u at $x\in(0,1)$ are defined by

$$D_+^s[u](x)=D_x\big(I_{0+}^{1-s}[u]\big)(x)=\frac{1}{\Gamma(1-s)}D_x\int_{-\infty}^x\frac{u(t)}{(x-t)^s}\,dt, \tag{2.4}$$

$$D_-^s[u](x):=-D_x\big(I_{1-}^{1-s}[u]\big)(x)=\frac{-1}{\Gamma(1-s)}D_x\int_x^{+\infty}\frac{u(t)}{(t-x)^s}\,dt, \tag{2.5}$$

$$D^s[u](x)=D_xI^{1-s}[u](x):=\frac{1}{2}\big(D_+^s[u](x)-D_-^s[u](x)\big). \tag{2.6}$$

We emphasize that, referring to the trivial extension of u ($u=0$ a.e. on $\mathbb{R}\setminus(0,1)$), all integrals in (2.1)–(2.5) are convolutions, though we consider their values restricted to the open set $(0,1)$ (see [25, 26]):

$$I_+^s[u]=u*\frac{H(x)}{\Gamma(s)|x|^{1-s}},\qquad I_-^s[u]=u*\frac{H(-x)}{\Gamma(s)|x|^{1-s}},\qquad I^s[u]=u*\frac{1}{2\Gamma(s)|x|^{1-s}},$$

namely

$$I_+^s[u](x)=\frac{1}{\Gamma(s)}\int_0^1\frac{u(t)H(x-t)}{|x-t|^{1-s}}\,dt\quad\text{for every } x\in\mathbb{R},$$

$$I_-^s[u](x)=\frac{1}{\Gamma(s)}\int_0^1\frac{u(t)H(t-x)}{|x-t|^{1-s}}\,dt\quad\text{for every } x\in\mathbb{R}.$$

Moreover, all the derivatives appearing in (2.4) and (2.5) are understood in the distributional sense in $\mathcal{D}'(\mathbb{R})$, though we will work in $\mathcal{D}'(0,1)$ by handling their restrictions to the open set $(0,1)$. We refer to [26] for the related theory.

We recall the Sobolev and Bounded Variation spaces associated to the Riemann–Liouville fractional derivative as introduced in [26].

Definition 2.3 Let $s \in (0,1)$:

$$W_+^{s,1}(0,1) := \left\{u \in L^1(0,1) \mid I_+^{1-s}[u] \in W^{1,1}(0,1)\right\}, \tag{2.7}$$

$$W_-^{s,1}(0,1) := \left\{u \in L^1(0,1) \mid I_-^{1-s}[u] \in W^{1,1}(0,1)\right\}. \tag{2.8}$$

Both $W_+^{s,1}$ and $W_-^{s,1}$ are Banach spaces, when endowed with their natural norms. Unfortunately, bounded subsets of Sobolev spaces $W_+^{s,1}$ and $W_-^{s,1}$ lack compactness properties in L^1 for fractional derivatives. This led us to introduce the fractional BV counterparts of these spaces in a symmetric framework.

Definition 2.4 (Riemann–Liouville fractional bounded variation spaces) For every $s \in (0,1)$ we define $BV^s(0,1)$, or shortly BV^s, by

$$BV^s = BV_+^s \cap BV_-^s, \tag{2.9}$$

where, referring to Definition 2.1,

$$BV_+^s = \left\{u \in L^1(0,1) \mid I_+^{1-s}[u] \in BV(0,1)\right\} = \left\{u \in L^1(0,1) \mid D_+^s[u] \in \mathcal{M}(0,1)\right\},$$
$$BV_-^s = \left\{u \in L^1(0,1) \mid I_-^{1-s}[u] \in BV(0,1)\right\} = \left\{u \in L^1(0,1) \mid D_-^s[u] \in \mathcal{M}(0,1)\right\}.$$

The next theorem shows that BV^s endowed with its natural norm is a Banach space too and that every bounded subset of BV^s is relatively compact both in the strong L^1 topology and the weak* topology of measures for the related derivatives, respectively, D_+^s, D_-^s.

Theorem 2.5 (see [26]) *Assume that the parameter s fulfills* $0 < s < 1$.

Then, the space $BV^s(0,1)$ *is a normed space endowed with the norm*

$$\|u\|_{BV^s(0,1)} := \|u\|_{L^1(0,1)} + \left\|D_+^s[u]\right\|_{\mathcal{M}(0,1)} + \left\|D_-^s[u]\right\|_{\mathcal{M}(0,1)}. \tag{2.10}$$

Moreover, $BV^s(0,1)$ *is a Banach space and for every* $q \in [1, 1/(1-s))$, *there is* $C = C(s,q)$, *such that*

$$\|u\|_{L^q(0,1)} \leq C(s,q)\|u\|_{BV^s(0,1)}. \tag{2.11}$$

Every $u \in BV^s(0,1)$ *can be represented by both*

$$u(x) = I_+^s\left[D_+^s[u]\right](x) + \frac{I_+^{1-s}[u](0_+)}{\Gamma(s)} x^{s-1} \quad a.e.\ x \in (0,1), \tag{2.12}$$

and

$$u(x) = I_-^s\left[D_-^s[u]\right](x) + \frac{I_-^{1-s}[u](1_-)}{\Gamma(s)} (1-x)^{s-1} \quad a.e.\ x \in (0,1). \tag{2.13}$$

Theorem 2.6 (Compactness in $BV^s(0,1)$ (see [26])) *Assume that* $0 < s < 1$ *and*

$$\|u_n\|_{BV^s(0,1)} \le C. \tag{2.14}$$

Then, there exist $u \in L^1(0,1)$ *and a subsequence such that, without relabeling,*

$$\begin{cases} \text{(i)} & u_n \rightharpoonup u \quad \textit{weakly in } L^q(0,1)), \forall q \in [1, 1/(1-s)), \\ \text{(ii)} & I_+^{1-s}[u_n] \to I_+^{1-s}[u] \quad \textit{strongly in } L^p(0,1), \forall p < +\infty, \\ \text{(iii)} & I_-^{1-s}[u_n] \to I_-^{1-s}[u] \quad \textit{strongly in } L^p(0,1), \forall p < +\infty, \end{cases}$$

$$I_+^{1-s}[u_n] \rightharpoonup I_+^{1-s}[u], \qquad I_-^{1-s}[u_n] \rightharpoonup I_-^{1-s}[u] \quad \textit{weakly in } BV(0,1). \tag{2.15}$$

We end this section by collecting some embeddings and continuity results related to the space of functions of bounded variation $BV(0,1)$, the space of special functions of bounded variation $SBV(0,1)$ (the bounded variation ones whose distributional derivative has no Cantor part), see [1, 2]) and the fractional bounded variation space $BV^s(0,1)$.

