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Helix surfaces for Berger-like metrics on the anti-de Sitter space

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Abstract

We consider the Anti-de Sitter space $\mathbb{H}_1^3$ equipped with Berger-like metrics, that deform the standard metric of $\mathbb{H}_1^3$ in the direction of the hyperbolic Hopf vector field. Helix surfaces are the ones forming a constant angle with such vector field. After proving that these surfaces have (any) constant Gaussian curvature, we achieve their explicit local description in terms of a one-parameter family of isometries of the space and some suitable curves. These curves turn out to be general helices, which meet at a constant angle the fibers of the hyperbolic Hopf fibration.

Keywords Helix surfaces · Constant angle surfaces · Anti-de Sitter Space

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1 Introduction

A *helix surface* (or *constant angle surface*) is an oriented surface, whose normal vector field forms a constant angle with a fixed field of directions in the ambient space. In recent years, many authors investigated helix surfaces in different ambient spaces. Several examples of the study of helix surfaces in Riemannian settings may be found in [3–6, 9, 10, 13–15] and references therein.

The investigation of helix surfaces also extended to other settings. On the one hand, higher codimensional Riemannian helix surfaces were studied (see for example [7, 8, 19]). On the other hand, Lorentzian ambient spaces were considered. Lorentzian settings allow to more possibilities, as both spacelike and timelike surfaces can be studied. Some examples of the study of the geometry of helix surfaces in Lorentzian spaces are given in [11, 12, 16, 17]. In particular, helix surfaces of the anti-de Sitter space $\mathbb{H}_1^3$ were studied in [12]. Equipping $\mathbb{H}_1^3$ with its canonical metric of constant curvature, all left-invariant vector fields are Killing (see also [2]), so no special directions emerge. Moreover, all helix surfaces turn out to be flat [12].

In [2], the first author and D. Perrone introduced and studied a new family of metrics $\tilde{g}_{\lambda\mu\nu}$ on $\mathbb{H}_1^3(\kappa/4)$. These metrics were induced in a natural way by corresponding metrics defined on the tangent sphere bundle $T_1\mathbb{H}^2(\kappa)$, after describing the covering map F from $\mathbb{H}_1^3(\kappa/4)$ to $T_1\mathbb{H}^2(\kappa)$ in terms of paraquaternions. A crucial role in this construction is played by the *hyperbolic Hopf map* and the *hyperbolic Hopf vector field*, that is, the hyperbolic counterparts of the Hopf map and vector field on $\mathbb{S}^3$, respectively.

In [13, 14, 16, 17], the second author et al. gave an explicit local classification of helix surfaces by means a one-parameter family of isometries of the ambient space and a suitable curve. A similar appproach will be used in this paper in order to find the explicit characterization of the helix surfaces in the anti-de Sitter space. We shall equip the anti-de Sitter space $\mathbb{H}_1^3(\kappa/4)$ with a special type of the Lorentzian metrics introduced in [2], that is, the ones that deform the standard metric of $\mathbb{H}_1^3(\kappa/4)$ only in the direction of the hyperbolic Hopf vector field, which is then a Killing vector field. Because of their analogies with the Berger metrics on $\mathbb{S}^3$, we shall refer to these metrics as *Berger-like metrics*. We shall consider the anti-de Sitter space $\mathbb{H}_1^3(\kappa/4)$ equipped with Berger-like metrics and completely describe their surfaces, whose normal vector field forms a constant angle with the hyperbolic Hopf vector field.

The paper is organized in the following way. In Sect. 2 we provide some needed information on the anti-de Sitter space $\mathbb{H}_1^3(\kappa/4)$. In Sect. 3 we describe the Levi-Civita connection and the curvature of Berger-like metrics g_τ on $\mathbb{H}_1^3(\kappa/4)$. The general equations for a surface of $(\mathbb{H}_1^3(\kappa/4), g_\tau)$ are given in Sect. 4 and then are applied in Sect. 5 to the case of helix surfaces. Three different cases occur, according to the sign of some constant B, which depends on the parameter τ of the Berger-like metric, the causal character of the surface and the constant angle. These three cases are completely described and characterized in Sect. 6, in terms of a 1-parameter family of isometries on $\mathbb{H}_{1,\tau}^3$ and some suitable curves. As we prove in Sect. 7, these curves are general helices, which meet at a constant angle the fibers of the hyperbolic Hopf fibration.

2 Preliminaries

Let $\mathbb{R}^4_2 = (\mathbb{R}^4, g_0)$ denote the 4-dimensional pseudo-Euclidean space, of neutral signature $(2, 2)$, equipped with the flat pseudo-Riemannian metric

$$g_0 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2.$$

For any real constant $\kappa > 0$, the *anti-de Sitter (three)-space* $\mathbb{H}^3_1(\kappa)$ is the hypersurface of $\mathbb{R}^4_2$ defined by

$$\mathbb{H}^3_1(\kappa) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_2 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1/\kappa\}.$$

We denote by $\langle \, , \, \rangle$ the Lorentzian metric induced from g_0 on $\mathbb{H}^3_1(\kappa)$. This metric, known as the canonical metric of $\mathbb{H}^3_1$, has constant sectional curvature $-\kappa < 0$.

Consider now the algebra $\mathbb{B}$ of paraquaternionic numbers over $\mathbb{R}$ generated by $\{1, i, j, k\}$, where $k = ij$, $-i^2 = j^2 = 1$ and $ij = -ji$. This is an associative, non-commutative and unitary algebra over $\mathbb{R}$. An arbitrary paraquaternonic number is given by $q = x_1 + x_2 i + x_3 j + x_4 k$. The conjugate of q is $\bar{q} = x_1 - x_2 i - x_3 j - x_4 k$ and the norm of q is given by

$$||q||^2 = q\bar{q} = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

The scalar product induced by such norm on $\mathbb{R}^4$ is exactly g_0, and $\{1, i, j, k\}$ is an orthonormal basis with $1, i$ spacelike and j, k timelike. In terms of paraquaternionic numbers, we have

$$\mathbb{H}^3_1(\kappa) = \{q \in \mathbb{B} : ||q||^2 = -1/\kappa\}.$$

The above presentation was used in [2] to describe a covering map from the anti-de Sitter space $\mathbb{H}^3_1(\kappa/4)$ to the unit tangent sphere bundle of the (Riemannian) hyperbolic two-space $\mathbb{H}^2(\kappa)$, which is embedded in $\mathbb{R}^3_1$ as

$$\mathbb{H}^2(\kappa) = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 : x_1^2 + x_2^2 - x_3^2 = -1/\kappa, x_3 > 0\}.$$

Now consider each paraquaternonic number q as a pair of complex number in the following way $q := (z, w) = (x_1 + ix_2, x_3 + ix_4)$, we obtain the covering map

$$F : \mathbb{H}^3_1(\kappa/4) \to T_1\mathbb{H}^2(\kappa)$$
$$(z, w) \mapsto \left(\frac{\sqrt{\kappa}}{4} \left(2z\bar{w}, |z|^2 + |w|^2 \right), -\frac{\kappa}{4} \left((z^2 + \bar{w}^2), 2\mathrm{Re}(z\,w) \right) \right).$$

The composition of F with the canonical projection

$$\pi : T_1\mathbb{H}^2(\kappa) \to \mathbb{H}^2(\kappa),$$
$$(x, u) \mapsto x$$

yields the *hyperbolic Hopf map* $h = \pi \circ F$, that is,

$$h : \mathbb{H}^3_1(\kappa/4) \to \mathbb{H}^2(\kappa)$$
$$(z, w) \mapsto \frac{\sqrt{\kappa}}{4}(2z\bar{w}, |z|^2 + |w|^2). \tag{2.1}$$

