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Critical metrics for quadratic curvature functionals on some solvmanifolds

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Abstract

We prove the existence of four-dimensional compact manifolds admitting some non-Einstein Lorentzian metrics, which are critical points for all quadratic curvature functionals. For this purpose, we consider left-invariant semi-direct extensions $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}\mathcal{S})$ of the Heisenberg Lie group H, for any $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$, equipped with a family g_a of left-invariant metrics. After showing the existence of lattices in all these four-dimensional solvable Lie groups, we completely determine when g_a is a critical point for some quadratic curvature functionals. In particular, some four-dimensional solvmanifolds raising from these solvable Lie groups admit non-Einstein Lorentzian metrics, which are critical for all quadratic curvature functionals.

Keywords Quadratic curvature functionals · Solvmanifolds · Heisenberg group · Semi-direct extensions

Mathematics Subject Classification 53C50 · 53C21 · 35A01

1 Introduction

Let M^n denote a closed oriented manifold and $\mathcal{M}_1$ the space of Riemannian metrics of volume one on M. A well known problem consists in determining within $\mathcal{M}_1$ the critical points for a given curvature functional.

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Let R, ϱ, τ respectively denote the Riemann curvature tensor, the Ricci tensor and the scalar curvature of a given metric g. The Euler-Lagrange equations associated to the Einstein-Hilbert functional $g \mapsto \int_M \tau dvolg$ are given by $\varrho = \lambda g$, for some real constant λ, so that critical metrics on $\mathcal{M}_1$ for the Einstein-Hilbert functional coincide with the Einstein ones.

The next natural step is to consider functionals defined by scalar quadratic curvature invariants. In Riemannian settings this topic started in [1], has been intensively studied and is a very active field of research (see for example [4, 10, 12, 17–22, 27, 28, 30] and references therein). We report here some essential information on the study of quadratic curvature invariants, referring to [10] and [30] for excellent surveys on the topic. A basis for the space of quadratic curvature invariants is given by $\{\Delta\tau, \tau^2, ||\varrho||^2, ||R||^2\}$ so that, correspondingly, an arbitrary quadratic curvature functional has the form

$$g \mapsto \int_M \left(a||R||^2 + b||\varrho||^2 + c\tau^2\right) dvol_g,$$

for some real constants a, b, c. In this general framework, the four-dimensional case carries some special features and interest, as these functionals also arise naturally in some gravitational field theories (see for example [11]). When $\dim M = 4$, the Gauss-Bonnet Theorem yields

$$32\pi^2\chi(M) = \int_M \left(||R||^2 - 4||\varrho||^2 + \tau^2\right) dvol_g.$$

Consequently, in dimension four, all quadratic curvature functionals are equivalent to

$$\mathcal{S}(g) = \int_M \tau^2 dvol_g, \qquad \mathcal{F}_t(g) = \int_M \left(||\varrho||^2 + t\tau^2\right) dvol_g, \quad t \in \mathbb{R}.$$

As already observed in [2], Einstein metrics are critical for $\mathcal{F}_t$ on $\mathcal{M}_1$ for every $t \in \mathbb{R}$. In general, critical metrics for quadratic curvature functionals need not be Einstein. For example, Bach-flat metrics are critical points for $\mathcal{F}_{-1/3}$, and Weyl metrics of vanishing scalar curvature are critical points for $\mathcal{F}_{-1/4}$ (and for $\mathcal{S}$). Moreover, it may be observed that although this problem has been extensively studied for Riemannian metrics, its formulation is also possible in different signatures, and it leads to the same Euler-Lagrange equations. These remarks lead naturally to the following questions:

(1) Do there exist conditions under which a critical metric for $\mathcal{F}_t$ is necessarily Einstein?
(2) Do there exist non-Einstein critical metrics for all quadratic curvature functionals?
(3) What happens considering metrics of different signatures?

The first two of the above questions have been studied by several authors. In particular, suitable curvature conditions forcing critical metrics to be Einstein were found in [10]. A positive answer to the second of the above questions was obtained in [4]: there exist four-dimensional Riemannian metrics which are critical for all quadratic curvature invariants but are not Einstein. The examples investigated in [4] are non-flat cones $R^+ \times_r N$, where N is a three-dimensional Einstein manifold of constant sectional curvature -3. We may observe that by their own construction, these examples are not compact.

On the other hand, up to our knowledge, the third question is just starting to attract the interest of researchers (see for example [5]) and has not been extensively investigated yet. The aim of this paper is to contribute to this general topic, with particular regard to the third of the above questions, proving that some compact four-dimensional manifolds admit non-Einstein Lorentzian metrics, which are critical points for all quadratic curvature functionals. These compact manifolds naturally arise as *solvmanifolds*, i.e., compact quotients of solvable Lie groups with respect to some lattices.

Following [26], where semi-direct extensions of the Heisenberg group (of arbitrary dimension) were introduced, let H denote the three-dimensional Heisenberg group and $\mathfrak{h} = \mathrm{span}\{X, Y, U\}$ its Lie algebra, with $[X, Y] = U$.

Each matrix $\mathcal{S}$ belonging to the Lie algebra $\mathfrak{sp}(1, \mathbb{R})$ of the symplectic group $Sp(1, \mathbb{R})$ on $\mathbb{R}^2$, defines a derivation S of $\mathfrak{h}$, given by

$$[S, (z, u)] = (\mathcal{S}(z), 0)$$

and so, a corresponding one-dimensional semi-direct extension $\mathfrak{g} = \mathfrak{h} \rtimes (\mathbb{R}S)$ of $\mathfrak{h}$.

We denote by $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ the connected, simply connected Lie group corresponding to $\mathfrak{h} \rtimes (\mathbb{R}S)$.

It is easy to check that $\mathcal{S} \in \mathbb{R}^{2,2}$ satisfies $\mathcal{S}^t \circ J + J \circ \mathcal{S} = 0$ if and only if

$$\mathcal{S} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \tag{1.1}$$

for some real constants α, β, γ and the Lie algebra $\mathfrak{h} \rtimes (\mathbb{R}S)$=span$\{U, X, Y, S\}$ is completely described by the following brackets:

$$[X, Y] = U, \qquad [S, X] = \alpha X + \gamma Y, \qquad [S, Y] = \beta X - \alpha Y. \tag{1.2}$$

In particular, for

$$\mathcal{S} = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}, \qquad \mu > 0,$$

we find the well known *oscillator algebra*. The corresponding (four-dimensional) *oscillator group* [29] admits a bi-invariant metric g_0, which has been generalized to a one-parameter family g_a, $a^2 < 1$, of left-invariant metrics. Setting $U = e_1$, $X = e_2$, $Y = e_3$ and $S = e_4$, this family of metrics is described by

$$\langle e_1, e_1\rangle = \langle e_4, e_4\rangle = a, \quad \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = \langle e_1, e_4\rangle = \langle e_4, e_1\rangle = 1, \langle e_i, e_j\rangle = 0 \text{ otherwise.} \tag{1.3}$$

Since its introduction in [16] as a generalization of previous examples studied in [24], this family of metrics has been intensively studied in different contexts (see for example [6], [8] and references therein). We remark that the above equations (1.3) define a metric for any real constant a satisfying $a^2 \neq 1$. In particular, these metrics are Lorentzian if $a^2 < 1$ and Riemannian when $a^2 > 1$.