By (74), (78), (91), and (93) of [26] we recall these space embeddings

$$SBV(0,1) \subset \bigcap_{s \in (0,1)} W^{s,1}(0,1),$$

$$BV(0,1) \subset W^{s,1}(0,1) \subset BV^s(0,1) \tag{2.16}$$

and several embeddings that are lacking

$$BV(0,1) \underset{\neq}{\subset} W^{s,1}(0,1), \qquad W^{s,1}(0,1) \not\subseteq BV(0,1), \quad \forall s \in (0,1),$$

$$W^{s,1}(0,1) \underset{\neq}{\subset} BV_+^s(0,1), \qquad W^{s,1}(0,1) \underset{\neq}{\subset} BV_-^s(0,1), \quad \forall s \in (0,1).$$

3 FTV : symmetrized fractional total variation model (1-dimensional)

In this section, we consider a 1D Variational Model designed for signal processing. Let us mention that while the fractional calculus was extensively used in optimal control for differential equations theory, only recently has this mathematical approach been considered in computer vision ([35]).

We introduce here a 1D model based on Fractional Total Variation of restored images: a variational principle where a suitably weighted fidelity term matches with a regularizing term corresponding to the total variation of left and right fractional derivatives.

Given

$$\lambda > 0, \qquad s \in (0,1), \tag{3.1}$$

$$g \in L^2(0,1), \tag{3.2}$$

where λ is weighting parameter for the fidelity term, s is the fractional order of derivatives and g represents the raw datum, we consider the following problem

$$\min\{\mathcal{F}(u) : u \in BV^s(0,1) \cap L^2(0,1)\}, \tag{$\mathcal{P}_{\mathcal{F}}$}$$

where

$$\mathcal{F}(u) := \frac{\lambda}{2}\int_0^1 \left|u(x)-g(x)\right|^2 dx + \left\|D^s_+[u]\right\|_{T_+} + \left\|D^s_-[u]\right\|_{T_-}, \tag{3.3}$$

and $D^s_{\pm}u$ denote, respectively, the left and right distributional Riemann–Liouville fractional derivatives of u as defined in [26], as convolutions, so that $D^s_+[u] \equiv 0$ on $(-\infty, 0)$ and $D^s_-[u] \equiv 0$ on $(1, +\infty)$; thus, we are adopting this notation

$$\left\|D^s_+[u]\right\|_{T_+} := \left\|D^s_+[u]\right\|_{\mathcal{M}(-\infty,1)} = \left\|D^s_+[u]\right\|_{\mathcal{M}(0,1)} + \left|I^{1-s}_+[u](0_+)\right|, \tag{3.4}$$

$$\left\|D^s_-[u]\right\|_{T_-} := \left\|D^s_-[u]\right\|_{\mathcal{M}(0,+\infty)} = \left\|D^s_-[u]\right\|_{\mathcal{M}(0,1)} + \left|I^{1-s}_-[u](1_-)\right|, \tag{3.5}$$

where $\|\mu\|_{\mathcal{M}(A)}$ denotes the total variation in A of a real-valued measure μ.

A reasonable model of signal analysis should fulfil invariance under reflection of the data. We emphasize that our fractional total variation model is symmetric in the sense that no direction is preferred, since both D^s_+ and D^s_- are taken into account here, as is made precise by the next Remarks.

Remark 3.1 To every mimimizer v of problem ($\mathcal{P}_{\mathcal{F}}$) with datum g, correspond a minimizer $\widetilde{v}$ of problem ($\mathcal{P}_{\mathcal{F}}$) with datum $\widetilde{g}$, with the notation $\widetilde{w}(x) = w(1-x)$.

Remark 3.2 If one drops the last summand (the one dependent on D^s_-) in definition ($\mathcal{P}_{\mathcal{F}}$), then $\mathcal{F}(x^{s-1})$ would reduce to $\frac{\lambda}{2}\|x^{s-1}-g\|^2_{L^2}$, due to $D^s_+[x^{s-1}] = 0$, whereas with the present definition $\mathcal{F}(x^{s-1}) = \frac{\lambda}{2}\|x^{s-1}-g\|^2_{L^2} + \|D^s_-[x^{s-1}]\|_T$, where $D^s_-[x^{s-1}]$ is nontrivial, since, by Lemma 8 in [26], $D^s_-[v] \equiv 0$ if and only if $v(x) = C(1-x)^{s-1}$.

Actually, by formula (A.1) in the Appendix with $\alpha = s$ and $\beta = s-1$,

$$D^s_-\left[x^{s-1}\right] = -D_x I^{1-s}_-\left[x^{s-1}\right] = -D_x\left(\frac{1}{\Gamma(1-s)}\int_x^1 w^{-1}(1-w)^{-s}\,dw\right) \not\equiv 0.$$

Moreover, $D^s_-[x^{s-1}]$ is not a measure of bounded variation (indeed it is an analytic function with a singularity asymptotic to $(\Gamma(1-s))^{-1}(1-x)^{-s}/x)$ at $x = 0_+$; hence,

$$x^{s-1} \in BV^s_+ \setminus BV^s_- \quad \forall s \in (0,1), \tag{3.6}$$

$$(1-x)^{s-1} \in BV^s_- \setminus BV^s_+ \quad \forall s \in (0,1); \tag{3.7}$$

thus, neither x^{s-1} nor $(1-x)^{s-1}$ belong to $BV^s(0,1)$, hence neither x^{s-1} nor $(1-x)^{s-1}$ belong to the domain of $\mathcal{F}$. This is a nice property, since in signal processing we would like to disregard unbounded outputs.

Summarizing, if one drops $\|D^s_+[u]\|$ (respectively, $\|D^s_-[u]\|$) in the definition of $\mathcal{F}$ then $(1-x)^{s-1}$ (respectively, x^{s-1}) belongs to the domain.

Example 3.3 We can refine the example in the previous Remark, by showing a function $v \in BV^s_+(0,1) \setminus BV^s_-(0,1)$ with nontrivial $D^s_+[v]$: set

$$v(x) = H(x-1/2)|x-1/2|^{s-1},$$

then, exploiting Examples 1 and 8 provided in [26] referring to the interval $(-1,1)$ in place of $(0,1)$ and exploiting the translation invariance of fractional integrals and derivatives, we obtain

$$I_+^{1-s}\big[H(x-1/2)|x-1/2|^{s-1}\big](x) = \begin{cases} 0 & \text{if } 0<x<1/2, \\ \Gamma(s) & \text{if } 1/2<x<1, \end{cases}$$

$D_+^s[H(x-1/2)|x-1/2|^{s-1}](x) = \Gamma(s)\delta(x-1/2)$, $H(x-1/2)|x-1/2|^{s-1} \in BV_+^s(0,1)$, while $I_-^{1-s}[H(x-1/2)|x-1/2|^{s-1}](x)$ is unbounded in the neighborhood of $x=1/2$, hence $I_-^{1-s}[H(x-1/2)|x-1/2|^{s-1}](x)$ does not belong to $BV(0,1)$, therefore function $H(x-1/2)|x-1/2|^{s-1}$ does not belong to $BV_-^s(0,1)$. For instance, if $s=1/2$:

$$I_+^{1/2}\big[H(x-1/2)|x-1/2|^{-1/2}\big](x) = \begin{cases} 0 & \text{if } 0<x<1/2, \\ \sqrt{\pi} & \text{if } 1/2<x<1, \end{cases}$$

while

$$I_-^{1/2}\big[H(x-1/2)|x-1/2|^{-1/2}\big](x) = \frac{2\ln(1+\sqrt{2-2x}) - \ln|1-2x|}{\sqrt{\pi}}.$$

Remark 3.4 By denoting V the Cantor–Vitali function ([1]) and using notation $\widetilde{V}(x) = V(1-x)$, we obtain (see (A.5)):

$$\begin{aligned} &D_+^s[\widetilde{V}](x) = -D_-^s[V](1-x), \\ &V(x) + \widetilde{V}(x) = 1 \end{aligned}$$

and computing D_x and D_+^s on both sides

$$\begin{aligned} &D_x[V](x) + D_x[\widetilde{V}](x) = 0, \\ &D_+^s[V](x) + D_+^s[\widetilde{V}](x) = \frac{x^{-s}}{\Gamma(1-s)}, \\ &D_+^s[V](x) = D_-^s[V](1-x) + \frac{x^{-s}}{\Gamma(1-s)}, \end{aligned}$$

explicitly: $D_+^s[V](x) \neq D_-^s[V](1-x)$.