In particular, h is a submersion with geodesic fibers, which can be defined as orbits of the $\mathbb{S}^1$-action $\left(e^{it}, (z, w) \right) \mapsto \left(e^{it}z, e^{it}w \right)$ on the anti-de Sitter space $\mathbb{H}^3_1(\kappa/4)$. The vector field

$$X_1(q) = \frac{\sqrt{\kappa}}{2} iq = \frac{\sqrt{\kappa}}{2}(-x_2, x_1, -x_4, x_3)$$

is tangent to fibers of h and satisfies $\langle X_1, X_1 \rangle = -1$ and is called the *hyperbolic Hopf vector field*, in analogy with the case of the Hopf vector field and the Hopf fibration. We also consider

$$X_2(q) = \frac{\sqrt{\kappa}}{2} jq = \frac{\sqrt{\kappa}}{2}(x_3, -x_4, x_1, -x_2), \quad X_3(q) = \frac{\sqrt{\kappa}}{2} kq = \frac{\sqrt{\kappa}}{2}(x_4, x_3, x_2, x_1).$$

Then, $\{X_1, X_2, X_3\}$ parallelize $\mathbb{H}_1^3(\kappa/4)$ and

$$\langle X_2, X_2 \rangle = \langle X_3, X_3 \rangle = 1, \qquad \langle X_l, X_m \rangle = 0, \quad \text{for } l \neq m.$$

3 Levi-Civita connection and curvature

In [2], the covering map F described by (2) was used to introduce a family of Lorentzian metrics on the anti-de Sitter space $\mathbb{H}_1^3(\kappa/4)$. These metrics were referred to as *metrics of Kaluza-Klein type*, because they correspond in a natural way to metrics defined on the unit tangent sphere bundle $T_1\mathbb{H}^2(\kappa)$. Their general description is given by

$$g = -\rho\,\theta^1 \otimes \theta^1 + \mu\,\theta^2 \otimes \theta^2 + \nu\,\theta^3 \otimes \theta^3,$$

where ρ, μ, ν are positive real numbers and $\{\theta^i\}$ denotes the basis of 1-forms dual to $\{X_i\}$.

Here we shall focus our attention on a special type of metrics of Kaluza-Klein type, requiring to have a deformation only in the direction of the hyperbolic Hopf vector field X_1, which in this case is a Killing vector field [2]. Up to homotheties, we may restrict to the case where $\mu = \nu = 1$ and $\rho = \tau^2$ and consider the one-parameter family of metrics

$$g_\tau = -\tau^2\,\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3, \quad \tau > 0. \tag{3.1}$$

These metrics can be clearly considered an hyperbolic analogue of *Berger metrics* on the sphere $\mathbb{S}^3$ and so we refer to them as *Berger-like metrics* on $\mathbb{H}_1^3(\kappa/4)$. Observe that with respect to the standard metric $\langle, \rangle$ on $\mathbb{H}_1^3(\kappa/4)$, these metrics can be described as

$$g_\tau(X, Y) = \langle X, Y \rangle + (1 - \tau^2)\langle X, X_1 \rangle\langle Y, X_1 \rangle$$

and $\{E_1 = \tau^{-1}X_1, E_2 = X_2, E_3 = X_3\}$ is an orthonormal basis for g_τ, with E_1 timelike and E_2, E_3 spacelike. Computing the Lie brackets $[E_i, E_j]$, we find

$$[E_1, E_2] = -\frac{\sqrt{\kappa}}{\tau}\,E_3, \qquad [E_2, E_3] = \tau\sqrt{\kappa}\,E_1, \qquad [E_1, E_3] = \frac{\sqrt{\kappa}}{\tau}\,E_2.$$

Then, by the Koszul formula, we obtain the description of the Levi-Civita connection of $\mathbb{H}_{1,\tau}^3 = (\mathbb{H}_1^3(\kappa/4), g_\tau)$ with respect to $\{E_1, E_2, E_3\}$. Explicitly, we get

$$\nabla_{E_1}^\tau E_1 = 0, \qquad \nabla_{E_1}^\tau E_2 = \frac{\sqrt{\kappa}\,(\tau^2 - 2)}{2\tau}E_3, \ \nabla_{E_1}^\tau E_3 = -\frac{\sqrt{\kappa}\,(\tau^2 - 2)}{2\tau}E_2,$$

$$\nabla_{E_2}^\tau E_1 = \frac{\tau\sqrt{\kappa}}{2}E_3, \quad \nabla_{E_2}^\tau E_2 = 0, \qquad\qquad\qquad \nabla_{E_2}^\tau E_3 = \frac{\tau\sqrt{\kappa}}{2}E_1, \tag{3.2}$$

$$\nabla_{E_3}^\tau E_1 = -\frac{\tau\sqrt{\kappa}}{2}E_2, \ \nabla_{E_3}^\tau E_2 = -\frac{\tau\sqrt{\kappa}}{2}E_1, \qquad \nabla_{E_3}^\tau E_3 = 0.$$

Observe that E_1 is an unit timelike vector field tangent to the fibers of h and the Levi-Civita connection satisfies the following geometric relation

$$\nabla_X^\tau E_1 = -\tau\frac{\sqrt{\kappa}}{2}X \wedge E_1, \tag{3.3}$$

for any tangent vector field X, where $E_1 \wedge E_2 = E_3$, $E_2 \wedge E_3 = -E_1$ e $E_3 \wedge E_1 = E_2$ completely determine the wedge product $U \wedge V$ (see [18]). We now consider the curvature tensor, taken with the sign convention

$$R^\tau(X, Y) = [\nabla^\tau_X, \nabla^\tau_Y] - \nabla^\tau_{[X,Y]}.$$

Using (3.2) we have that the non vanishing components of the Riemannian curvature tensor R^τ are

$$R^\tau(E_1, E_2)E_1 = -\frac{\kappa}{4}\tau^2 E_2, \quad R^\tau(E_1, E_3)E_1 = -\frac{\kappa}{4}\tau^2 E_3,$$

$$R^\tau(E_3, E_2)E_3 = \frac{\kappa}{4}(4 - 3\tau^2)E_2 \tag{3.4}$$

and we can prove the following result.

Proposition 3.1 *The curvature tensor of $\mathbb{H}^3_{1,\tau}$ is given by*

$$\begin{aligned} R^\tau(X, Y)Z = {} &\frac{\kappa}{4}(3\tau^2 - 4)[g_\tau(Y, Z)X - g_\tau(X, Z)Y] \\ &+ \kappa(\tau^2 - 1)[g_\tau(Y, E_1)g_\tau(Z, E_1)X - g_\tau(X, E_1)g_\tau(Z, E_1)Y \\ &+ g_\tau(Y, Z)g_\tau(X, E_1)E_1 - g_\tau(X, Z)g_\tau(Y, E_1)E_1], \end{aligned} \tag{3.5}$$

for all tangent vector fields X, Y, Z.

Proof Consider three arbitrary vector fields X, Y, Z on $\mathbb{H}^3_{1,\tau}$ and their decompositions as

$$X = \bar{X} + xE_1, \quad Y = \bar{Y} + yE_1, \quad Z = \bar{Z} + zE_1,$$

with $\bar{X}, \bar{Y}, \bar{Z}$ orthogonal to E_1 and $x = g_\tau(X, E_1)$ and so on. Observe that using (3.4) we have that all terms of $g_\tau(R^\tau(X, Y)Z, W)$ where E_1 occurs either one, three or four times necessarily vanish. Therefore, we have

$$\begin{aligned} g_\tau(R^\tau(X, Y)Z, W) = {} &g_\tau(R^\tau(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) \\ &+ yzg_\tau(R^\tau(\bar{X}, E_1)E_1, \bar{W}) + xzg_\tau(R^\tau(E_1, \bar{Y})E_1, \bar{W}) \\ &+ wxg_\tau(R^\tau(E_1, \bar{Y})\bar{Z}, E_1) + wyg_\tau(R^\tau(\bar{X}, E_1)\bar{Z}, E_1). \end{aligned}$$

From the decompositions of $\bar{X}, \bar{Y}, \bar{Z}$ and $\bar{W}$ follows that

$$g_\tau(R^\tau(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) = \frac{\kappa}{4}(3\tau^2 - 4)(g_\tau(\bar{X}, \bar{W})g_\tau(\bar{Y}, \bar{Z}) - g_\tau(\bar{X}, \bar{Z})g_\tau(\bar{Y}, \bar{W})).$$