Generalizing this example, one can consider $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$, equipped with an arbitrary left-invariant metric g_a of the form (1.3) (see [7], [9]).

In this paper we shall completely classify left-invariant metrics g_a on $G_{\mathcal{S}}$, which are critical points for some quadratic curvature functionals. In particular, we will prove that whatever the form of the defining matrix $\mathcal{S}$, the Lorentzian metric g_0 is a critical point for all quadratic curvature functionals. We will also show that all Lie groups $G_{\mathcal{S}}$ admit some lattices, that is, discrete co-compact subgroups, so that they give rise to solvmanifolds. As a consequence, we prove the existence of some non-Einstein Lorentzian critical metrics for all quadratic curvature invariants on some compact four-manifolds.

The paper is organized in the following way. In Sect. 2 we shall provide some basic information on Lie groups $G_{\mathcal{S}}$ and their left-invariant metrics (1.3). In Sect. 3 we shall investigate the existence of lattices on all these solvable Lie groups. In Sect. 4 we shall consider the Euler-Lagrange equations for the quadratic curvature functionals and solve them for metrics of the form (1.3).

2 Preliminaries

Let $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ denote the semi-direct extension of the three-dimensional Heisenberg group corresponding to an Hamiltonian matrix $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$. The following result holds (see [7] for more details).

Proposition 1 *[7] Given $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$ described as in (1.1), the semi-direct extension $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ of the Heisenberg group determined by $\mathcal{S}$ can be realized as the following four-dimensional subgroup of* $\mathrm{GL}(4, \mathbb{R})$*:*

$$G_{\mathcal{S}} = \{M_{\mathcal{S}}(x_1, x_2, x_3, x_4) \in \mathrm{GL}(4, \mathbb{R}) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\},$$

where

$$M_{\mathcal{S}}(x_i) = \begin{pmatrix} 1 & x_2\, w(x_4) - x_3\, u(x_4) & x_2\, z(x_4) - x_3 v\,(x_4) & 2x_1 \\ 0 & u(x_4) & v(x_4) & x_2 \\ 0 & w(x_4) & z(x_4) & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and, depending on whether $\Delta = -\det(\mathcal{S}) = \alpha^2 + \beta\gamma$ is positive, null or negative, we have:

$$u(x_4) = \begin{cases} \cosh(\sqrt{\Delta} x_4) + \frac{\alpha}{\sqrt{\Delta}} \sinh(\sqrt{\Delta} x_4) & \textit{if } \Delta > 0, \\ 1 + \alpha x_4 & \textit{if } \Delta = 0, \\ \cos(\sqrt{-\Delta} x_4) + \frac{\alpha}{\sqrt{-\Delta}} \sin(\sqrt{-\Delta} x_4) & \textit{if } \Delta < 0. \end{cases} \tag{2.1}$$

$$v(x_4) = \begin{cases} \frac{\beta}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \textit{if } \Delta > 0, \\ \beta x_4 & \textit{if } \Delta = 0, \\ \frac{\beta}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \textit{if } \Delta < 0, \end{cases} \tag{2.2}$$

$$w(x_4) = \begin{cases} \frac{\gamma}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \textit{if } \Delta > 0, \\ \gamma x_4 & \textit{if } \Delta = 0, \\ \frac{\gamma}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \textit{if } \Delta < 0, \end{cases} \tag{2.3}$$

$$z(x_4) = \begin{cases} \cosh(\sqrt{\Delta}x_4) - \frac{\alpha}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \textit{if } \Delta > 0, \\ 1 - \alpha x_4 & \textit{if } \Delta = 0, \\ \cos(\sqrt{-\Delta}x_4) - \frac{\alpha}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \textit{if } \Delta < 0, \end{cases} \tag{2.4}$$

Let $\partial_j := \partial/\partial_{x_j}$ denote the coordinate vector field corresponding to the x_j-coordinate. Then, vector fields

$$\begin{aligned} e_1 &= \partial_1, \\ e_2 &= \tfrac{x_2 w(x_4) - x_3 u(x_4)}{2}\partial_1 + u(x_4)\partial_2 + w(x_4)\partial_3, \\ e_3 &= \tfrac{x_2 z(x_4) - x_3 v(x_4)}{2}\partial_1 + v(x_4)\partial_2 + z(x_4)\partial_3, \\ e_4 &= \partial_4 \end{aligned} \tag{2.5}$$

determine a basis of left-invariant vector fields on $G_{\mathcal{S}}$, with $(e_j)_I = (\partial_{x_j})_I$. Explicitly, we find

$$[e_2, e_3] = e_1, \qquad [e_2, e_4] = -\alpha e_2 - \gamma e_3, \qquad [e_3, e_4] = -\beta e_2 + \alpha e_3 \tag{2.6}$$

and so, setting $U = e_1$, $X = e_2$, $Y = e_3$, $S = e_4$, we see that the Lie algebra spanned by $\{e_1, e_2, e_3, e_4\}$ coincides with $\mathfrak{g} = \mathfrak{h} \rtimes (\mathbb{R}S)$. Equations (1.3) describe the family of left-invariant metrics g_a, $a^2 \neq 1$ on $\mathfrak{g}$ with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

We observe that there is a natural distinction between the properties of the Lie algebra $\mathfrak{g}$, and the ones of the left-invariant metric g_a we are equipping $\mathfrak{g}$ with. At the Lie algebra level, $\mathfrak{g} = \mathfrak{h} \rtimes (\mathbb{R}S)$ is uniquely determined up to isomorphisms, and different choices of $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$ can lead to isomorphic Lie algebras. On the other hand, the specific $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$ plays a fundamental role, via the *Koszul formula*, in determining the connection and curvature properties of (G_S, g_a). We shall take advantage of this simple remark in the following sections.

3 Lattices on $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$

It is easy to check that for any prescribed $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$, the corresponding four-dimensional Lie group $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ is solvable. C. Bock [3] classified solvable Lie groups, up to dimension six, which admit some lattices. We shall now prove that any Lie group G_S appears in Bock's classification and so, it gives rise to some solvmanifolds.

We start from the three-dimensional Heisenberg Lie algebra $\mathfrak{h}$ =span$\{X, Y, U\}$, with $[X, Y] = U$. Following [7], we remark that for any real constants a, b, c, d such that $ad - bc \neq 0$, if we consider the linearly independent vectors $\tilde{X} = aX + bY$, $\tilde{Y} = cX + dY$ and we set $\tilde{U} = (ad - bc)U$, we have $[\tilde{X}, \tilde{Y}] = \tilde{U}$, so that the description of the Heisenberg Lie algebra $\mathfrak{h}$ is exactly the same with respect to both bases $\{X, Y, U\}$ and $\{\tilde{X}, \tilde{Y}, \tilde{U}\}$.

Substitution of $\{X, Y\}$ by $\{\tilde{X}, \tilde{Y}\}$ corresponds to substitute a given $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$ by a similar matrix $\tilde{\mathcal{S}} \in \mathfrak{sp}(1, \mathbb{R})$, and Lie algebras $\mathfrak{h} \rtimes (\mathbb{R}S)$ and $\mathfrak{h} \rtimes (\mathbb{R}\tilde{S})$ (whence, the corresponding simply connected Lie groups $G_{\mathcal{S}}$ and $G_{\tilde{\mathcal{S}}}$) are isomorphic.