3.1 Analysis of 1-dimensional FTV model

Here, we prove the existence and uniqueness of a solution to ($\mathcal{P}_{\mathcal{F}}$).

Lemma 3.5 (Lower semicontinuity) *Assume u_n is a uniformly bounded sequence in $BV^s(0,1)$ and $u_n \rightharpoonup u$ weakly in $L^2(0,1)$. Then,*

$$\liminf \big\| D_+^s u_n \big\|_{T_+} \ge \big\| D_+^s u \big\|_{T_+}, \qquad \liminf \big\| D_-^s u_n \big\|_{T_-} \ge \big\| D_-^s u \big\|_{T_-}. \tag{3.8}$$

Proof The sequence u_n is uniformly bounded in $L^2(0,1)$ and hence in $L^1(0,1)$. Set

$$z_n(x) = I\big[D_+^s[u_n]\big](x) \quad \text{and} \quad z(x) = I\big[D_+^s[u]\big](x).$$

Here, $I[\mu]$ denotes the primitive vanishing on $x < 0$ of measure μ: $I[\mu](x) = \mu([0,x])$. Thus, z_n is uniformly bounded in $BV(0,1)$ and up to subsequences $z_n \to z$ strongly in L^1, then

$$\liminf \left\| D^s_+[u_n] \right\|_{T_+} = \liminf \|Dz_n\|_{T_+} \geq \|Dz\|_{T_+} = \left\| D^s_+[u] \right\|_{T_+}.$$

The same analysis can be used to deal with $D^s_-[u]$. Thus, (3.8) is proved.

We emphasize that z_n is allowed to be different from u_n and z is allowed to be different from u, since:

$$z_n(x) = I\big[D^s_+[u_n]\big](x) = I^{1-s}_+\big[I^s_+\big[D^s_+[u_n]\big]\big](x) = I^{1-s}_+\left[u_n - \frac{I^{1-s}_+[u_n]}{\Gamma(s)} x^{s-1}\right](x),$$
$$z(x) = I\big[D^s_+[u]\big](x) = I^{1-s}_+\big[I^s_+\big[D^s_+[u]\big]\big](x) = I^{1-s}_+\left[u - \frac{I^{1-s}_+[u]}{\Gamma(s)} x^{s-1}\right](x). \qquad \square$$

Theorem 3.6 *Assume $g \in L^2(0,1)$. Then, problem ($\mathcal{P}_{\mathcal{F}}$) has a unique minimizer.*

Proof We apply the direct method of calculus of variations. The domain of functional $\mathcal{F}$ is nonempty, since $\mathcal{F}(0) = \|g\|^2_{L^2} < +\infty$. Moreover, $\mathcal{F}$ is always nonnegative, hence it is bounded from below.

Let u_n be a minimizing sequence for $\mathcal{F}$; then u_n is bounded in $L^2(0,1)$, in $L^1(0,1)$ and in $BV^s(0,1)$. Therefore, up to subsequences and without relabeling, u_n weakly converges to some $\widetilde{u}$ in $L^2(0,1)$. In addition, $D^s_+ u_n$ and $D^s_- u_n$ are bounded in $\mathcal{M}$. Thus, due to compactness (Theorem 2.6) there exists $\widetilde{u} \in BV^s((0,1))$, $D^s_+ u_n \to D^s_+ \widetilde{u}$, $D^s_- u_n \to D^s_- \widetilde{u}$ (weak* convergences in $\mathcal{M}$) and $u_n \to \widetilde{u}$ strongly in L^1 and weakly in L^2.

The existence of a minimizer follows by the lower semicontinuity of positive quadratic forms in L^2 and the lower semicontinuity of total variation of $D^s_+ u_n$ and $D^s_- u_n$ with respect to the weak* convergence (see Lemma 3.5).

The uniqueness of the minimizer is due to the fact that functional $\mathcal{F}$ is strictly convex, since it is the sum of a strictly convex term and two convex terms. $\square$

We also introduce the problems

$$\min\big\{\mathcal{E}_+(v) : v \in \mathcal{M}_+\big\}, \qquad (\mathcal{P}^+_{\mathcal{E}})$$
$$\min\big\{\mathcal{E}_-(w) : w \in \mathcal{M}_-\big\}, \qquad (\mathcal{P}^-_{\mathcal{E}})$$

where $\mathcal{M}_+ := \{\mu \in \mathcal{M}(-\infty,1) : \operatorname{spt}\mu \subset [0,1)\}$, $\mathcal{M}_- := \{\mu \in \mathcal{M}(0,+\infty) : \operatorname{spt}\mu \subset (0,1]\}$ and

$$\mathcal{E}_+(v) := \frac{\lambda}{2}\left\| I^s_+[v] - g \right\|^2_{L^2} + \|v\|_{T_+} + \left\| D^s_-\big[I^s_+[v]\big] \right\|_{T_-}, \qquad (3.9)$$
$$\mathcal{E}_-(w) := \frac{\lambda}{2}\left\| I^s_-[w] - g \right\|^2_{L^2} + \|w\|_{T_-} + \left\| D^s_+\big[I^s_-[w]\big] \right\|_{T_+}. \qquad (3.10)$$

Theorem 3.7 *Assume that $g \in L^2(0,1)$ and $s \in (0,1)$.*

Then, both problems ($\mathcal{P}^+_{\mathcal{E}}$) and ($\mathcal{P}^-_{\mathcal{E}}$) have a unique minimizer.

Precisely, the problems ($\mathcal{P}_{\mathcal{F}}$), ($\mathcal{P}^+_{\mathcal{E}}$), and ($\mathcal{P}^-_{\mathcal{E}}$) are equivalent problems. Moreover, if $\widetilde{u}$, $\widetilde{v}$, and $\widetilde{w}$ are the unique minimizers, respectively, of ($\mathcal{P}_{\mathcal{F}}$), ($\mathcal{P}^+_{\mathcal{E}}$), and ($\mathcal{P}^-_{\mathcal{E}}$), then

$$\widetilde{u} = I^s_+[\widetilde{v}] = I^s_-[\widetilde{w}].$$

Proof Indeed, we can perform these changes of variables on generic competing functions:

$$u = I_+^s[v], \qquad u = I_-^s[w]. \tag{3.11}$$

Let us check that the changes of variables are bijective.