Moreover, we have

$$R^\tau(\bar{X}, E_1)E_1 = \frac{\kappa}{4}\tau^2 \bar{X}, \quad R^\tau(E_1, \bar{Y})E_1 = -\frac{\kappa}{4}\tau^2 \bar{Y}.$$

Thus, we conclude that

$$\begin{aligned} g_\tau(R^\tau(X, Y)Z, W) = {} &\frac{\kappa}{4}(3\tau^2 - 4)(g_\tau(\bar{X}, \bar{W})g_\tau(\bar{Y}, \bar{Z}) - g_\tau(\bar{X}, \bar{Z})g_\tau(\bar{Y}, \bar{W})) \\ &+ \frac{\kappa}{4}\tau^2(yzg_\tau(\bar{X}, \bar{W}) - xzg_\tau(\bar{Y}, \bar{W}) + wxg_\tau(\bar{Y}, \bar{Z}) - ywg_\tau(\bar{X}, \bar{Z})) \\ = {} &g_\tau\Big(\frac{\kappa}{4}(3\tau^2 - 4)[g_\tau(\bar{Y}, \bar{Z})\bar{X} - g_\tau(\bar{X}, \bar{Z})\bar{Y}] \\ &+ \frac{\kappa}{4}\tau^2[g_\tau(Y, E_1)g_\tau(Z, E_1)\bar{X} - g_\tau(X, E_1)g_\tau(Z, E_1)\bar{Y} \end{aligned}$$

$$+ g_\tau(X, E_1)g_\tau(\bar{Y}, \bar{Z})E_1 - g_\tau(Y, E_1)g_\tau(\bar{X}, \bar{Z})E_1], \bar{W}\Big)$$

$$= g_\tau\Big(\frac{\kappa}{4}(3\tau^2 - 4)[g_\tau(Y, Z)X - g_\tau(X, Z)Y]$$

$$+ \kappa(\tau^2 - 1)[g_\tau(Y, E_1)g_\tau(Z, E_1)X - g_\tau(X, E_1)g_\tau(Z, E_1)Y$$

$$+ g_\tau(X, E_1)g_\tau(Y, Z)E_1 - g_\tau(Y, E_1)g_\tau(X, Z)E_1], W\Big),$$

which ends the proof since W is arbitrary. $\square$

We end this section describing the isometries of $\mathbb{H}^3_{1,\tau}$. Following the idea used in [14] and [17], we observe that the isometry group of $\mathbb{H}^3_{1,\tau}$ is the four-dimensional pseudo-unitary group $U_1(2)$, that can be identified with

$$U_1(2) = \{A \in O_2(4) | A J_1 = \pm J_1 A\},$$

where J_1 is the complex structure of $\mathbb{R}^4$ corresponding to i, i.e., defined by

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.6}$$

while

$$O_2(4) = \{A \in GL(4, \mathbb{R}) | A^t = \epsilon A^{-1} \epsilon\}, \qquad \epsilon = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is the pseudo-orthogonal group, i.e., the group of 4×4 real matrices preserving the semi-definite inner product of $\mathbb{R}^4_2$.

We now consider a 1-parameter family $A(v)$, $v \in (a, b) \subset \mathbb{R}$, consisting of 4×4 pseudo-orthogonal matrices commuting (anticommuting, respectively) with J_1. In order to describe explicitly the family $A(v)$, we consider the two product structures J_2 and J_3 of $\mathbb{R}^4$ corresponding to j and k respectively, that is,

$$J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $A(v)$ is a pseudo-orthogonal matrix, the first row must be a unit vector $\mathbf{r}_1(v)$ of $\mathbb{R}^4_2$ for all $v \in (a, b)$. Thus, without loss of generality, we can take

$$\mathbf{r}_1(v) = (\cosh \xi_1(v) \cos \xi_2(v), -\cosh \xi_1(v) \sin \xi_2(v),$$
$$\sinh \xi_1(v) \cos \xi_3(v), -\sinh \xi_1(v) \sin \xi_3(v)),$$

for some real functions ξ_1, ξ_2 and ξ_3 defined in (a, b). Since $A(v)$ commutes (anticommutes, respectively) with J_1 the second row of $A(v)$ must be $\mathbf{r}_2(v) = \pm J_1 \mathbf{r}_1(v)$. Now, the four vectors $\{\mathbf{r}_1, J_1\mathbf{r}_1, J_2\mathbf{r}_1, J_3\mathbf{r}_1\}$ form an orthonormal basis of $\mathbb{R}^4_2$, thus the third row $\mathbf{r}_3(v)$ of $A(v)$ must be a linear combination of them. Since $\mathbf{r}_3(v)$ is unit and it is orthogonal to both $\mathbf{r}_1(v)$ and $J_1\mathbf{r}_1(v)$, there exists a function $\xi(v)$ such that

$$\mathbf{r}_3(v) = \sin \xi(v) J_2\mathbf{r}_1(v) + \cos \xi(v) J_3\mathbf{r}_1(v).$$

Finally, the fourth row of $A(v)$ is $\mathbf{r}_4(v) = \pm J_1 \mathbf{r}_3(v) = \mp \sin \xi(v) J_2 \mathbf{r}_1(v) \pm \cos \xi(v) J_3 \mathbf{r}_1(v)$. This means that any 1-parameter family $A(v)$ of 4×4 pseudo-orthogonal matrices commuting (anticommuting, respectively) with J_1 can be described by four functions ξ_1, ξ_2, ξ_3 and ξ as

$$A(\xi, \xi_1, \xi_2, \xi_3)(v) = \begin{pmatrix} \mathbf{r}_1(v) \\ \pm J_1 \mathbf{r}_1(v) \\ \sin \xi(v) J_2 \mathbf{r}_1(v) + \cos \xi(v) J_3 \mathbf{r}_1(v) \\ \mp \sin \xi(v) J_2 \mathbf{r}_1(v) \pm \cos \xi(v) J_3 \mathbf{r}_1(v) \end{pmatrix}. \tag{3.7}$$

4 Structure equations for surfaces in $\mathbb{H}^3_{1,\tau}$

Consider a pseudo-Riemannian oriented surface M immersed in $\mathbb{H}^3_{1,\tau}$. Let N denote the unit vector field normal to M in $\mathbb{H}^3_{1,\tau}$. We set $\lambda := g_\tau(N, N)$, so that M is spacelike if $\lambda = -1$ and timelike if $\lambda = 1$.

In the following, we compute the Gauss and Codazzi equations for M, using the metric induced by g_τ on M, the shape operator A, the tangent projection of E_1 on M and the *angle function* $v := g_\tau(N, E_1) g_\tau(N, N) = \lambda g_\tau(N, E_1)$. The vector field E_1 decomposes as

$$E_1 = T + vN,$$

where T is tangent to M, whence,

$$g_\tau(T, T) = -(1 + \lambda v^2). \tag{4.1}$$

Denoting by X and Y two vector fields tangent to M we have,

$$\nabla^\tau_X Y = \nabla_X Y + \alpha(X, Y), \qquad \nabla^\tau_X N = -A(X),$$

where ∇ is the Levi-Civita connection of M and α the second fundamental form with respect to the immersion in $\mathbb{H}^3_{1,\tau}$. Thus, we conclude that

$$\begin{aligned} \alpha(X, Y) &= \lambda g_\tau(\alpha(X, Y), N) N = \lambda g_\tau(\nabla^\tau_X Y, N)N \\ &= -\lambda g_\tau(Y, \nabla^\tau_X N)N = \lambda g_\tau(Y, A(X))N. \end{aligned}$$

Observe that

$$\begin{aligned} \nabla^\tau_X E_1 &= \nabla^\tau_X T + X(v)N + v\nabla^\tau_X N \\ &= \nabla_X T + \alpha(X, T) + X(v)N - vA(X) \\ &= \nabla_X T + \lambda g_\tau(T, A(X))N + X(v)N - vA(X). \end{aligned} \tag{4.2}$$

On the other hand, by (3.3) we have

$$\begin{aligned} \nabla^\tau_X E_1 &= -\frac{\sqrt{\kappa}}{2} \tau X \wedge E_1 = -\frac{\sqrt{\kappa}}{2} \tau X \wedge (T + vN) \\ &= -\frac{\sqrt{\kappa}}{2} \tau (X \wedge T) - \frac{\sqrt{\kappa}}{2} v\tau(X \wedge N) \\ &= -\frac{\sqrt{\kappa}}{2} \lambda\tau g_\tau(X \wedge T, N)N + \frac{\sqrt{\kappa}}{2} v\tau(N \wedge X) \\ &= -\frac{\sqrt{\kappa}}{2} \lambda\tau g_\tau(JX, T)N + \frac{\sqrt{\kappa}}{2} v\tau \, JX, \end{aligned} \tag{4.3}$$

where $JX := N \wedge X$ satisfies

$$g_\tau(JX, JY) = -\lambda \, g_\tau(X, Y), \qquad J^2 X = \lambda \, X. \tag{4.4}$$

Comparing the above expressions for $\nabla_X^\tau E_1$, we find

$$\begin{cases} \nabla_X T = \nu \left(A(X) + \dfrac{\sqrt{\kappa}}{2} \tau J X \right), \\[2mm] X(\nu) = -\lambda \, g_\tau \left(A(X) + \dfrac{\sqrt{\kappa}}{2} \tau J X, T \right). \end{cases} \tag{4.5}$$

We now prove the following.