As $\mathrm{trace}(\mathcal{S}) = 0$, the characteristic equation of $\mathcal{S}$ is completely determined by $\Delta = -\det(\mathcal{S})$. It is then natural to consider separately three cases, depending on the sign of Δ. By standard linear algebra arguments one then concludes that for an arbitrary $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$, the corresponding one-dimensional extension $\mathfrak{h} \rtimes (\mathbb{R}S)$ is isomorphic to the Lie algebra $\mathfrak{g} = \mathrm{span}\{e_1 = \tilde{U}, e_2 = \tilde{X}, e_3 = \tilde{Y}, e_4 = \tilde{S}\}$, completely described by $[e_2, e_3] = e_1$ and

$$
\begin{array}{lll}
(A)\ [e_4, e_2] = \mu e_2,\ [e_4, e_3] = -\mu e_3, & \mu > 0 & \text{if } 1 > 0; \\
(B)\ [e_4, e_2] = \mu e_3,\ [e_4, e_3] = 0, & \mu \geq 0 & \text{if } 1 = 0; \\
(C)\ [e_4, e_2] = \mu e_3,\ [e_4, e_3] = -\mu e_2, & \mu > 0 & \text{if } 1 < 0.
\end{array}
$$

Remark 1 The Lie algebra described in Case (C) corresponds to the oscillator group. Case (A) is known is literature as *Boidol's group* or *split oscillator group*. Case (B) is nilpotent.

In case (A) we consider the new basis $\{X_1, X_2, X_3, X_4\}$ of $\mathfrak{g}$, defined by

$$X_1 = e_1, \quad X_2 = e_2, \quad X_3 = e_3, \quad X_4 = -\frac{1}{\mu} e_4$$

Then, with respect to $\{X_1, X_2, X_3, X_4\}$, the Lie algebra $\mathfrak{g}$ is completely determined by brackets

$$[X_2, X_3] = X_1, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = -X_3,$$

which is exactly case $\mathfrak{g}_{4.8}^{-1}$ in Table A.1 of [3].

With regard to case (B), if $\mu = 0$ then we already have case $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ in Table A.1 of [3] (simply renaming $e_i = X_i$, $i = 1, .., 4$). If $\mu > 0$ we set

$$X_1 = -e_1, \quad X_2 = e_3, \quad X_3 = \frac{1}{\mu} e_4, \quad X_4 = e_2.$$

Then, with respect to $\{X_1, X_2, X_3, X_4\}$, the Lie algebra $\mathfrak{g}$ is described by

$$[X_2, X_4] = X_1, \quad [X_3, X_4] = X_2,$$

which is case $\mathfrak{g}_{4.1}$ in Table A.1 of [3]. Finally, in case (C) we set

$$X_1 = e_1, \quad X_2 = e_2, \quad X_3 = e_3, \quad X_4 = \frac{1}{\mu} e_4$$

and the Lie algebra $\mathfrak{g}$ takes the form

$$[X_2, X_3] = X_1, \quad [X_2, X_4] = -X_3, \quad [X_3, X_4] = X_2,$$

which is case $\mathfrak{g}^0_{4.9}$ in Table A.1 of [3].

Thus, we have the following.

Proposition 2 *For any $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$, the corresponding one-dimensional extension $\mathfrak{g} = \mathfrak{h} \rtimes (\mathbb{R}S)$ of the Heisenberg Lie algebra $\mathfrak{h}$ admits some lattices and so, it gives rise to some solvmanifolds.*

We now report some explicit examples of lattices and corresponding solvmanifolds for the connected, simply connected Lie groups corresponding to the Lie algebras described above with respect to bases of the form $\{X_1, .., X_4\}$.

Case (A). Following [3], elements

$$\gamma_1 = \big(1, 1, -\tfrac{1+\sqrt{5}}{3+\sqrt{5}}\big), \quad \gamma_2 = \big(-\tfrac{2(2+\sqrt{5})}{3+\sqrt{5}}, \tfrac{1+\sqrt{5}}{3+\sqrt{5}}, -\tfrac{11+5\sqrt{5}}{7+3\sqrt{5}}\big), \quad \gamma_3 = \big(0, 0, \sqrt{5}\big)$$

satisfy $[\gamma_1, \gamma_2] = \gamma_3$ (with γ_3 central) and generate a lattice Γ in the Heisenberg group H. Consider the one-parameter group $\rho_A(t)(x, y, z) = \big(e^{-tt_1}x, e^{tt_1}y, z\big)$, with $t_1 = \ln(\frac{3+\sqrt{5}}{2})$. Then, the connected, simply connected Lie group corresponding to case (A) can be described as $H \rtimes_{\rho_A} \mathbb{R}$. Since $\rho_A(1)$ preserves Γ, one obtains a lattice $\Gamma \rtimes_{\rho_A} \mathbb{Z}$ and the corresponding solvmanifold $H \rtimes_{\rho_A} \mathbb{R}/\, \Gamma \rtimes_{\rho_A} \mathbb{Z}$ for case (A).

Case (B)-1: $\mu = 0$ As shown in [3], the corresponding connected, simply connected Lie group can be described as $\mathbb{R} \ltimes_{\rho_{B_0}} \mathbb{R}^3$, where

$$\rho_{B_0}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

A lattice is then given by $\mathbb{Z} \ltimes_{\rho_{B_0}} \mathbb{Z}^3$, giving rise to solvmanifold $\mathbb{R} \ltimes_{\rho_{B_0}} \mathbb{R}^3/\mathbb{Z} \ltimes_{\rho_{B_0}} \mathbb{Z}^3$ for case (B) with $\mu = 0$.

Case (B)-2: $\mu > 0$ In this case, again by [3], the connected, simply connected Lie group can be described as $\mathbb{R} \ltimes_{\rho_B} \mathbb{R}^3$, where ρ_B is the one-parameter group

$$\rho_B(t) = \begin{pmatrix} 1 & t & \frac{1}{2}t(t-1) \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

A lattice is given by $\mathbb{Z} \rtimes_{\rho_B} \mathbb{Z}^3$, giving rise to solvmanifold $\mathbb{R} \ltimes_{\rho_B} \mathbb{R}^3/\mathbb{Z} \ltimes_{\rho_B} \mathbb{Z}^3$ for case (B) with $\mu > 0$.

Case (C). Following [13] and referring to the notations we used in Sect. 2, for the oscillator group $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, the set

$$\Gamma_r = \begin{cases} \mathbb{Z}^3 \times 0 & \text{for } r \text{ even} \\ \{(z, u, 0) : z \in \mathbb{Z}^2, u \in \frac{1}{2}z_1 z_2 + \mathbb{Z}\} & \text{for } r \text{ odd} \end{cases}$$

defines a lattice for any $r \in \mathbb{N}$, determining a corresponding solvmanifold $H \rtimes \exp(\mathbb{R}S)/\Gamma_r$ for case (C).

Remark 2 The classification of lattices of a given solvable Lie group is a very interesting topic, which has been investigated by several authors. With regard to the Lie groups we considered above: complete classifications of lattices of the oscillator Lie group are known (see [13], [25]) and have been recently applied to harmonic analysis of solvmanifolds [14]; lattices for Lie group corresponding to case (A) have been recently classified [15].