For every $v \in \operatorname{dom} \mathcal{E}_+ = \{v \in \mathcal{M}^+ : I_+^s[v] \in L^2(0,1)\}$ there exists a uniquely defined function $u = I_+^s[v] \in L^2(0,1)$, which also fulfils $I_+^{1-s}[u](x) = I_+^{1-s}[I_+^s[v]](x) = \int_0^x v \in BV$, hence u belongs to $\operatorname{dom} \mathcal{F} = BV^s \cap L^2$.

In exactly the same way, one can verify that for every $w \in \operatorname{dom} \mathcal{E}_- = \{v \in \mathcal{M}^- : I_-^s[v] \in L^2(0,1)\}$ there exists a uniquely defined $u = I_-^s[w] \in \operatorname{dom} \mathcal{F}$.

Vice versa, for all $u \in \operatorname{dom} \mathcal{F} = BV^s \cap L^2 = \{u \in L^2(0,1) : I_+^{1-s}[u], I_-^{1-s}[u] \in BV(0,1)\}$, due to Propositions 3 and Corollary 2 in [26] we can solve in the distributional framework the forward and backward Abel integral equations (3.11), finding a unique Laplace-transformable solution v and a unique solution w among the Laplace-transformable distribution computed in the variable $1 - x$, explicitly given by:

$$v(x) = D_+^s[u](x) + I_+^{1-s}[u](0_+)\delta(x), \tag{3.12}$$

$$w(x) = D_-^s[u](x) + I_-^{1-s}[u](1_-)\delta(x-1). \tag{3.13}$$

The equivalence follows by plain substitution: taking into account the identities of $D_+^s[I_+^s[u]] = u$, $D_-^s[I_-^s[u]] = u$ (see (117), (121) in [26]), we obtain

$$\mathcal{F}(u) = \mathcal{E}_+(v) = \mathcal{E}_-(w).$$

Thus, the equivalences are shown. Then, by equivalence with problem ($\mathcal{P}_\mathcal{F}$) we obtain the existence for ($\mathcal{P}_\mathcal{E}^+$) and ($\mathcal{P}_\mathcal{E}^-$); uniqueness follows by the bijectivity of variable changes. □

3.2 Optimality conditions

We provide explicit representations for adjoint operators of fractional integral and fractional derivatives, in order to state some necessary conditions for candidate minimizers of problem ($\mathcal{P}_\mathcal{F}$).

Lemma 3.8 *The adjoint operators* $(I_+^s)^*$, $(I_-^s)^*$, $(I^s)^*$ *are defined by*

$$\int_0^1 \left(I_+^s\right)^*[v](x)u(x)\,dx = \int_0^1 v(x)I_+^s[u](x)\,dx, \quad \forall u, v \in L^1(0,1),$$

$$\int_0^1 \left(I_-^s\right)^*[v](x)u(x)\,dx = \int_0^1 v(x)I_-^s[u](x)\,dx, \quad \forall u, v \in L^1(0,1),$$

$$\int_0^1 \left(I^s\right)^*[v](x)u(x)\,dx = \int_0^1 v(x)I^s[u](x)\,dx, \quad \forall u, v \in L^1(0,1)$$

and fulfil

$$\left(I_+^s\right)^* = I_-^s, \qquad \left(I_-^s\right)^* = I_+^s, \qquad \left(I^s\right)^* = I^s. \tag{3.14}$$

The adjoint operators $(D^s_+)^*$, $(D^s_-)^*$, $(D^s)^*$ *are defined by*:

$$\int_0^1 \big(D^s_+\big)^*[v](x)u(x)\,dx = \int_0^1 v(x)D^s_+[u](x)\,dx, \quad \forall u \in BV^s, v \in W^{1,1}_0(0,1),$$
$$\int_0^1 \big(D^s_-\big)^*[v](x)u(x)\,dx = \int_0^1 v(x)D^s_-[u](x)\,dx, \quad \forall u \in BV^s, v \in W^{1,1}_0(0,1),$$
$$\int_0^1 \big(D^s\big)^*[v](x)u(x)\,dx = \int_0^1 v(x)D^s[u](x)\,dx, \quad \forall u \in BV^s, v \in W^{1,1}_0(0,1),$$

where $W^{1,1}_0(0,1)$ *is the completion of* $C^\infty_0(0,1)$ *in norm* $W^{1,1}_G(0,1)$, *and fulfil*

$$\big(D^s_+\big)^*[v] = -I^{1-s}_-[D_x v] \quad \forall v \in W^{1,1}_0(0,1), \tag{3.15}$$
$$\big(D^s_-\big)^*[v] = -I^{1-s}_+[D_x v] \quad \forall v \in W^{1,1}_0(0,1). \tag{3.16}$$

Proof For every $u, v \in L^1(0,1)$ we obtain:

$$\begin{aligned}\int_0^1 \big(I^s_+\big)^*[v](x)u(x)\,dx &= \int_0^1 v(x)I^s_+[u](x)\,dx\\ &= \frac{1}{\Gamma(s)}\int_0^1 v(x)\bigg(\int_0^x \frac{u(t)}{(x-t)^{1-s}}\,dt\bigg)\,dx\\ &= \frac{1}{\Gamma(s)}\int_0^1 u(t)\bigg(\int_t^1 \frac{v(x)}{(x-t)^{1-s}}\,dx\bigg)\,dt\\ &= \int_0^1 I^s_-[v](t)u(t)\,dt,\end{aligned}$$

thus the first equality in (3.14) is proved. The second one follows by analogous computations. Together, they entail the third one, since $I^s = 1/2(I^s_+ + I^s_-)$.

For every $u \in W^{s,1}(0,1)$ and $v \in W^{1,1}_0(0,1)$, by $v(0) = v(1) = 0$ and (3.14), we obtain:

$$\begin{aligned}\int_0^1 D^s_+[u](x)v(x)\,dx &= \int_0^1 D_x I^{1-s}_+[u](x)v(x)\,dx\\ &= -\int_0^1 I^{1-s}_+[u](x)D_x v(x)\,dx + \big[I^{1-s}_+[u](x)v(x)\big]^{x=0}_{x=1}\\ &= -\int_0^1 I^{1-s}_+[u](x)D_x v(x)\,dx = -\int_0^1 u(x)I^{1-s}_-[D_x v](x)\,dx.\end{aligned}$$

Thus, (3.15) is proved, and (3.16) can be shown by analogous computations. □

Lemma 3.9 *The unique minimizer z of problem* ($\mathcal{P}_{\mathcal{F}}$) *fulfils in* $\mathcal{D}'(0,1)$

$$\lambda(g-z) \in \big(D^s_+\big)^*\big[\partial T_+\big(D^s_+[z]\big)\big] + \big(D^s_-\big)^*\big[\partial T_-\big(D^s_-[z]\big)\big], \tag{3.17}$$

explicitly

$$\lambda(z-g) \in I^{1-s}_-\big[D_x\big[\partial T_+\big(D^s_+[z]\big)\big]\big] + I^{1-s}_+\big[D_x\big[\partial T_-\big(D^s_-[z]\big)\big]\big], \tag{3.18}$$

where T_+, T_- are defined in (3.4) and (3.5), ∂J denotes the subdifferential of a convex functional J, and the right-hand sides in both inclusions (3.17) and (3.18) have to be understood with restriction to the selections of ∂T leading to local integrability for both $D^s_+[\partial T_+[z]]$ and $D^s_-[\partial T_-[z]]$.