Proposition 4.1 *Denoting by X and Y two vector fields tangent to M, with K the Gaussian curvature of M and with $\bar{K}$ the sectional curvature in $\mathbb{H}_{1,\tau}^3$ of the plane tangent to M, we have*

$$K = \bar{K} + \lambda \det A = -\frac{\kappa}{4}\tau^2 + \lambda \left[\det A + \kappa \nu^2 (1 - \tau^2)\right] \tag{4.6}$$

and

$$\nabla_X A(Y) - \nabla_Y A(X) - A[X, Y] = -\kappa \lambda \nu (1 - \tau^2) \left[g_\tau(X, T)Y - g_\tau(Y, T)X\right]. \tag{4.7}$$

Proof Recall that for a pseudo-Riemannian surface, one has

$$K = \bar{K} + \lambda \, \frac{g_\tau(A(X), X)g_\tau(A(Y), Y) - g_\tau(A(X), Y)^2}{g_\tau(X, X)g_\tau(Y, Y) - g_\tau(X, Y)^2}.$$

Consider a local orthonormal basis $\{X, Y\}$ on M, i.e., $g_\tau(X, X) = 1$, $g_\tau(X, Y) = 0$ and $g_\tau(Y, Y) = -\lambda$. Then, by Proposition 3.1, we have

$$\begin{aligned} R^\tau(X, Y)Y = {}& \frac{\kappa}{4}(3\tau^2 - 4)[g_\tau(Y, Y)X - g_\tau(X, Y)Y] \\ & + \kappa \, (\tau^2 - 1)[g_\tau(Y, E_1)g_\tau(Y, E_1)X - g_\tau(X, E_1)g_\tau(Y, E_1)Y \\ & + g_\tau(Y, Y)g_\tau(X, E_1)E_1 - g_\tau(X, Y)g_\tau(Y, E_1)E_1] \end{aligned}$$

and so,

$$\begin{aligned} g_\tau(R^\tau(X, Y)Y, X) = {}& -\frac{\kappa}{4}\lambda(3\tau^2 - 4) + \kappa(\tau^2 - 1)[g_\tau(Y, E_1)^2 - \lambda g_\tau(X, E_1)^2] \\ = {}& -\frac{\kappa}{4}\lambda(3\tau^2 - 4) + \kappa(\tau^2 - 1)[g_\tau(Y, T)^2 - \lambda g_\tau(X, T)^2]. \end{aligned}$$

Taking into account $g_\tau(X, T)^2 - \lambda g_\tau(Y, T)^2 = g_\tau(T, T) = -(1 + \lambda \nu^2)$, we have

$$R^\tau(X, Y, Y, X) = -\frac{\kappa}{4}\lambda(3\tau^2 - 4) + \kappa\lambda(\tau^2 - 1)(1 + \lambda\nu^2)$$

that gives

$$\bar{K} = -\lambda R^\tau(X, Y, Y, X) = \frac{\kappa}{4}(3\tau^2 - 4) - \kappa(\tau^2 - 1)(1 + \lambda\nu^2) = -\frac{\kappa}{4}\tau^2 + \kappa\lambda\nu^2(1 - \tau^2),$$

whence Eq. (4.6) follows.

Consider the Codazzi equation

$$g_\tau(R^\tau(X, Y)Z, N) = g_\tau(\nabla_X A(Y) - \nabla_Y A(X) - A[X, Y], Z).$$

From Proposition 3.1, we have

$$R^{\tau}(X, Y)N = \kappa\,(\tau^2 - 1)\lambda\nu[g_{\tau}(Y, E_1)X - g_{\tau}(X, E_1)Y]$$
$$= \kappa\lambda\nu\,(\tau^2 - 1)[g_{\tau}(Y, T)X - g_{\tau}(X, T)Y],$$

that leads to Eq. (4.7) by the arbitrariness of Z. □

5 Basic properties of helix surfaces in $\mathbb{H}^3_{1,\tau}$

We start with the following.

Definition 5.1 Let M be an oriented pseudo-Riemannian surface immersed in $\mathbb{H}^3_{1,\tau}$ and N the unit vector field normal to M, with $g_{\tau}(N, N) = \lambda$. The surface M is called a *helix surface* (or *a constant angle surface*) if the angle function $\nu = \lambda\,g_{\tau}(N, E_1)$ is constant on M.

Remark 5.1 If M is spacelike (respectively, timelike), then T is spacelike (respectively, timelike) and JT is spacelike. From (4.1) and (4.4) we get

$$g_{\tau}(T, T) = -(1 + \lambda\,\nu^2), \qquad g_{\tau}(JT, JT) = \lambda + \nu^2 > 0, \qquad g_{\tau}(T, JT) = 0.$$

Observe that in the case $\lambda = 1$, if $\nu = 0$ then E_1 is tangent to M at each point. Therefore, M is a *Hopf tube*. On the other hand, if $\lambda = -1$, then $|\nu| > 1$ and so $\nu \neq 0$.

Taking into account Remark 5.1 from now on we assume $\nu \neq 0$.

Proposition 5.1 *Let M be a helix surface in $\mathbb{H}^3_{1,\tau}$ and N the unit vector field normal to M. Then the following hold:*

(i) *with respect to the tangent basis $\{T, JT\}$, the matrix of the shape operator is given by*

$$A = \begin{pmatrix} 0 & -\dfrac{\sqrt{\kappa}}{2}\lambda\tau \\ \dfrac{\sqrt{\kappa}}{2}\tau & \mu \end{pmatrix},$$

for some smooth function μ on M;

(ii) *the Levi-Civita connection ∇ of M is described by*

$$\nabla_T T = \nu\tau\sqrt{\kappa}\,JT, \quad \nabla_{JT}T = \mu\nu\,JT,$$
$$\nabla_T JT = \lambda\nu\tau\sqrt{\kappa}\,T, \quad \nabla_{JT}JT = \lambda\mu\nu\,T;$$

(iii) *the Gaussian curvature of M is given by*

$$K = \lambda\kappa\nu^2(1 - \tau^2);$$

(iv) *the function μ satisfies equation*

$$T(\mu) + \nu\mu^2 + \kappa\nu B = 0, \tag{5.1}$$

where $B := \nu^2(\tau^2 - 1) - \lambda$.

Proof Taking into account Remark 5.1 for the tangent basis $\{T, JT\}$ we have

$$g_{\tau}(A(T), T) = -\frac{\sqrt{\kappa}}{2}\tau g_{\tau}(JT, T) - \lambda\,T(\nu) = 0;$$

$$g_\tau(A(JT), T) = -\lambda\, JT(\nu) - \frac{\sqrt{\kappa}}{2}\tau g_\tau(J^2 T, T) = -\frac{\sqrt{\kappa}}{2}\lambda\tau g_\tau(T, T),$$

which, by the symmetry of the shape operator, yields *(i)*.