4 Critical metrics on $G_{\mathcal{S}}$ for quadratic curvature functionals

The Euler-Lagrange equations of quadratic curvature functional are well known. They have been calculated in the Riemannian case ([2], [17]). As the argument does not depend on the signature of the involved metrics, they extend to pseudo-Riemannian settings.

The gradients of functionals $\mathcal{S}(g) = \int_M \tau^2 dvol_g$ and $\mathcal{F}_t(g) = \int_M \left(||\varrho||^2 + t\tau^2\right) dvol_g$ are respectively given by

$$\begin{aligned}
(\nabla\mathcal{S})_{ij} &= 2\nabla^2_{ij}\tau - 2(\Delta\tau)g_{ij} - 2\tau\varrho_{ij} + \frac{1}{2}\tau^2 g_{ij},\\
(\nabla\mathcal{F}_t)_{ij} &= -\Delta\varrho_{ij} + (1+2t)\nabla^2_{ij}\tau - \frac{1+4t}{2}(\Delta\tau)g_{ij}\\
&\quad -2t\tau\varrho_{ij} - 2\varrho_{kl}R_{ikjl} + \frac{1}{2}\left(||\varrho||^2 + t\tau^2\right)g_{ij},
\end{aligned}$$

and g is critical for $\mathcal{F}_t$ if and only if $(\nabla\mathcal{F}_t) = cg$ for some real constant c. Since the trace of this equation yields

$$(n-4)\left(||\varrho||^2 + t\tau^2\right) - (n+4(n-1)t)\Delta\tau = 2nc,$$

g is critical for $\mathcal{F}_t$ if and only if

$$\begin{aligned}
-\Delta\varrho_{ij} &+ (1+2t)\nabla^2_{ij}\tau - \frac{2t}{n}(\Delta\tau)g_{ij} - 2\varrho_{kl}R_{ikjl} - 2t\tau\varrho_{ij}\\
&+ \frac{2}{n}\left(||\varrho||^2 + t\tau^2\right)g_{ij} = 0
\end{aligned} \tag{4.1}$$

and

$$(n-4)\left(||\varrho||^2 + t\tau^2 - \lambda\right) = (n+4(n-1)t)\Delta\tau, \tag{4.2}$$

with $\lambda = \mathcal{F}_t(g)$ (see [10]). It follows that Einstein metrics are critical for $\mathcal{F}_t$ for all values of t. Moreover, by the above expression of $\nabla\mathcal{S}$, metrics which are either Einstein or of vanishing scalar curvature are critical points for $\mathcal{S}$.

It is worthwhile to observe that the above Euler-Lagrange equations simplify remarkably in the case of a four-dimensional metric of constant scalar curvature, which is the case for any four-dimensional homogeneous metric (in particular, for left-invariant metrics on four-dimensional Lie groups). In fact, in such a case, $(\nabla\mathcal{S})_{ij} = -2\tau\left(\varrho_{ij} - \frac{1}{4}\tau g_{ij}\right)$, equation (4.2) is automatically satisfied, while equation (4.1) reduces to

$$\Delta\varrho + 2R\left[\varrho\right] + 2t\tau\varrho - \frac{1}{2}\left(||\varrho||^2 + t\tau^2\right)g = 0, \tag{4.3}$$

where $R\left[\varrho\right]$ denotes the tensor defined by components $\varrho_{kl}R_{ikjl}$. We shall now completely classify solutions to equation (4.3) among left-invariant metrics g_a, $a^2 \neq 1$ on $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$.

The description of the Levi-civita connection and curvature of $(G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S), g_a)$ was obtained in [7]. Since the metrics are left-invariant, it suffices to work at the Lie algebra level. With respect to the basis $\{e_1 = U, e_2 = X, e_3 = Y, e_4 = S\}$ of left-invariant vector fields, the Levi-Civita connection ∇ is completely determined by the following possibly non-vanishing covariant derivatives:

$$\begin{array}{ll}
\nabla_{e_1}e_2 = -\frac{a}{2}e_3, & \nabla_{e_1}e_3 = \frac{a}{2}e_2, \\
\nabla_{e_2}e_1 = -\frac{a}{2}e_3, & \nabla_{e_2}e_2 = -\frac{\alpha}{a^2-1}e_1 + \frac{a\alpha}{a^2-1}e_4, \\
\nabla_{e_2}e_3 = \frac{a^2-1-\beta-\gamma}{2(a^2-1)}e_1 + \frac{a(\beta+\gamma)}{2(a^2-1)}e_4, & \nabla_{e_2}e_4 = -\alpha e_2 - \frac{1}{2}(\beta+\gamma+1)e_3, \\
\nabla_{e_3}e_1 = \frac{a}{2}e_2, & \nabla_{e_3}e_2 = -\frac{a^2-1+\beta+\gamma}{2(a^2-1)}e_1 + \frac{a(\beta+\gamma)}{2(a^2-1)}e_4, \\
\nabla_{e_3}e_3 = \frac{\alpha}{a^2-1}e_1 - \frac{a\alpha}{a^2-1}e_4, & \nabla_{e_3}e_4 = -\frac{1}{2}(\beta+\gamma-1)e_1 + \alpha e_3, \\
\nabla_{e_4}e_2 = -\frac{1}{2}(\beta-\gamma+1)e_3, & \nabla_{e_4}e_3 = \frac{1}{2}(\beta-\gamma+1)e_2.
\end{array} \tag{4.4}$$

The Riemann-Christoffel curvature tensor R of $(G_{\mathcal{S}}, g_a)$ is then completely determined by components $R_{ijkh} = g_a(R(e_i, e_j)e_k, e_h)$, where $R(e_i, e_j)e_k = \nabla_{e_i}\nabla_{e_j}e_k - \nabla_{e_j}\nabla_{e_i}e_k - \nabla_{[e_i,e_j]}e_k$ for all indices i, j, k. Explicitly, we find

$$\begin{array}{ll}
R_{1212} = R_{1313} = \frac{a^2}{4}, & R_{1224} = -\frac{a}{4}(\beta+\gamma+1), \\
R_{1234} = R_{1324} = \frac{a}{2}\alpha, & R_{1334} = \frac{a}{4}(\beta+\gamma-1), \\
R_{2323} = \frac{a\left(4\alpha^2-3a^2+3+(\beta+\gamma)^2\right)}{4(a^2-1)}, & R_{2424} = \frac{1}{4}\left((\beta+1)^2 - 3\gamma^2 - 2\beta\gamma + 2\gamma - 4\alpha^2\right), \\
R_{2434} = -(\beta-\gamma+1)\alpha, & R_{3434} = \frac{1}{4}\left((\gamma-1)^2 - 3\beta^2 - 2\beta\gamma - 2\beta - 4\alpha^2\right).
\end{array} \tag{4.5}$$

The Ricci tensor $\varrho(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y)$ of g_a is then described, with respect to the basis $\{e_i\}$, by the symmetric matrix

$$\varrho=\begin{pmatrix}\frac{1}{2}a^2 & 0 & 0 & \frac{1}{2}a\\ 0 & -\frac{a\left(a^2-1-\beta^2+\gamma^2\right)}{2(a^2-1)} & -\frac{a\alpha(\beta-\gamma)}{a^2-1} & 0\\ 0 & -\frac{a\alpha(\beta-\gamma)}{a^2-1} & -\frac{a\left(a^2-1+\beta^2-\gamma^2\right)}{2(a^2-1)} & 0\\ \frac{1}{2}a & 0 & 0 & -\frac{1}{2}(4\alpha^2+(\beta+\gamma)^2-1)\end{pmatrix}. \tag{4.6}$$

In particular, it is easily seen that $(G_{\mathcal{S}}=H\rtimes\exp(\mathbb{RS}), g_a)$ is Einstein if and only if $a=0$ and its Lie algebra, as described in (1.2), satisfies $4\alpha^2=1-(\beta+\gamma)^2$ (in this case, g_a is Ricci-flat [7]).