Proof The subdifferential of a proper convex functional on a Banach space, as is the case for $\mathcal{F}$ on $BV^s(0,1)$, is a (possibly multivalued) maximal monotone operator with values in the dual space ([9]), it is coincident with the Frechet derivative for C^1 functionals and if there are minimizers then the null element of dual space belongs to the subdifferential evaluated at every minimizer.

By performing smooth variations φ times a real constant ε around a minimizer z we obtain $0 \in \partial\mathcal{F}(z)$ and

$$\langle \partial\mathcal{F}(z), \varphi\rangle = \int_0^1 \lambda(z-g)\varphi\, dx + \langle \partial T_+\big[D^s_+[z]\big], D^s_+[\varphi]\rangle + \langle \partial T_-\big[D^s_-[z]\big], D^s_-[\varphi]\rangle$$

and (3.17) is proved. Hence, (3.15) and (3.16) entail (3.18). □

Remark 3.10 We emphasize that if $\mathcal{F}(z) < +\infty$ and $D^s_+[z] \equiv 0$ then $D^s_-[z] \equiv 0$ too (due to Remark 3.2). Analogously, $\mathcal{F}(z) < +\infty$ and $D^s_-[z] \equiv 0$ entail $D^s_+[z] \equiv 0$.

Summarizing, if $\mathcal{F}(z) < +\infty$ and, either $D^s_+[z] \equiv 0$ or $D^s_-[z] \equiv 0$, then $z = 0$.

Remark 3.11 If $z \in BV^s(0,1)$ solves (3.17), $I^{1-s}_+[z] \in C^1(-\infty,1)$, $I^{1-s}_-[z] \in C^1(0,+\infty)$ and z fulfils both conditions $D^s_+[z](x) \neq 0$ and $D^s_-[z](x) \neq 0$ for all $x \in (0,1)$, then

$$\lambda(z-g) \in I^{1-s}_-\left[D_x\left[\frac{D^s_+[z]}{|D^s_+[z]|}\right]\right] + I^{1-s}_+\left[D_x\left[\frac{D^s_-[z]}{|D^s_-[z]|}\right]\right].$$

Lemma 3.12 *The unique minimizer w of problem $(\mathcal{P}^+_{\mathcal{E}})$ fulfils*

$$\lambda\big(I^s_+\big)^*\big[g - I^s_+[w]\big] \in \partial T_+(w) + \big(I^s_+\big)^*\big(D^s_-\big)^*\partial T_-\big(D^s_-\big[I^s_+[w]\big]\big), \tag{3.19}$$

or, equivalently

$$\lambda I^s_-\big[g - I^s_+[w]\big] \in \partial T_+(w) - I^s_-\big[I^{1-s}_+\big[D_x\big(\partial T_-\big(D^s_-\big[I^s_+[w]\big]\big)\big)\big]\big].$$

Proof The same argument is applied as in the proof of Lemma 3.9. □

4 FTV : symmetrised fractional total variation model (2-dimensional)

We consider an image in the domain $\Omega = (0,1) \times (0,1)$. About notation, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively, denote the distributional partial derivatives with respect to x and y.

4.1 Fractional integral and fractional derivative in 2D

Definition 4.1 (Fractional partial integrals) Set $\Omega = (0,1) \times (0,1)$, $u \in L^1(\Omega)$. For every $0 < s < 1$, we define the partial left-side and right-side Riemann–Liouville fractional integrals, by setting, respectively,

$$I^s_{x,+}[u](x,y) := \frac{1}{\Gamma(s)} \int_0^x \frac{u(t,y)}{(x-t)^{1-s}}\, dt, \quad 0 \le x \le 1,$$

$$I^s_{x,-}[u](x,y) := \frac{1}{\Gamma(s)} \int_x^1 \frac{u(t,y)}{(t-x)^{1-s}}\, dt, \quad 0 \le x \le 1,$$
$$I^s_{y,+}[u](x,y) := \frac{1}{\Gamma(s)} \int_0^y \frac{u(x,t)}{(y-t)^{1-s}}\, dt, \quad 0 \le y \le 1,$$
$$I^s_{y,-}[u](x,y) := \frac{1}{\Gamma(s)} \int_y^1 \frac{u(x,t)}{(t-y)^{1-s}}\, dt, \quad 0 \le y \le 1.$$

Next, starting from the 1D case, see [25, 26] and [33], we introduce a new definition of left and right Riemann–Liouville fractional partial derivatives.

Definition 4.2 (Distributional fractional partial derivative) Assume that $u \in L^1(\Omega)$ and $0 < s < 1$, then the left and right fractional partial derivatives of u at $(x,y) \in \Omega$ are defined by

$$\left(\frac{\partial}{\partial x}\right)^s_+ [u](x,y) := \frac{\partial}{\partial x}\big(I^{1-s}_{x,+}[u]\big)(x,y) = \frac{1}{\Gamma(1-s)} \frac{\partial}{\partial x} \int_{-\infty}^x \frac{u(t,y)}{(x-t)^s}\, dt, \tag{4.1}$$
$$\left(\frac{\partial}{\partial x}\right)^s_- [u](x,y) := -\frac{\partial}{\partial x}\big(I^{1-s}_{x,-}[u]\big)(x,y) = \frac{-1}{\Gamma(1-s)} \frac{\partial}{\partial x} \int_x^{+\infty} \frac{u(t,y)}{(t-x)^s}\, dt, \tag{4.2}$$
$$\left(\frac{\partial}{\partial y}\right)^s_+ [u](x,y) := \frac{\partial}{\partial y}\big(I^{1-s}_{y,+}[u]\big)(x,y) = \frac{1}{\Gamma(1-s)} \frac{\partial}{\partial y} \int_{-\infty}^y \frac{u(x,t)}{(y-t)^s}\, dt, \tag{4.3}$$
$$\left(\frac{\partial}{\partial y}\right)^s_- [u](x,y) := -\frac{\partial}{\partial y}\big(I^{1-s}_{y,-}[u]\big)(x,y) = \frac{-1}{\Gamma(1-s)} \frac{\partial}{\partial y} \int_y^{+\infty} \frac{u(x,t)}{(t-y)^s}\, dt. \tag{4.4}$$

4.2 Description and analysis of the 2-dimensional FTV model

Set

$$\|\cdot\|_{T_{x,+}} = \|\cdot\|_{\mathcal{M}([0,1)_x)}, \qquad \|\cdot\|_{T_{x,-}} = \|\cdot\|_{\mathcal{M}((0,1]_x)},$$
$$\|\cdot\|_{T_{y,+}} = \|\cdot\|_{\mathcal{M}([0,1)_y)}, \qquad \|\cdot\|_{T_{y,-}} = \|\cdot\|_{\mathcal{M}((0,1]_y)} \quad \text{and}$$
$$\Omega_{x,+} = [0,1) \times (0,1), \qquad \Omega_{x,-} = (0,1] \times (0,1),$$
$$\Omega_{y,+} = (0,1) \times [0,1), \qquad \Omega_{y,-} = (0,1) \times (0,1].$$