For the Levi-Civita connection ∇ of M, using (4.5), we have

$$\nabla_T T = \nu\left(A(T) + \frac{\sqrt{\kappa}}{2}\tau\, JT\right) = \sqrt{\kappa}\nu\tau\, JT;$$

$$\nabla_{JT} T = \nu\left(A(JT) + \frac{\sqrt{\kappa}}{2}\tau\, J^2 T\right) = \nu\mu\, JT,$$

and using the compatibility of ∇ with the metric g_τ, follows

$$g_\tau(\nabla_T JT, T) = -g_\tau(JT, \nabla_T T), \qquad g_\tau(\nabla_{JT} JT, T) = -g_\tau(JT, \nabla_{JT} T).$$

So,

$$\nabla_T JT = \sqrt{\kappa}\lambda\nu\tau\, T, \qquad \nabla_{JT} JT = \lambda\nu\mu\, T.$$

From (4.6) we have that Gaussian curvature is given by

$$K = -\frac{\kappa}{4}\tau^2 + \lambda\,[\det A + \kappa\nu^2(1 - \tau^2)]$$
$$= \lambda\kappa\nu^2\,(1 - \tau^2).$$

Remark 5.2 It is worthwhile to remark that the Gaussian curvature K is a constant, which depends on the causal character of the surface and on the sign of $(1 - \tau^2)$ and it can assume any real value. In the special case of $\tau = 1$ the Gaussian curvature vanishes and helix surfaces for the standard metric on the anti-de Sitter space are flat, coherently with the results obtained in [12] for surfaces forming a constant angle with an unit Killing vector field.

Finally, we calculate

$$\nabla_T A(JT) = -\frac{\kappa}{2}\lambda\tau^2\nu\, JT + T(\mu)\, JT + \sqrt{\kappa}\lambda\tau\mu\nu\, T,$$

$$\nabla_{JT} A(T) = \frac{\sqrt{\kappa}}{2}\lambda\tau\nu\mu\, T,$$

$$A[T, JT] = A(\nabla_T JT - \nabla_{JT} T) = \frac{\sqrt{\kappa}}{2}\lambda\tau\nu\mu\, T - \mu^2\nu\, JT + \frac{\kappa}{2}\lambda\tau^2\nu\, JT.$$

By Proposition 4.1, we get

$$\nabla_T A(JT) - \nabla_{JT} A(T) - A[T, JT] = \kappa(\tau^2 - 1)\lambda\nu(g_\tau(T, T)JT)$$
$$= -\kappa(\tau^2 - 1)\lambda\nu(1 + \lambda\nu^2)JT$$

and so, by comparing,

$$T(\mu) + \mu^2\nu - \kappa\lambda\nu[1 - \lambda\nu^2(\tau^2 - 1)] = 0$$

which ends the proof. $\square$

We recall that E_1 is a timelike vector field and $g_\tau(E_1, N) = \nu\lambda$. Thus, there exists a smooth function φ on M, such that

$$N = -\nu\lambda\, E_1 + \sqrt{\lambda + \nu^2}\cos\varphi\, E_2 + \sqrt{\lambda + \nu^2}\sin\varphi\, E_3,$$

whence,

$$T = E_1 - \nu N = (1 + \nu^2\lambda) E_1 - \nu\sqrt{\lambda + \nu^2} \cos\varphi\, E_2 - \nu\sqrt{\lambda + \nu^2} \sin\varphi\, E_3$$

and we can calculate

$$JT = N \wedge T = N \wedge (E_1 - \nu N) = N \wedge E_1 = \sqrt{\lambda + \nu^2} (\sin\varphi\, E_2 - \cos\varphi\, E_3).$$

Moreover, we have

$$\nabla_T^\tau E_1 = -\frac{\sqrt{\kappa}}{2} \tau\, T \wedge E_1 = \frac{\sqrt{\kappa}}{2} \nu\tau\, JT,$$

$$\nabla_T^\tau E_2 = \frac{\sqrt{\kappa}}{2} (1 + \nu^2\lambda) \frac{\tau^2 - 2}{\tau} E_3 + \nu\tau \frac{\sqrt{\kappa(\lambda + \nu^2)}}{2} \sin\varphi\, E_1,$$

$$\nabla_T^\tau E_3 = -\frac{\sqrt{\kappa}}{2} (1 + \nu^2\lambda) \frac{\tau^2 - 2}{\tau} E_2 - \nu \frac{\sqrt{\kappa(\lambda + \nu^2)}}{2} \tau \sin\varphi\, E_1.$$

Therefore,

$$\begin{aligned}
A(T) &= -\nabla_T^\tau N \\
&= \nu\lambda \nabla_T^\tau E_1 - \sqrt{\lambda + \nu^2} [-\sin\varphi\, T(\varphi)\, E_2 + \cos\varphi\, \nabla_T^\tau E_2 \\
&\quad + \cos\varphi\, T(\varphi)\, E_3 + \sin\varphi\, \nabla_T^\tau E_3] \\
&= \frac{\sqrt{\kappa}}{2} \lambda\nu^2\tau\, JT + T(\varphi)\, JT + \frac{\sqrt{\kappa}}{2} (1 + \nu^2\lambda) \frac{\tau^2 - 2}{\tau}\, JT.
\end{aligned}$$

But $A(T) = \frac{\sqrt{\kappa}}{2} \tau\, JT$ and so, we conclude that

$$\frac{\sqrt{\kappa}}{2} \lambda\nu^2\tau + T(\varphi) + \frac{\sqrt{\kappa}}{2} (1 + \lambda\nu^2) \frac{\tau^2 - 2}{\tau} = \frac{\sqrt{\kappa}}{2} \tau,$$

that is,

$$T(\varphi) = -\frac{\sqrt{\kappa}}{\lambda\tau} B.$$

Moreover,

$$\nabla_{JT}^\tau E_1 = \sqrt{\lambda + \nu^2} (\sin\varphi \nabla_{E_2}^\tau E_1 - \cos\varphi \nabla_{E_3}^\tau E_1) = \frac{\sqrt{\kappa(\lambda + \nu^2)}}{2} \tau (\sin\varphi\, E_3 + \cos\varphi\, E_2),$$

$$\nabla_{JT}^\tau E_2 = -\sqrt{\lambda + \nu^2} \cos\varphi \nabla_{E_3}^\tau E_2 = \frac{\sqrt{\kappa(\lambda + \nu^2)}}{2} \tau \cos\varphi\, E_1,$$

$$\nabla_{JT}^\tau E_3 = \sqrt{\lambda + \nu^2} \tau \sin\varphi \nabla_{E_2}^\tau E_3 = \frac{\sqrt{\kappa(\lambda + \nu^2)}}{2} \tau \sin\varphi\, E_1$$

and so,

$$\begin{aligned}
A(JT) &= -\nabla_{JT}^\tau N \\
&= \nu\lambda \nabla_{JT}^\tau E_1 - \sqrt{\lambda + \nu^2} [-\sin\varphi\, JT(\varphi) E_2 + \cos\varphi \nabla_{JT}^\tau E_2 \\
&\quad + \cos\varphi\, JT(\varphi)\, E_3 + \sin\varphi \nabla_{JT}^\tau E_3] \\
&= JT(\varphi)\, JT - \frac{\sqrt{\kappa}}{2} \lambda\tau\, T.
\end{aligned}$$

But $A(JT) = -\dfrac{\sqrt{\kappa}}{2} \tau \lambda\, T + \mu\, JT$, so that we obtain

$$JT(\varphi) = \mu.$$

In this way, we determined the following system

$$
\begin{cases}
T(\varphi) = -\dfrac{\sqrt{\kappa}}{\lambda\tau} B, \\[2mm]
JT(\varphi) = \mu,
\end{cases}
$$

whose compatibility condition is given by

$$[T, JT] = \nabla_T JT - \nabla_{JT} T = \sqrt{\kappa}\,\nu\lambda\tau\, T - \mu\nu\, JT.$$

Recalling that

$$[T, JT](\varphi) = T(\mu),$$

by comparison we obtain

$$T(\mu) + \mu^2 \nu + \kappa\nu B = 0,$$

that is, Eq. (5.1), which we already established, so that the system is compatible.