Again from (4.6), it is easy to check that the scalar curvature of g_a is given by

$$\tau=-\frac{a\left(a^2+4\alpha^2+(\beta+\gamma)^2-1\right)}{2(a^2-1)}. \tag{4.7}$$

Finally, the above descriptions of R, ϱ and τ yield that *Weyl conformal curvature tensor W* is completely determined by the following possibly non-vanishing components W_{ijkh} with respect to $\{e_i\}$:

$$\begin{aligned}
W_{1212}&=\frac{a^2}{6(a^2-1)}\left(a^2-2\alpha^2-2\beta^2+\gamma^2-\gamma\beta-1\right),\\
W_{1213}&=\frac{a^2}{2(a^2-1)}\alpha(\beta-\gamma),\\
W_{1224}&=\frac{-a}{12(a^2-1)}\left(a^2(3\beta+3\gamma+2)-4(\alpha^2+\beta^2)\right.\\
&\qquad\left.-2(1-\gamma^2+\gamma\beta)-3(\beta+\gamma)\right),\\
W_{1234}&=W_{1324}=\frac{a}{2(a^2-1)}\alpha(a^2+\gamma-\beta-1),\\
W_{1313}&=\frac{a^2}{6(a^2-1)}\left(a^2+\beta^2-2(\gamma^2+\alpha^2)-\gamma\beta-1\right),\\
W_{1334}&=\frac{a}{12(a^2-1)}\left(a^2(3\beta+3\gamma-2)+4(\alpha^2+\gamma^2)\right.\\
&\qquad\left.+2(1-\beta^2+\gamma\beta)-3(\beta+\gamma)\right),\\
W_{1414}&=\frac{a}{6}(2(1-a^2)+4\alpha^2+(\beta+\gamma)^2),\\
W_{2323}&=\frac{a}{6(a^2-1)}\left(2(1-a^2)+4\alpha^2+(\beta+\gamma)^2\right),\\
W_{2424}&=\frac{1}{6(a^2-1)}\left(a^4+a^2(-2\alpha^2+\beta^2-2\gamma^2-\gamma\beta+3\beta+3\gamma-1)\right.\\
&\qquad\left.-3(\beta+\gamma)(\beta-\gamma+1)\right),
\end{aligned}$$

$$W_{2434} = \frac{-1}{2(a^2-1)}\alpha(a^2(\beta-\gamma+2)+2(\gamma-\beta-1)),$$
$$W_{3434} = \frac{1}{6(a^2-1)}\Big(a^4+a^2(-2\alpha^2+\gamma^2-2\beta^2-\gamma\beta-3\beta-3\gamma-1)$$
$$+3(\beta^2-\gamma^2+\gamma+\beta)\Big). \tag{4.8}$$

In particular, as the vanishing of the Weyl tensor characterizes the (locally) conformally flat metrics in dimension four, from the above components we easily conclude that $(G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S), g_a)$ is locally conformally flat if and only if $a = 0$ and its Lie algebra, as described in (1.2), satisfies either $\beta = \gamma - 1$ or $\alpha = \beta + \gamma = 0$ (which is the well-known case of the bi-invariant metric of the oscillator group).

We are now ready to determine when metrics g_a are critical for some curvature invariants.

First, g_a is a critical point for functional $\mathcal{S}$ if and only if either g_a is Einstein, or its scalar curvature vanishes. Since the scalar curvature also vanishes when g_a is Einstein (indeed, Ricci-flat), by (4.7) we directly obtain the following characterization.

Theorem 1 *For any $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$ and real constant a with $a^2 \neq 1$, consider the semi-direct extension $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ of the Heisenberg group, equipped with the left-invariant metric g_a described by (1.3). Then, the following properties are equivalent:*

(i) g_a is a critical point for functional $\mathcal{S}(g) = \int_M \tau^2 dvol_g$;
(ii) The scalar curvature of g_a vanishes;
(iii) Either $a = 0$ or $a = \pm\sqrt{1-4\alpha^2-(\beta+\gamma)^2}$.

We now turn our attention to equation (4.3), which, by our previous analysis, completely characterizes g_a being a critical point for functional $\mathcal{F}_t$.

We first compute the components $\Delta\varrho_{ij}$ with respect to $\{e_i\}$ by means of (4.4) and (4.6). We find:

$$\begin{aligned}
\Delta\varrho_{11} &= -a^3,\\
\Delta\varrho_{14} &= -\tfrac{a^2}{2(a^2-1)}\left(2(a^2-1)+(\beta-\gamma)(4\alpha^2+(\beta+\gamma)^2)\right),\\
\Delta\varrho_{22} &= \tfrac{a^2}{2(a^2-1)^2}\big((a^2-1)^2+2(a^2-1-4\alpha^2-3\beta^2-\gamma^2-4\beta\gamma)\alpha^2+(2\gamma^2-\beta^2+\beta\gamma)a^2\\
&\quad -\beta\gamma-2\gamma^2-3\beta^2\gamma^2+\beta^2+\gamma^4-2\beta^4-\beta^3\gamma-3\beta\gamma^3\big),\\
\Delta\varrho_{23} &= \tfrac{a^2(\beta-\gamma)\alpha}{2(a^2-1)^2}\left(4\alpha^2+3(a^2-1)+3\beta^2-2\beta\gamma+3\gamma^2\right),\\
\Delta\varrho_{33} &= \tfrac{a^2}{2(a^2-1)^2}\big((a^2-1)^2+2(a^2-1-4\alpha^2-\beta^2-3\gamma^2-4\beta\gamma)\alpha^2+(2\beta^2-\gamma^2+\beta\gamma)a^2\\
&\quad -3\beta^2\gamma^2-\beta\gamma+\gamma^2-2\beta^2-2\gamma^4+\beta^4-3\beta^3\gamma-\beta\gamma^3\big),\\
\Delta\varrho_{44} &= -\tfrac{a}{2(a^2-1)}\big(4(a^2-1-4\alpha^2-2(\beta+\gamma)^2+2(\beta-\gamma))\alpha^2+((\beta+\gamma)^2+2)a^2\\
&\quad -2-4(\beta^2+\gamma^2+\beta\gamma)\beta\gamma+2(\beta+\gamma)^2(\beta-\gamma)-(\beta+\gamma)^2-(\beta^2+\gamma^2)^2\big).
\end{aligned} \tag{4.9}$$