Find u minimizing

$$\begin{aligned} \mathcal{F}(u) := {}& \frac{\lambda}{2} \int_\Omega \big|u(x,y) - g(x,y)\big|^2\, dx\, dy \\ & + \int_0^1 \left\| \left(\frac{\partial}{\partial x}\right)^s_+ [u] \right\|_{T_{x,+}} dy + \int_0^1 \left\| \left(\frac{\partial}{\partial y}\right)^s_+ [u] \right\|_{T_{y,+}} dx \\ & + \int_0^1 \left\| \left(\frac{\partial}{\partial x}\right)^s_- [u] \right\|_{T_{x,+}} dy + \int_0^1 \left\| \left(\frac{\partial}{\partial y}\right)^s_- [u] \right\|_{T_{y,+}} dx \\ = {}& \frac{\lambda}{2} \|u-g\|^2_{L^2(\Omega)} \\ & + \left| \left(\frac{\partial}{\partial x}\right)^s_+ \right|_{T(\Omega_{x,+})} + \left| \left(\frac{\partial}{\partial y}\right)^s_+ \right|_{T(\Omega_{x,-})} + \left| \left(\frac{\partial}{\partial x}\right)^s_- \right|_{T(\Omega_{y,+})} + \left| \left(\frac{\partial}{\partial y}\right)^s_- \right|_{T(\Omega_{y,-})} \end{aligned} \tag{4.5}$$

among $u \in L^2(\Omega)$ s.t. $(\frac{\partial}{\partial x})^s_+[u](\cdot,y)$ and $(\frac{\partial}{\partial x})^s_-[u](\cdot,y)$ are measures of bounded variation for a.e. y, and $(\frac{\partial}{\partial y})^s_+[u](x,\cdot)$ and $(\frac{\partial}{\partial x})^s_-[u](x,\cdot)$ are measures of bounded variation for a.e. x, hence $(\frac{\partial}{\partial x})^s_+[u]$, $(\frac{\partial}{\partial x})^s_-[u]$, $(\frac{\partial}{\partial y})^s_+[u]$ and $(\frac{\partial}{\partial x})^s_-[u]$ are measures of bounded variation, respectively, on $\Omega_{x,+}$, $\Omega_{x,-}$, $\Omega_{y,+}$, and $\Omega_{y,-}$.

Here, $|\mu|_{T(A)}$ denote the total variation of μ on A, for $A \subset \mathbb{R}^2$ Borel set.

First, we note that the sublevels of the functional $\mathcal{F}$ are sequentially compact with respect to the natural topology of the domain.

Lemma 4.3 *Assume that $\mathcal{F}(v_k) \le C < +\infty$. Then, there is $v \in L^2(\Omega)$ with $(\frac{\partial}{\partial x})^s_\pm[v]$ and $(\frac{\partial}{\partial y})^s_\pm[v]$ that are measures of bounded variation, respectively, on $\Omega_{x,\pm}$ and $\Omega_{y,\pm}$, and there is a subsequence v_{h_k} such that*

$$v_{h_k} \rightharpoonup v \quad \text{weakly in } L^2(\Omega), \tag{4.6}$$

$$\left(\frac{\partial}{\partial x}\right)^s_+ [v_{h_k}] \rightharpoonup \left(\frac{\partial}{\partial x}\right)^s_+ [v] \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega), \tag{4.7}$$

$$\left(\frac{\partial}{\partial x}\right)^s_- [v_{h_k}] \rightharpoonup \left(\frac{\partial}{\partial x}\right)^s_- [v] \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega), \tag{4.8}$$

$$\left(\frac{\partial}{\partial y}\right)^s_+ [v_{h_k}] \rightharpoonup \left(\frac{\partial}{\partial y}\right)^s_+ [v] \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega), \tag{4.9}$$

$$\left(\frac{\partial}{\partial y}\right)^s_- [v_{h_k}] \rightharpoonup \left(\frac{\partial}{\partial y}\right)^s_- [v] \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega). \tag{4.10}$$

Proof Due to standard compactness properties, by subsequent extractions, we find v_{h_k}, v, $\mu_{x,+}$, $\mu_{x,-}$, $\mu_{y,+}$, $\mu_{y,-}$ s.t. (4.6) holds true together with

$$\left(\frac{\partial}{\partial x}\right)^s_+ [v_{h_k}] \rightharpoonup \mu_{x,+} \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega),$$

$$\left(\frac{\partial}{\partial x}\right)^s_- [v_{h_k}] \rightharpoonup \mu_{x,-} \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega),$$

$$\left(\frac{\partial}{\partial y}\right)^s_+ [v_{h_k}] \rightharpoonup \mu_{y,+} \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega),$$

$$\left(\frac{\partial}{\partial y}\right)^s_- [v_{h_k}] \rightharpoonup \mu_{y,-} \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega).$$

$v_{h_k} \rightharpoonup v$ in $L^2(\Omega)$ entails $v_{h_k} \to v$ in $\mathcal{D}'(\Omega)$, hence $I_+^{1-s}[v_{h_k}] \to I_+^{1-s}[v]$ in $\mathcal{D}'(\Omega)$ and

$$\left(\frac{\partial}{\partial x}\right)^s_+ [v_{h_k}] = \frac{\partial}{\partial x} I_+^{1-s}[v_{h_k}] \to \frac{\partial}{\partial x} I_+^{1-s}[v] = \mu_{x,+} \quad \text{in } \mathcal{D}'(\Omega),$$

hence (4.7) is proved.

Convergences (4.8), (4.9), and (4.10) can be shown by the same argument. □

Theorem 4.4 *Assume $g \in L^2(\Omega)$. Then, the functional (4.5) achieves a finite minimum and the minimizer is unique.*

Proof It is a consequence of Lemma 4.3, since all terms are lower semicontinuous, respectively, in the convergences warranted by the lemma. Uniqueness follows by convexity of the functional and strict convexity of the fidelity term in L^2. □

5 Comparison of FTV model with the Zhang–Chen model

A few authors have introduced and studied isotropic fractional models for signal filtering.

In particular Zhang–Chen in [35] chose the domain for the competing functions as the space $V^s_+ \cap V^s_- \cap L^2$, where

$$V^s_\pm := \left\{ v \in L^1 : \sup_{\varphi \in C^1_0, |\phi| \le 1} \int v \operatorname{div}^s_+ \phi \right\}, \quad \text{with } \operatorname{div}^s_\pm \phi = \sum_j (D_j)^s_\pm \phi_j,$$

whereas in the present article we chose $BV^s \cap L^2$, where

$$BV^s = \left\{ v \in L^1 : \text{both } I^{1-s}_+[v] \text{ and } I^{1-s}_-[v] \text{ belong to } BV(0,1) \right\}.$$

Comparison of domains is quite technical, since pointwise values (defined by integral averages and distributional derivatives) could be undefined at boundary points.

However, the space $W^{s,1}$ is contained in both BV^s and V^s.