We can choose a system of local coordinates (x, y) on M, such that

$$
\begin{cases}
\partial_x = T, \\
\partial_y = a\, T + b\, JT,
\end{cases}
\tag{5.2}
$$

for some smooth functions $a = a(x, y)$, $b = b(x, y)$ on M. Requiring that $[\partial_x, \partial_y] = 0$, we get

$$
\begin{cases}
a_x = -\sqrt{\kappa}\lambda\nu\tau b, \\
b_x = b\mu\nu.
\end{cases}
$$

So, Eq. (5.1) can be rewritten as follows:

$$\mu_x = -\nu(\kappa B + \mu^2).
\tag{5.3}$$

In order to integrate equation (5.3), we need to consider separately the following cases

(i) If $B > 0$, we find

$$\mu(x, y) = \sqrt{\kappa B} \tan(\eta(y) - \nu\sqrt{\kappa B}\, x).$$

(ii) If $B = 0$, we have

$$\mu(x, y) = \dfrac{1}{\nu x + \eta(y)}.$$

(iii) If $B < 0$, we get

$$\mu(x, y) = \sqrt{-\kappa B} \tanh(\eta(y) + \nu\sqrt{-\kappa B}\, x).$$

In all the above cases, $\eta(y)$ is an arbitrary smooth function.

As we are interested in only one coordinate system (x, y) on the surface M, we only need one admissible solution for a and b in each case.

Case $B > 0$

Since $b_x = b\mu v$, we deduce

$$b_x = b\sqrt{\kappa B}\tan(\eta(y) - v\sqrt{\kappa B}\,x)\,v,$$

which admits as solution $b = \cos(\eta(y) - v\sqrt{\kappa B}\,x)$. Then, we have

$$a_x = -\lambda v\sqrt{\kappa}\tau b = -\sqrt{\kappa}\lambda v\tau\cos(\eta(y) - v\sqrt{\kappa B}\,x)$$

and so, we can take $a = \dfrac{\lambda\tau}{\sqrt{B}}\sin(\eta(y) - v\sqrt{\kappa B}\,x)$. Now, from

$$
\begin{cases}
\varphi_x = -\dfrac{\sqrt{\kappa}}{\lambda\tau}B, \\[2mm]
\varphi_y = -\dfrac{a\sqrt{\kappa}}{\lambda\tau}B + b\mu = 0,
\end{cases}
$$

we get $\varphi(x, y) = -\dfrac{\sqrt{\kappa}}{\lambda\tau}Bx + c$, for some real constant c.

Case $B = 0$

From $b_x = b\mu v$, we now have

$$b_x = b\frac{1}{vx + \eta(y)}v$$

and a solution is given by $b = vx + \eta(y)$. Moreover, we have:

$$a_x = -\sqrt{\kappa}\,\lambda v\tau b = -\sqrt{\kappa}\,\lambda v^2\tau x + \eta(y),$$

which holds for $a = -v\frac{\sqrt{\kappa}}{2}\lambda\tau x(vx + 2\eta(y))$. Then,

$$
\begin{cases}
\varphi_x = -\dfrac{\sqrt{\kappa}}{\lambda\tau}B = 0, \\[2mm]
\varphi_y = -\dfrac{a\sqrt{\kappa}}{\lambda\tau}B + b\mu = 1,
\end{cases}
$$

whence, $\varphi(x, y) = y + c$ for some real constant c.

Case $B < 0$

Recalling that $b_x = b\mu v$, we have

$$b_x = b\sqrt{-\kappa B}\tanh(\eta(y) + v\sqrt{-\kappa B}\,x)\,v,$$

which is satisfied by $b = \cosh(\eta(y) + v\sqrt{-\kappa B}\,x)$. Moreover, we find:

$$a_x = -\lambda v\sqrt{\kappa}\,\tau b = -\lambda v\sqrt{\kappa}\,\tau\cosh(\eta(y) + v\sqrt{-\kappa B}\,x)$$

and so, we take $a = -\dfrac{\lambda\tau}{\sqrt{-B}}\sinh(\eta(y) + v\sqrt{-\kappa B}\,x)$. Finally,

$$
\begin{cases}
\varphi_x = -\dfrac{\sqrt{\kappa}}{\lambda\tau}B, \\[2mm]
\varphi_y = -\dfrac{a\sqrt{\kappa}}{\lambda\tau}B + b\mu = 0,
\end{cases}
$$

and we obtain $\varphi(x, y) = -\dfrac{\sqrt{\kappa}}{\lambda\tau}Bx + c$, for some real constant c.

Using the above results, we have the following.

Proposition 5.2 *Let M be a helix surface in $\mathbb{H}^3_{1,\tau}$ with constant angle function v. With respect to the local coordinates (x, y) defined above, the position vector F of M in $\mathbb{R}^4_2$ satisfies the following equation:*

(a) *if $B = 0$,*

$$
\frac{\partial^2 F}{\partial x^2} = 0, \tag{5.4}
$$

(b) *if $B \neq 0$,*

$$
\frac{\partial^4 F}{\partial x^4} + (\tilde{b}^2 + 2\tilde{a})\frac{\partial^2 F}{\partial x^2} + \tilde{a}^2 F = 0, \tag{5.5}
$$

where

$$
\tilde{a} = \frac{\kappa}{4}\frac{B}{\tau^2}(\lambda + v^2), \qquad \tilde{b} = -\sqrt{\kappa}\frac{B}{\lambda\tau}.
$$

Proof Let M be a helix surface and let

$$
F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x, y)).
$$

denote the position vector of M in $\mathbb{R}^4_2$, described with respect to the local coordinates (x, y) defined before. By definition of position vector, we get

$$
\partial_x F = (\partial_x F_1, \partial_x F_2, \partial_x F_3, \partial_x F_4) = T
$$
$$
= \sqrt{\lambda + v^2}\left[\lambda\sqrt{\lambda + v^2}E_{1|F} - v\cos\varphi\, E_{2|F} - v\sin\varphi\, E_{3|F}\right]. \tag{5.6}
$$

Then, if we consider the expressions of E_1, E_2 and E_3 with respect to the coordinates of $\mathbb{R}^4_2$, we can express the above equation as

$$
\begin{cases}
\partial_x F_1 = \dfrac{\sqrt{\kappa(\lambda + v^2)}}{2}\left[-\dfrac{\lambda}{\tau}\sqrt{\lambda + v^2}\,F_2 - v\cos\varphi\, F_3 - v\sin\varphi\, F_4\right], \\[3mm]
\partial_x F_2 = \dfrac{\sqrt{\kappa(\lambda + v^2)}}{2}\left[\dfrac{\lambda}{\tau}\sqrt{\lambda + v^2}\,F_1 + v\cos\varphi\, F_4 - v\sin\varphi\, F_3\right], \\[3mm]
\partial_x F_3 = \dfrac{\sqrt{\kappa(\lambda + v^2)}}{2}\left[-\dfrac{\lambda}{\tau}\sqrt{\lambda + v^2}\,F_4 - v\cos\varphi\, F_1 - v\sin\varphi\, F_2\right], \\[3mm]
\partial_x F_4 = \dfrac{\sqrt{\kappa(\lambda + v^2)}}{2}\left[\dfrac{\lambda}{\tau}\sqrt{\lambda + v^2}\,F_3 + v\cos\varphi\, F_2 - v\sin\varphi\, F_1\right].
\end{cases} \tag{5.7}
$$

Therefore, if $B = 0$ taking the derivative of (5.7) with respect to x, we get (5.4).