Next, by means of (4.5) and (4.8), we compute the components of tensor $R[\varrho]$ with respect to $\{e_i\}$. We explicitly get

$$
\begin{aligned}
R\left[\varrho\right]_{11} &= -\tfrac{1}{4}a^3, \\
R\left[\varrho\right]_{14} &= -\tfrac{a^2}{4(a^2-1)}\left(a^2-(\beta-\gamma)(4\alpha^2+(\beta+\gamma)^2)-1\right), \\
R\left[\varrho\right]_{22} &= \tfrac{a^2}{4(a^2-1)^2}\Big(2a^4-(2\alpha^2-\beta^2+2\gamma^2+\gamma\beta+4)a^2+8\alpha^4+2(5\gamma^2-\beta^2+1+4\gamma\beta)\alpha^2 \\
&\quad -\beta^4-\beta^3\gamma+(3\gamma^2-1)\beta^2+(5\gamma^2+1)\beta\gamma+2(1+\gamma^4+\gamma^2)\Big), \\
R\left[\varrho\right]_{23} &= \tfrac{3a^2\alpha(\beta-\gamma)}{4(a^2-1)^2}\left(-a^2+4\alpha^2+(\beta+\gamma)^2+1\right), \\
R\left[\varrho\right]_{33} &= \tfrac{a^2}{4(a^2-1)^2}\Big(2a^4+(-2\alpha^2-\gamma\beta-2\beta^2+\gamma^2-4)a^2+8\alpha^4+2(5\beta^2-\gamma^2+4\gamma\beta+1)\alpha^2 \\
&\quad +2\beta^4+5\beta^3\gamma+(2+3\gamma^2)\beta^2-(\gamma^2-1)\beta\gamma+2-\gamma^4-\gamma^2\Big), \\
R\left[\varrho\right]_{44} &= \tfrac{a}{4(a^2-1)}\Big((4\alpha^2+(\beta+\gamma)^2-1)a^2+4(2\beta^2+2(1-2\gamma)\beta+2\gamma^2-2\gamma-1)\alpha^2 \\
&\quad +2\beta^4+2\beta^3-(1-2\gamma+4\gamma^2)\beta^2-2(\gamma+1)\beta\gamma+1-\gamma^2-2\gamma^3+2\gamma^4\Big).
\end{aligned}
\tag{4.10}
$$

Finally, we use (1.3), (4.6), (4.7), (4.9) and (4.10) and we compute the components of tensor

$$F_t = \Delta\varrho + 2R\left[\varrho\right] + 2t\tau\varrho - \frac{1}{2}\left(||\varrho||^2 + t\tau^2\right)g,$$

appearing in equation (4.3), with respect to $\{e_i\}$. We find that the symmetric tensor F_t is completely determined by the following possibly nonvanishing components:

$$
\begin{aligned}
F_{t\,11} = aF_{t\,14} &= -\tfrac{a^3}{8(a^2-1)^2}\Big(5(t+3)a^4+2((12\alpha^2+3(\beta+\gamma)^2-5)t-15)a^2+16(t+1)\alpha^4 \\
&\quad +8((t+2)(\beta^2+\gamma^2)+2t\beta\gamma-3t)\alpha^2+(t+3)\beta^4+4(t+1)\gamma\beta^3, \\
&\quad +(2(3t+1)\gamma^2-3t)\beta^2+(4((t+1)\gamma^2-3t)\beta\gamma+(t+3)\gamma^4-6t\gamma^2+5t+15\Big), \\
F_{t\,22} &= \tfrac{a^2}{8(a^2-1)^2}\Big(3(t+3)a^4+2((4\alpha^2+3\gamma^2-3+2\gamma\beta-\beta^2)t-9)a^2-16(t+1)\alpha^4 \\
&\quad +8((t+2)\gamma^2-3(t+2)\beta^2-2t\gamma\beta-t)\alpha^2-5(t+3)\beta^4-12(t+1)\gamma\beta^3 \\
&\quad -2((1+3t)\gamma^2-t)\beta^2+4((t+1)\gamma^2-t)\beta\gamma+3(t+3)\gamma^4-6t\gamma^2+3t+9\Big), \\
F_{t\,23} &= \tfrac{a^2\alpha(\beta-\gamma)}{(a^2-1)^2}\Big((a^2+4\alpha^2+(\beta+\gamma)^2-1)t+8\alpha^2+3\beta^2+3\gamma^2+2\beta\gamma\Big), \\
F_{t\,33} &= \tfrac{a^2}{8(a^2-1)^2}\Big(3(t+3)a^4+2((4\alpha^2+3\beta^2-\gamma^2+2\gamma\beta-3)t-9)a^2-16(t+1)\alpha^4 \\
&\quad +8((t+2)\beta^2-2t\gamma\beta-3(t+2)\gamma^2-t)\alpha^2+3(t+3)\beta^4+4(t+1)\gamma\beta^3 \\
&\quad -2((3t+1)\gamma^2+3t)\beta^2-4(3(t+1)\gamma^3+t\gamma)\beta-5(t+3)\gamma^4+2t\gamma^2+3t+9\Big), \\
F_{t\,44} &= -\tfrac{a}{8(a^2-1)^2}\Big((t+3)a^6+2(t-2t\gamma\beta-t\gamma^2-t\beta^2+3-4t\alpha^2)a^4 \\
&\quad +\Big(-48(t+1)\alpha^4-8(3(t+2)(\beta^2+\gamma^2)+6t\gamma\beta-5t)\alpha^2-3(t+3)\beta^4-12(t+1)\beta^3\gamma \\
&\quad +2(5t-3(1+3t)\gamma^2)\beta^2+4(5t-3(t+1)\gamma^2)\beta\gamma-3(t+3)\gamma^4+10t\gamma^2-7t-21\Big)a^2 \\
&\quad +64(t+1)\alpha^4+32((t+2)(\beta^2+\gamma^2)+2t\gamma\beta-t)\alpha^2+4(t+3)\beta^4+16(t+1)\beta^3\gamma \\
&\quad +8((3t+1)\gamma^2-t)\beta^2+16((t+1)\gamma^3-t\gamma)\beta+4(t+3)\gamma^4-8t\gamma^2+12+4t\Big).
\end{aligned}
\tag{4.11}
$$

Thus, for an arbitrary metric g_a, equation (4.3) holds for some t if and only if all the corresponding components listed in (4.11) vanish. In particular, it easily follows from the above expression of component $F_{t\,23}$ that we must consider the following possible cases:

(i) $a = 0$;
(ii) $\beta = \gamma$;
(iii) $\alpha = 0$;
(iv) $t = -\frac{8\alpha^2+3\beta^2+3\gamma^2+2\beta\gamma}{a^2+4\alpha^2+(\beta+\gamma)^2-1}$.

Case (i): $a = 0$ It follows at once from (4.11) that in this case $F_t = 0$ for all values of t. Therefore, g_0 is critical for all quadratic curvature operators for any value of α, β, γ, that is, on all semi-direct extensions of the Heisenberg group. On the other hand, unless the additional condition $4\alpha^2 = 1 - (\beta + \gamma)^2$ holds, this metric is not Einstein.

In all the remaining cases we shall always assume that $a \neq 0$.