Indeed, if $v \in W^{s,1}$ then one can evaluate its classical Caputo derivatives obtaining (rephrasing (3.2) of [35] in 1D, for simplicity, say considering D^s_+ in place of div^s_+) with notation ${}_{RL}D^s_\pm$ in place of $D^s_\pm$ to emphasize the Riemann–Liouville versus Caputo fractional operators:

$${}_C D^s_+[v](x) = {}_{RL}D^s_+[v](x) - \frac{v(0)}{\Gamma(1-s)}(x-a)^{-s}, \quad 0 < s < 1, v \in W^{s,1}, \tag{5.1}$$

$${}_C D^s_-[v](x) = {}_{RL}D^s_-[v](x) - \frac{v(1)}{\Gamma(1-s)}(b-x)^{-s}, \quad 0 < s < 1, v \in W^{s,1}, \tag{5.2}$$

and, by exploiting (3.15) and (3.16), we obtain

$$-\int_0^1 v\big({}_C D^s_+[\varphi]\big) = \int_0^1 \big({}_{RL}D^s_-[v]\big)\varphi, \quad 0 < s < 1, v \in W^{s,1}, \varphi \in C^1_0, \tag{5.3}$$

$$-\int_0^1 v\big({}_C D^s_-[\varphi]\big) = \int_0^1 \big({}_{RL}D^s_+[v]\big)\varphi, \quad 0 < s < 1, v \in W^{s,1}, \varphi \in C^1_0. \tag{5.4}$$

Hence, for every $v \in W^{s,1}$,

$$\sup\left\{ \int_0^1 v\big({}_{RL}D^s_+[\varphi]\big) : \varphi \in C^1_0, |\varphi| \le 1 \right\} = \left| {}_C D^s_-[v] \right|_{\mathcal{M}(0,1)}, \tag{5.5}$$

$$\sup\left\{ \int_0^1 v\big({}_{RL}D^s_-[\varphi]\big) : \varphi \in C^1_0, |\varphi| \le 1 \right\} = \left| {}_C D^s_+[v] \right|_{\mathcal{M}(0,1).} \tag{5.6}$$

Note that all dualities above actually are Lebesgue integrals, due to the regularity of v.

Remark 5.1 Nearly all previous models have an intrinsic anisotropic formulation; however, we already emphasized examples and theoretical motivations for the quest of symmetrical formulations (see Remarks 3.1, 3.2, and 3.4 and Example 3.3).

Numerical experiments show asymmetric reconstruction of symmetric data. Indeed, also Zhang–Chen who introduced another isotropic approach, showed numerical experiments of comparison with anisotropic models: for instance, referring to TGV in the caption of Fig. 1(b) of [35], one can see how TGV, though achieving a good fidelity to the datum, exhibits an asymmetric image reconstruction of the reversed parabola synthetic datum, when exploiting D^s_+ only.

6 Conclusions

In this paper, we introduce and study a variational model for signal and image analysis based on Riemann–Liouville fractional derivatives and aiming to denoise images with textures.

We call it the isotropic FTV model, since it is based on Fractional Total Variation of restored images: it relies on the minimization of a suitably weighted quadratic fidelity term with a regularizing term corresponding to the sum of total variations of both left and right fractional derivatives. The presence of both derivatives provides an orientation-independent analysis. We emphasize that, as long as fractional derivatives are involved, the isotropy is not for free but must be imposed, since both right and left derivatives have different nontrivial kernels.

Model analysis is performed first in one dimension, then in two dimensions.

We prove that there exists a unique minimizer in the space of bilateral Fractional Bounded Variation space BV^s that we have introduced in a previous paper, and we show several optimality conditions fulfilled by the minimizers.

Appendix

We collect here some useful identities that were exploited within the paper.

Assume $0 < x < 1$, $0 < \alpha < 1$, $0 < \beta < 1$. Then,

$$I_{1-}^{1-\alpha}\left[x^{\beta}\right](x) = \frac{1}{\Gamma(1-\alpha)} x^{1+\beta-\alpha} \int_x^1 w^{\alpha-\beta-2}(1-w)^{-\alpha}\, dw. \tag{A.1}$$

It is worthwhile noting that, as $x \to 0_+$, the infinitesimal $x^{1+\beta-\alpha}$ is multiplied by a divergent integral. Thus, in the limit as $x \to 0_+$, formula (A.1) recovers

$$I_{1-}^{1-\alpha}\left[x^{\beta}\right](0) = \frac{1}{(1+\beta-\alpha)\Gamma(1-\alpha)}. \tag{A.2}$$

Proof

$$\begin{aligned}
I_{1-}^{1-\alpha}\left[x^{\beta}\right] &= \frac{1}{\Gamma(1-\alpha)} \int_x^1 \frac{t^{\beta}}{(t-x)^{\alpha}}\, dt \\
&=^{\{\tau=1-t,\ dt=-d\tau\}} \frac{1}{\Gamma(1-\alpha)} \int_{1-x}^0 \frac{(1-\tau)^{\beta}}{(1-x-\tau)^{\alpha}} (-1)\, d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^{1-x} (1-\tau)^{\beta} \big((1-\tau)-x\big)^{-\alpha}\, d\tau \\
&= \frac{x^{\beta-\alpha}}{\Gamma(1-\alpha)} \int_0^{1-x} \left(\frac{1-\tau}{x}\right)^{\beta} \left(\left(\frac{1-\tau}{x}\right)-1\right)^{-\alpha} d\tau
\end{aligned}$$

$$= _{\left\{\substack{y=(1-\tau)/x,\\ d\tau=-x\,dy,\\ 1<y<1/x}\right\}} \frac{x^{1+\beta-\alpha}}{\Gamma(1-\alpha)} \int_1^{1/x} y^{\beta}(y-1)^{-\alpha}\,dy$$

$$= _{\{y=1/w,\ dy=(-1/w^2)\,dw\}} \frac{x^{1+\beta-\alpha}}{\Gamma(1-\alpha)} \int_x^{1} w^{\alpha-\beta-2}(1-w)^{-\alpha}\,dw. \qquad \square$$

By choosing $\alpha = s$, $\beta = s-1$ we obtain

$$D_-^s\big[x^{s-1}\big](x) = \frac{1}{\Gamma(1-\alpha)} D_x \int_x^1 w^{-1}(1-w)^{-s}\,dw \not\equiv 0, \tag{A.3}$$

hence $D_-^s[x^{s-1}] \notin \mathcal{M}$, that is $x^{s-1} \notin BV^s$, whereas $\ker D_-^s = K(1-x)^{s-1}$.

Lemma A.1 *By setting* $\widetilde{w}(x) = w(1-x)$, *we have*

$$D_-^s[\widetilde{w}](x) = -D_+^s[w](1-x), \tag{A.4}$$

$$D_+^s[\widetilde{w}](x) = -D_-^s[w](1-x). \tag{A.5}$$

Proof We show (A.4):

$$D_-^s[\widetilde{w}](x) = -\frac{1}{\Gamma(1-s)} D_x \int_x^1 \frac{w(1-t)}{(t-x)^s}\,dt$$

$$= -\frac{1}{\Gamma(1-s)} D_x \int_{1-x}^0 \frac{w(\tau)}{((1-x)-\tau)^s}(-d\tau) = -D_+^s[w](1-x).$$

Equation (A.5) can be proved by an analogous change of variable. $\square$

Acknowledgements
We thank Maïtine Bergounioux for many helpful discussions.