If we suppose $B \neq 0$, we obtain

$$\begin{cases} (F_1)_{xx} = -\tilde{a}\, F_1 - \tilde{b}\,(F_2)_x, \\ (F_2)_{xx} = -\tilde{a}\, F_2 + \tilde{b}\,(F_1)_x, \\ (F_3)_{xx} = -\tilde{a}\, F_3 - \tilde{b}\,(F_4)_x, \\ (F_4)_{xx} = -\tilde{a}\, F_4 + \tilde{b}\,(F_3)_x, \end{cases} \tag{5.8}$$

where

$$\tilde{a} = \frac{\kappa}{4}\frac{B}{\tau^2}(\lambda + \nu^2), \qquad \tilde{b} = -\sqrt{\kappa}\frac{B}{\lambda\tau}.$$

In conclusion, taking twice the derivative of (5.8) with respect to x and using the previous relations we find (5.5). $\qquad\square$

Remark 5.3 From $\langle F, F \rangle = -4/\kappa$ and relations (5.7), (5.8), we get:

$$\langle F, F \rangle = -\frac{4}{\kappa}, \qquad \langle F_x, F_x \rangle = \frac{4}{\kappa}\tilde{a}, \quad \langle F, F_x \rangle = 0,$$

$$\langle F_x, F_{xx} \rangle = 0, \qquad \langle F_{xx}, F_{xx} \rangle = D, \quad \langle F, F_{xx} \rangle = -\frac{4}{\kappa}\tilde{a},$$

$$\langle F_x, F_{xxx} \rangle = -D, \quad \langle F_{xx}, F_{xxx} \rangle = 0, \quad \langle F, F_{xxx} \rangle = 0, \tag{5.9}$$

$$\langle F_{xxx}, F_{xxx} \rangle = E,$$

where

$$D = \frac{4}{\kappa}(\tilde{a}\tilde{b}^2 + 3\tilde{a}^2), \qquad E = (\tilde{b}^2 + 2\tilde{a})D - \frac{4}{\kappa}\tilde{a}^3.$$

6 Characterization theorems for helix surfaces in $\mathbb{H}^3_{1,\tau}$

In order to give conditions under which an immersion defines a helix surface in $\mathbb{H}^3_{1,\tau}$ we observe that, if F is a position vector of a helix surface in $\mathbb{H}^3_{1,\tau}$ we have that:

$$J_1 F = \frac{2}{\sqrt{\kappa}}X_{1|F(x,y)} = \tau\frac{2}{\sqrt{\kappa}}E_{1|F(x,y)}$$

and, using the (5.9), we have the following identities:

$$\langle J_1 F, F_x \rangle = -\frac{2\lambda(\lambda + \nu^2)}{\tau\sqrt{\kappa}}, \qquad\qquad \langle J_1 F, F_{xx} \rangle = 0,$$

$$\langle J_1 F_{xx}, F_x \rangle = \tilde{a}\left[\frac{2\lambda(\lambda + \nu^2)}{\tau\sqrt{\kappa}} - \frac{4}{\kappa}\tilde{b}\right] = L, \ \langle J_1 F_x, F_{xxx} \rangle = 0, \tag{6.1}$$

$$\langle J_1 F_x, F_{xx} \rangle + \langle J_1 F, F_{xxx} \rangle = 0, \qquad \langle J_1 F_{xx}, F_{xxx} \rangle + \langle J_1 F_x, F_{xxxx} \rangle = 0.$$

We now use these relations to prove the following key result.

Proposition 6.1 *Let $F : \Omega \to \mathbb{H}^3_{1,\tau} \subset \mathbb{R}^4_2$ be an immersion from an open set $\Omega \subset \mathbb{R}^2$, with local coordinates (x, y) such that the projection of $E_1 = \frac{\sqrt{\kappa}}{2\tau}J_1 F$ is F_x. Then $F(\Omega) \subset \mathbb{H}^3_{1,\tau}$ defines a helix surface of constant angle function ν if and only if*

$$g_\tau(F_x, F_x) = g_\tau(E_1, F_x) = -\lambda(\lambda + \nu^2) \tag{6.2}$$

and

$$g_\tau(F_x, F_y) - g_\tau(F_y, E_1) = 0. \tag{6.3}$$

Proof Suppose that $F(\Omega)$ is a helix surface in $\mathbb{H}^3_{1,\tau}$ of constant angle function ν, then we have:

$$g_\tau(F_x, F_x) = \langle F_x, F_x \rangle + (1 - \tau^2)\langle F_x, X_1 \rangle^2 = \langle F_x, F_x \rangle + (1 - \tau^2)\frac{\kappa}{4}\langle F_x, J_1 F \rangle^2$$

$$= \frac{4}{\kappa}\tilde{a} + (1 - \tau^2)\left(-\frac{2\lambda(\lambda + \nu^2)}{\tau\sqrt{\kappa}}\right)^2 = -\lambda(\lambda + \nu^2).$$

In a similar way we find:

$$g_\tau(F_x, E_1) = \frac{1}{\tau}\left[\langle F_x, X_1 \rangle - (1 - \tau^2)\langle F_x, X_1 \rangle\right]$$

$$= \tau\langle F_x, X_1 \rangle = -\lambda(\lambda + \nu^2),$$

that leads to the equation (6.2). In addition, we have

$$g_\tau(F_y, E_1) = g_\tau(F_y, F_x + \nu N) = g_\tau(F_y, F_x)$$

that is equation (6.3).

To prove the converse, as vector fields F_x, F_y are tangent to $F(\Omega)$ (and so, orthogonal to normal vector field N), we consider

$$\tilde{T} = F_y - \frac{g_\tau(F_y, F_x)F_x}{g_\tau(F_x, F_x)}$$

and we get the orthogonal basis $\{F_x, \tilde{T}, N\}$ for the tangent space to $\mathbb{H}^3_{1,\tau}$ along $F(\Omega)$. Moreover, from (6.2) and (6.3) we have:

$$g_\tau(E_1, \tilde{T}) = g_\tau(E_1, F_y) - g_\tau(E_1, F_x)\frac{g_\tau(F_y, F_x)}{g_\tau(F_x, F_x)} = 0.$$

This leads to $E_1 = c_1 F_x + c_2 N$. Next, from (6.2) we have $c_1 = 1$ that means $(E_1)^T = F_x$ and also

$$-1 = g_\tau(E_1, E_1) = -(1 + \lambda\nu^2) + c_2^2\lambda$$

whence, $g_\tau(E_1, N) = \lambda|\nu|$ is constant and so, $F(\Omega)$ is a helix surface. $\qquad\square$

In order to get explicit solutions of Eqs. (5.5) and (5.4) we consider three different cases, depending on the different possibilities for B.

6.1 Helix surfaces of $\mathbb{H}^3_{1,\tau}$ in the case $B > 0$

Integrating (5.5) we prove the following.

Proposition 6.2 *Let M be a helix surface in $\mathbb{H}^3_{1,\tau}$ with constant angle function ν such that $B > 0$. Then, with respect to the local coordinates (x, y) defined above, the position vector F of M in $\mathbb{R}^4_2$ is explicitly given by*

$$F(x, y) = \cos(\alpha_1 x)w^1(y) + \sin(\alpha_1 x)w^2(y)$$
$$+ \cos(\alpha_2 x)w^3(y) + \sin(\alpha_2 x)w^4(y),$$

where

$$\alpha_1 = \frac{\sqrt{\kappa}}{2}\left(|\nu|\sqrt{B} + \frac{B}{\tau}\right), \qquad \alpha_2 = \frac{\sqrt{\kappa}}{2}\left(|\nu|\sqrt{B} - \frac{B}{\tau}\right)$$

are real constants and $w^i(y)$, $i = 1, 2, 3, 4$, *are mutually orthogonal vector fields in* $\mathbb{R}^4_2$, *depending only on* y, *such that, setting* $w_{ij} = \langle w^i(y), w^j(y)\rangle$ *for all indices* i, j, *we have*

$$w_{11} = w_{22} = \frac{4\tau}{\kappa^{\frac{3}{2}}B}\alpha_2, \qquad w_{33} = w_{44} = -\frac{4\tau}{\kappa^{\frac{3}{2}}B}\alpha_1. \tag{6.4}$$

Proof If $B > 0$, then $\tilde{b}^2 + 2\tilde{a} > 0$ and $\tilde{b}^2 + 4\tilde{a} > 0$. Integrating Eq. (5.5), we then obtain

$$F(x, y) = \cos(\alpha_1 x)w^1(y) + \sin(\alpha_1 x)w^2(y)$$
$$+ \cos(\alpha_2 x)w^3(y) + \sin(\alpha_2 x)w^4(y)$$

where

$$\alpha_{1,2} = \sqrt{\frac{\tilde{b}^2 + 2\tilde{a} \pm \sqrt{\tilde{b}^2(\tilde{b}^2 + 4\tilde{a})}}{2}}$$

are two real constants and $w^i(y)$, $i = 1, 2, 3, 4$, are vector fields in $\mathbb{R}^4_2$, depending only on y.