Case (ii): $\beta = \gamma$ We substitute condition $\gamma = \beta$ in (4.11). In particular, taking into account $a \neq 0$, requiring that $F_{t11} = 0$ and $F_{t22} = 0$ we obtain the following equations, written down as polynomials in t:

$$\begin{aligned}&\big(5(a^2-1)^2 + 24a^2(\alpha^2+\beta^2) - 24(\alpha^2+\beta^2) + 16(\alpha^2+\beta^2)^2\big)\, t\\ &+15(a^2-1)^2 + 16(\alpha^2+\beta^2)^2 = 0\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\big(3(a^2-1)^2 + 8a^2(\alpha^2+\beta^2) - 8(\alpha^2+\beta^2) - 16(\alpha^2+\beta^2)^2\big)\, t\\ &+9(a^2-1)^2 - 16(\alpha^2+\beta^2)^2 = 0.\end{aligned} \tag{4.13}$$

We sum up equations (4.12) and (4.13) and we get

$$8(a^2-1)\left((a^2+4(\alpha^2+\beta^2)-1)t + 3(a^2-1)\right) = 0.$$

Observe that as $a^2 \neq 1$, the above equation necessarily yields that $a^2+4(\alpha^2+\beta^2)-1 \neq 0$ and so,

$$t = -\frac{3(a^2-1)}{a^2+4(\alpha^2+\beta^2)-1}. \tag{4.14}$$

We now substitute t from (4.14) into (4.11). A straightforward calculation then shows that now $F_{tij} = 0$ for all indices i, j if and only if

$$(\alpha^2+\beta^2)\left(3a^2 - 4(\alpha^2+\beta^2) - 3\right) = 0. \tag{4.15}$$

We may observe that the case $\alpha = \beta (= \gamma) = 0$ will be included as a special case of Subcase (iii)-A below. For any prescribed value of α and β (not both vanishing), (4.15) admits two solutions, namely, $a = \pm\frac{1}{3}\sqrt{12(\alpha^2+\beta^2)+9}$. In particular, this implies that $a^2 > 1$ and so, g_a is Riemannian. If a and α (respectively, β) are prescribed, then from (4.15) we get values of β (respectively, α) for which (4.15) holds when $3(a^2-1) - 4\alpha^2 \geq 0$ (respectively, $3(a^2-1) - 4\beta^2 \geq 0$). Finally, from (4.14) and (4.15) we conclude that $t = -\frac{3}{4}$.

Case (iii): $\alpha = 0$ Taking into account $a \neq 0$, by F_{t22} and F_{t33} in (4.11) we now respectively get

$$\Big(3a^4+2(3\gamma^2-\beta^2-3+2\beta\gamma)a^2+3-6\gamma^2+2\beta^2+3\gamma^4-5\beta^4-6\beta^2\gamma^2-4\beta\gamma-12\beta^3\gamma+4\beta\gamma^3\Big)t$$
$$+9a^4-18a^2-12\beta^3\gamma+4\beta\gamma^3+9\gamma^4-15\beta^4-2\beta^2\gamma^2+9=0.$$

and

$$\Big(3a^4+2(3\beta^2-\gamma^2-3+2\beta\gamma)a^2+3-6\beta^2+2\gamma^2+3\beta^4-5\gamma^4-6\beta^2\gamma^2-4\beta\gamma-12\beta\gamma^3+4\beta^3\gamma\Big)t$$
$$+9a^4-18a^2-12\beta\gamma^3+4\beta^3\gamma+9\beta^4-15\gamma^4-2\beta^2\gamma^2+9=0.$$

We substract the second of the above equations by the first one and we find

$$8(\beta-\gamma)(\beta+\gamma)\Big((1-(\beta+\gamma)^2-a^2)t-2\beta\gamma-3\gamma^2-3\beta^2\Big)=0. \quad (4.16)$$

We already treated the case $\beta = \gamma$ in general (without the assumption that $\alpha = 0$). So, by (4.16) we are now left with two possible subcases: either $\gamma = -\beta$ or $t = \frac{3\gamma^2+3\beta^2+2\beta\gamma}{(1-(\beta+\gamma)^2-a^2)}$.

Subcase (iii)-A: $\alpha = \beta + \gamma = 0$ In this case, the components of F_t reduce to:

$$F_{t11} = aF_{t14} = -\tfrac{5}{8}a^3(t+3), \quad F_{t22} = F_{t33} = \tfrac{3}{8}a^2(t+3), \quad F_{t44} = -\tfrac{1}{8}a(a^2+4)(t+3),$$

whence we conclude that g_a (for all a with $a^2 \neq 1$) is critical for functional $\mathcal{F}_{-3}$.

Subcase (iii)-B: $\alpha = t - \frac{3\gamma^2+3\beta^2+2\beta\gamma}{(1-(\beta+\gamma)^2-a^2)} = 0$ We substitute the value of t in the components (4.11) of tensor F_t and write them as polynomials in the variable a. In particular, taking into account $a \neq 0$, we find that $F_{t11} = 0$ and $F_{t22} = 0$ if and only if

$$15a^4-5(3\beta^2+3\gamma^2+2\beta\gamma+6)a^2+15-4\beta^3\gamma-4\beta\gamma^3-8\beta^2\gamma^2+10\beta\gamma+15\beta^2+15\gamma^2=0 \quad (4.17)$$

and

$$9a^4-3(3\beta^2+3\gamma^2+2\beta\gamma+6)a^2+9+4\beta^3\gamma+4\beta\gamma^3+8\beta^2\gamma^2+6\beta\gamma+9\gamma^2+9\beta^2=0, \quad (4.18)$$

respectively. We multiply (4.17) by 9 and (4.18) by 15 and substract from one another. We get

$$0=-96\beta^3\gamma-96\beta\gamma^3-192\beta^2\gamma^2=-96\beta\gamma(\beta+\gamma)^2$$

so that either $\beta = 0$ or $\gamma = 0$, as we already treated the previous subcase $\beta + \gamma = 0$.

If $\beta = 0$ then the components of F_t reduce to:

$$
\begin{aligned}
F_{t11} &= aF_{t14} = -\tfrac{15a^3}{8(a^2-1)}(a^2-1-\gamma^2),\\
F_{t22} &= F_{t33} = \tfrac{9a^2}{8(a^2-1)}(a^2-1-\gamma^2),\\
F_{t44} &= -\tfrac{3a}{8(a^2-1)}(a^2+4)(a^2-1-\gamma^2).
\end{aligned}
$$

Therefore, g_a is critical if and only if $a^2-1-\gamma^2 = 0$. Observe that for $\beta = a^2-1-\gamma^2 = 0$, we get $t = -\frac{3}{2}$. We may also remark that $a^2-1-\gamma^2 = 0$ yields $a = \pm\sqrt{1+\gamma^2}$, for any real value of γ. On the other hand, it also yields $\gamma = \pm\sqrt{a^2-1}$, which impies that $a^2 > 1$, that is, g_a is Riemannian.

In the case where $\gamma = 0$, the argument and calculations are very similar to the above ones for case $\beta = 0$ and we shall omit them. In this case we conclude that g_a is critical for $\mathcal{F}_t$ if and only if $t = -\frac{3}{2}$ and $a^2-1-\beta^2 = 0$.

Remark 3 We observe that equation (4.16) is also satisfied, for all values of t, if $1-(\beta+\gamma)^2-a^2 = -2\beta\gamma-3\gamma^2-3\beta^2 = 0$. However, starting from (4.11), a standard calculation yields that in this case necessarily $a = 0$, which we already considered in the above Case (i). So, this concludes subcase (iii)-B.