Funding
The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This research was partially funded by the Italian M.U.R. PRIN: grant number 2017BTM7SN "Variational Methods for stationary and evolution problems with singularities and interfaces" and by the Italian M.U.R. PRIN: grant number 2017KC8WMB "Development of a UV imaging system in liquid argon detectors for neutrino, particle, and medical physics applications."

Availability of data and materials
This manuscript has no associated data or the data will not be deposited.

Declarations

Ethics approval and consent to participate
The authors approve the ethics of the journal and give the consent to participate.

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce, Italy.
[2]Politecnico di Milano, Dipartimento di Matematica, Milano, Italy.

Publisher's Note

Received: 15 November 2022 Accepted: 19 February 2023 Published online: 07 March 2023

References

1. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs. Oxford University Press, Oxford (2000)
2. Attouch, H., Buttazzo, G., Michaille, G.: Variational Analysis in Sobolev and BV Spaces. SIAM Series on Optimization. SIAM and MPS, Philadelphia (2006)
3. Bergounioux, M.: Mathematical analysis of a inf-convolution model for image processing. J. Optim. Theory Appl. **168**, 1–21 (2016)
4. Bergounioux, M., Leaci, A., Nardi, G., Tomarelli, F.: Fractional Sobolev spaces and functions of bounded variation of one variable. Fract. Calc. Appl. Anal. **20**(4), 936–962 (2017)
5. Bergounioux, M., Piffet, L.: A second-order model for image denoising. Set-Valued Var. Anal. **18**, 277–306 (2010)
6. Bessas, K., Stefani, G.: Non-local *BV* functions and a denoising model with L^1 fidelity. Preprint (2022). http://cvgmt.sns.it/paper/5768/, https://arxiv.org/abs/2210.11958
7. Boccellari, T., Tomarelli, F.: Generic uniqueness of minimizer for Blake & Zisserman functional. Rev. Mat. Complut. **26**, 361–408 (2013)
8. Bredies, K., Kunisch, K., Pock, T.: Total generalized variation. SIAM J. Imaging Sci. **3**(3), 492–526 (2010)
9. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)
10. Carbotti, A., Comi, G.E.: A note on Riemann–Liouville fractional Sobolev spaces. Commun. Pure Appl. Anal. **20**(1), 17–54 (2021)
11. Carriero, M., Leaci, A., Tomarelli, F.: A second order model in image segmentation: Blake & Zisserman functional. In: Serapioni, R., Tomarelli, F. (eds.) Variational Methods for Discontinuous Structures. Progress in Nonlinear Differential Equations and Their Applications, vol. 25, pp. 57–72. Birkäuser, Basel (1996)
12. Carriero, M., Leaci, A., Tomarelli, F.: Necessary conditions for extremals of Blake & Zisserman functional. C. R. Math. Acad. Sci. Paris **334**(4), 343–348 (2002)
13. Carriero, M., Leaci, A., Tomarelli, F.: Euler equations for Blake & Zisserman functional. Calc. Var. Partial Differ. Equ. **32**, 81–110 (2008). https://doi.org/10.1007/s00526-007-0129-2
14. Carriero, M., Leaci, A., Tomarelli, F.: A candidate local minimizer of Blake & Zisserman functional. J. Math. Pures Appl. **96**, 58–87 (2011). https://doi.org/10.1016/j.matpur.2011.01.005
15. Carriero, M., Leaci, A., Tomarelli, F.: Uniform density estimates for Blake & Zisserman functional. Discrete Contin. Dyn. Syst., Ser. A **31**(4), 1129–1150 (2011)
16. Carriero, M., Leaci, A., Tomarelli, F.: A survey on Blake–Zisserman functional. Milan J. Math. **83**(2), 397–420 (2015)
17. Carriero, M., Leaci, A., Tomarelli, F.: Segmentation and inpainting of color images. J. Convex Anal. **25**, 435–458 (2018)
18. Chan, R.H., Lanza, A., Morigi, S., Sgallari, F.: An adaptive strategy for the restoration of textured images using fractional order regularization. Numer. Math., Theory Methods Appl. **6**(4), 276–296 (2013)
19. Chen, D., Chen, Y., Xue, D.: Fractional-order total variation image denoising based on proximity algorithm. Appl. Math. Comput. **257**, 537–545 (2015)
20. Colombo, F., Gantner, J.: Quaternionic Closed Operators, Fractional Powers and Fractional Diffusion Processes. Operator Theory: Advances and Applications, vol. 274. Birkhäuser/Springer, Cham (2019)
21. Comi, G.E., Spector, D., Stefani, G.: The fractional variation and the precise representative of $BV^{\alpha,p}$ functions. Fract. Calc. Appl. Anal. **25**, 520–558 (2022)
22. De Giorgi, E., Carriero, M., Leaci, A.: Existence theorem for a minimum problem with free discontinuity set. Arch. Ration. Mech. Anal. **108**, 195–218 (1989)
23. Ghamisi, P., Couceiro, M.S., Benediktsson, J.A., Ferreira, N.M.F.: An efficient method for segmentation of images based on fractional calculus and natural selection. Expert Syst. Appl. **39**(16), 12404–12417 (2012)
24. Hinterberger, W., Scherzer, O.: Variational methods on the space of functions of bounded Hessian for convexification and denoising. Computing **76**, 109–133 (2006)
25. Leaci, A., Tomarelli, F.: Bilateral Riemann–Liouville fractional Sobolev spaces. Note Mat. **41**(2), 61–83 (2021)
26. Leaci, A., Tomarelli, F.: Riemann–Liouville fractional Sobolev and bounded variation spaces. Axioms **11**(1), 30 (2022)
27. Nakib, A., Oulhadj, H., Siarry, P.: A thresholding method based on two-dimensional fractional differentiation. Image Vis. Comput. **27**, 1343–1357 (2009)
28. Oldham, K., Myland, J., Spanier, J.: An Atlas of Functions, 2nd edn. Springer, Berlin (2009)
29. Pu, Y., Siarry, P., Zhou, J., Zhang, N.: A fractional partial differential equation based multiscale denoising model for texture image. Math. Methods Appl. Sci. **37**(12), 1784–1806 (2014)
30. Pu, Y., Zhou, J., Siarry, P., Zhang, N., Liu, Y.: Fractional partial differential equation: fractional total variation and fractional steepest descent approach-based multiscale denoising model for texture image. Abstr. Appl. Anal. **2013**, Article ID 483791 (2013). https://doi.org/10.1155/2013/483791
31. Pu, Y., Zhou, J., Yuan, X.: Fractional differential mask: a fractional differential based approach for multiscale texture enhancement. IEEE Trans. Image Process. **19**(2), 491–511 (2010)
32. Rudin, L.I., Osher, S., Fatemi, E.: Nonlinear total variation based noise removal algorithms. Phys. D, Nonlinear Phenom. **60**(14), 259–268 (1992)
33. Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives—Theory and Applications. Gordon & Breach, New York (1993)
34. You, J., Hungnahally, S., Sattar, A.: Fractional discrimination for texture image segmentation. In: Proceedings of International Conference on Image Processing, vol. 3 (1997)
35. Zhang, J., Chen, K.: A total fractional-order variation model for image restoration with nonhomogeneous boundary conditions and its numerical solution. SIAM J. Imaging Sci. **8**(4), 2487–2518 (2015)