Case (iv): $t = -\frac{8\alpha^2+3\beta^2+3\gamma^2+2\beta\gamma}{a^2+4\alpha^2+(\beta+\gamma)^2-1}$

We preliminarily observe that by (4.11), the component F_{t23} also vanishes, for all values of t, when $8\alpha^2+3\beta^2+3\gamma^2+2\beta\gamma = a^2+4\alpha^2+(\beta+\gamma)^2-1 = 0$. However, under these conditions, if we require that $F_{tij} = 0$ for all indices i, j, a long but straightforward calculation proves that necessarily $a = 0$. As this possibility has already been investigated in the previous Case (i), we are left to consider the case when $t = -\frac{8\alpha^2+3\beta^2+3\gamma^2+2\beta\gamma}{a^2+4\alpha^2+(\beta+\gamma)^2-1}$.

We start substituting the value of t in equations (4.11) and writing them as polynomials in the variable a. In particular, conditions $F_{t11} = 0$ and $F_{t22} = 0$ now read

$$
\begin{aligned}
&15a^4-5(8\alpha^2+3\beta^2+3\gamma^2+2\beta\gamma+6)a^2-16\alpha^4+4(10-6\beta\gamma-\beta^2-\gamma^2)\alpha^2\\
&-4(\beta+\gamma)^2\beta\gamma+5(3\beta^2+3\gamma^2+2\beta\gamma+3) = 0
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
&9a^4-3(8\alpha^2+6+3\gamma^2+3\beta^2+2\beta\gamma)a^2+16\alpha^4+4(6+6\beta\gamma+\beta^2+\gamma^2)\alpha^2\\
&+4(\beta+\gamma)^2\beta\gamma+3(3\beta^2+3\gamma^2+2\beta\gamma+3) = 0,
\end{aligned}
\tag{4.20}
$$

respectively. We multiply (4.19) by 9, (4.20) by 15 and substract from one another. We get

$$
-96(\alpha^2+\beta\gamma)((\beta+\gamma)^2+4\alpha^2) = 0.
$$

We already discussed the solution $\alpha = \beta+\gamma = 0$ in the subcase (iii)-A in full generality. So, we are now left to consider solutions $\alpha = \pm\sqrt{-\beta\gamma}$, when $\beta\gamma \leq 0$.

We substitute this expression of α into (4.11) and we find that $F_{tij} = 0$ for all indices i, j if ad only if

$$a^2 - (\beta - \gamma)^2 - 1 = 0,$$

whence, $a = \pm\sqrt{(\beta-\gamma)^2+1}$, for all values of $\beta \neq \gamma$. Substituting the expressions of a and α in the above expression of t, we conclude that in this case $t = -\frac{3}{2}$. We remark that this case includes subcase (iii)-B, which is the special case obtained when $\beta\gamma = 0$.

The above calculations and results are summarized in the following main result.

Theorem 2 *For any $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$ and real constant a with $a^2 \neq 1$, consider the semi-direct extension $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ of the Heisenberg group, equipped with the left-invariant metric g_a described by (1.3). Then, g_a is a critical point for functional $\mathcal{F}_t(g) = \int_M \left(||\varrho||^2 + t\tau^2\right) dvol_g$ if and only if one of the following cases occurs:*

(1) $a = 0$. In this case, g_a is a critical point for $\mathcal{F}_t$ for all $t \in \mathbb{R}$.
(2) $\beta = \gamma$ and $a = \pm\frac{1}{3}\sqrt{12(\alpha^2+\beta^2)+9}$, with $(\alpha,\beta) \neq (0,0)$. Then, g_a (necessarily Riemannian) is critical for $\mathcal{F}_{-\frac{3}{4}}$.
(3) $\alpha = \beta + \gamma = 0$. In this case, all metrics g_a are critical for $\mathcal{F}_{-3}$.
(4) $\alpha = \pm\sqrt{-\beta\gamma}$ and $a = \pm\sqrt{(\beta-\gamma)^2+1}$ (with $\beta\gamma \leq 0$ and $\beta \neq \gamma$). Then g_a (necessarily Riemannian) is critical for $\mathcal{F}_{-\frac{3}{2}}$.

The complete classification of metrics g_a which are critical for some quadratic curvature invariants is obtained in Theorems 1 and 2. We reported such classification in Table 1.

We may observe that case (3) in the above Theorem 2 corresponds to the oscillator group. So, *all left-invariant metrics g_a (both Riemannian and Lorentzian) on the oscillator group are critical for the quadratic curvature functional $\mathcal{F}_{-3}$.*

Remark 4 Metric g_0 carries several special properties. In fact, for any $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$, among left-invariant metrics g_a, $a^2 \neq 1$ on the corresponding semi-direct extension $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ of the Heisenberg group, the metric g_0 is the only one:

- Admitting a null parallel vector field (namely, e_1) and so, being a *Walker metric* [7]. In particular, g_0 is also a *pp*-wave, as it follows from [23] because g_0 admits the null parallel vector field e_1 and, by (4.5), is *transversally flat* ($R(x, y) = 0$ whenever x, y are orthogonal to e_1);

Table 1 Critical metrics g_a for quadratic curvature functionals

Value of a	Conditions on (α,β,γ)	Critical for $\mathcal{S}$	Critical for $\mathcal{F}_t$
0	None	✓	✓
$\pm\frac{1}{3}\sqrt{12(\alpha^2+\beta^2)+9}$	$\beta=\gamma$ and $(\alpha,\beta) \neq (0,0)$	✗	$t=-\frac{3}{4}$
Arbitrary	$\alpha=\beta+\gamma=0$	Only if $a=0$	$t=-3$
$a=\pm\sqrt{(\beta-\gamma)^2+1}$	$\alpha=\pm\sqrt{-\beta\gamma}, \beta\gamma \leq 0, \beta \neq \gamma$	✗	$t=-\frac{3}{2}$
$a=\pm\sqrt{1-4\alpha^2-(\beta+\gamma)^2}$	$4\alpha^2+(\beta+\gamma)^2<1$	✓	✗

- Being Ricci-parallel [7];
- Satisfying the *Ricci soliton equation* $\mathcal{L}_X g + \varrho = \lambda g$ for all real values of λ, and defining a *Yamabe soliton* [6], [7];
- Conformally Einstein [9].

Investigation of critical metrics for quadratic curvature functionals emphasizes once more the special role of g_0. In fact, Proposition 2 and Theorems 1 and 2 yield the following consequence.

Corollary 1 *For any* $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$*, the left-invariant Lorentzian metric* g_0 *on* $G_{\mathcal{S}} = H \rtimes \exp(\mathbb{R}S)$ *is critical for all quadratic curvature functionals. Unless* $4\alpha^2 + (\beta + \gamma)^2 = 1$*, the Lorentzian manifold* $(G_{\mathcal{S}}, g_0)$ *is not Einstein.*

The examples we studied suggest the possible existence of some interplays between properties listed above and critical metrics for quadratic curvature functionals. It would be very interesting to investigate further these possible interplays.

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Declarations

Conflict of interest The Authors declare that they do not have any potential conflicts of interest (financial or non-financial).

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