Using the expressions of $\tilde{a}$ and $\tilde{b}$, we have

$$\alpha_{1,2} = \frac{\sqrt{\kappa}}{2}\left(|\nu|\sqrt{B} \pm \frac{B}{\tau}\right).$$

Setting $w_{ij} = \langle w^i(y), w^j(y)\rangle$ and evaluating relations (5.9) at $(0, y)$, we get

$$w_{11} + 2w_{13} + w_{33} = -\frac{4}{\kappa}, \tag{6.5}$$

$$\alpha_1^2 w_{22} + 2\alpha_1\alpha_2 w_{24} + \alpha_2^2 w_{44} = \frac{4}{\kappa}\tilde{a}, \tag{6.6}$$

$$\alpha_1 w_{12} + \alpha_2 w_{14} + \alpha_1 w_{23} + \alpha_2 w_{34} = 0, \tag{6.7}$$

$$\alpha_1^3 w_{12} + \alpha_2\alpha_1^2 w_{14} + \alpha_1\alpha_2^2 w_{23} + \alpha_2^3 w_{34} = 0, \tag{6.8}$$

$$\alpha_1^4 w_{11} + 2\alpha_1^2\alpha_2^2 w_{13} + \alpha_2^4 w_{33} = D, \tag{6.9}$$

$$\alpha_1^2 w_{11} + (\alpha_1^2 + \alpha_2^2)w_{13} + \alpha_2^2 w_{33} = \frac{4}{\kappa}\tilde{a}, \tag{6.10}$$

$$\alpha_1^4 w_{22} + (\alpha_1^3\alpha_2 + \alpha_1\alpha_2^3)w_{24} + \alpha_2^4 w_{44} = D, \tag{6.11}$$

$$\alpha_1^5 w_{12} + \alpha_2^3\alpha_1^2 w_{14} + \alpha_1^3\alpha_2^2 w_{23} + \alpha_2^5 w_{34} = 0, \tag{6.12}$$

$$\alpha_1^3 w_{12} + \alpha_2^3 w_{14} + \alpha_1^3 w_{23} + \alpha_2^3 w_{34} = 0, \tag{6.13}$$

$$\alpha_1^6 w_{22} + 2\alpha_1^3\alpha_2^3 w_{24} + \alpha_2^6 w_{44} = E. \tag{6.14}$$

From (6.7), (6.8), (6.12) and (6.13), it follows that

$$w_{12} = w_{14} = w_{23} = w_{34} = 0.$$

Moreover, (6.5), (6.9) and (6.10) yield

$$w_{11} = \frac{4\tau}{\kappa^{\frac{3}{2}}B}\alpha_2, \qquad w_{13} = 0, \qquad w_{33} = -\frac{4\tau}{\kappa^{\frac{3}{2}}B}\alpha_1.$$

Next, by (6.6), (6.11) and (6.14) we get

$$w_{22} = w_{11} > 0, \qquad w_{24} = 0, \qquad w_{44} = w_{33} < 0.$$

□

We now prove the following result.

Theorem 6.1 (Of characterization for $B > 0$) *Let M be a helix surface in $\mathbb{H}^3_{1,\tau} \subset \mathbb{R}^4_2$ with constant angle function v so that $B > 0$. Then, locally, the position vector of M in $\mathbb{R}^4_2$, with respect to the local coordinates (u, v) on M defined in (5.2), is*

$$F(x, y) = A(y)\,\gamma(x),$$

where

$$\gamma(x) = (\sqrt{w_{11}}\,\cos(\alpha_1 x), -\lambda\sqrt{w_{11}}\,\sin(\alpha_1 x), \sqrt{-w_{33}}\,\cos(\alpha_2 x), \lambda\sqrt{-w_{33}}\,\sin(\alpha_2 x)),$$

is a twisted geodesic of the Lorentzian torus $\mathbb{S}^1(\sqrt{w_{11}}) \times \mathbb{S}^1(\sqrt{-w_{33}}) \subset \mathbb{H}^3_{1,\tau}$, w_{11}, w_{33}, α_1, α_2 are the four constants given in Proposition 6.2, and $A(y) = A(\xi, \xi_1, \xi_2, \xi_3)(y)$ is a 1-parameter family of 4×4 pseudo-orthogonal matrices commuting with J_1, as described in (3.6), where ξ is a constant and

$$\cosh^2(\xi_1(y))\,\xi_2'(y) + \sinh^2(\xi_1(y))\,\xi_3'(y) = 0. \tag{6.15}$$

Conversely, a parametrization $F(x, y) = A(y)\,\gamma(x)$, with $\gamma(x)$ and $A(y)$ as above, defines a helix surface in $\mathbb{H}^3_{1,\tau}$ with constant angle function v.

Proof With respect to the local coordinates (x, y) on M defined in (5.2), Proposition 6.2 implies that the position vector of the helix surface in $\mathbb{R}^4_2$ is given by

$$F(x, y) = \cos(\alpha_1 x)w^1(y) + \sin(\alpha_1 x)w^2(y) + \cos(\alpha_2 x)w^3(y) + \sin(\alpha_2 x)w^4(y),$$

where the vector fields $\{w^i(y)\}_{i=1}^4$ are mutually orthogonal and

$$\|w^1(y)\| = \|w^2(y)\| = \sqrt{w_{11}} = \text{constant},$$
$$\|w^3(y)\| = \|w^4(y)\| = \sqrt{-w_{33}} = \text{constant}.$$

Thus, if we put $e_i(y) = w^i(y)/\|w^i(y)\|$, $i \in \{1, \ldots, 4\}$, we can write:

$$F(x, y) = \sqrt{w_{11}}\,(\cos(\alpha_1 x)e_1(y) + \sin(\alpha_1 x)e_2(y)) + \sqrt{-w_{33}}\,(\cos(\alpha_2 x)e_3(y)$$
$$+ \sin(\alpha_2 x)e_4(y)). \tag{6.16}$$

The identities (6.1), evaluated in $(0, y)$, yield:

$$\alpha_1 w_{11}\langle J_1 e_1, e_2\rangle - \alpha_2 w_{33}\langle J_1 e_3, e_4\rangle + \sqrt{-w_{11}w_{33}}\,(\alpha_1\langle J_1 e_3, e_2\rangle + \alpha_2\langle J_1 e_1, e_4\rangle)$$
$$= -\frac{2\lambda(\lambda + v^2)}{\tau\sqrt{\kappa}}, \tag{6.17}$$

$$\langle J_1 e_1, e_3\rangle = 0, \tag{6.18}$$

$$w_{11}\alpha_1^3\langle J_1 e_1, e_2\rangle + \sqrt{-w_{11}w_{33}}(\alpha_1^2\alpha_2\langle J_1 e_1, e_4\rangle + \alpha_2^2\alpha_1\langle J_1 e_3, e_2\rangle)$$
$$- w_{33}\alpha_2^3\langle J_1 e_3, e_4\rangle = -L, \tag{6.19}$$

$$\langle J_1 e_2, e_4\rangle = 0, \tag{6.20}$$

$$\alpha_1\langle J_1 e_2, e_3\rangle + \alpha_2\langle J_1 e_1, e_4\rangle = 0, \tag{6.21}$$

$$\alpha_2 \langle J_1 e_2, e_3 \rangle + \alpha_1 \langle J_1 e_1, e_4 \rangle = 0. \tag{6.22}$$

We point out that to obtain the previous identities we have divided by $\alpha_1^2 - \alpha_2^2 = \kappa \tau^{-1} |\nu| / \sqrt{B^3}$, which, by the assumption on ν, is always different from zero. From (6.21) and (6.22), taking into account $\alpha_1^2 - \alpha_2^2 \neq 0$, we find

$$\langle J_1 e_3, e_2 \rangle = 0, \qquad \langle J_1 e_1, e_4 \rangle = 0. \tag{6.23}$$

Therefore,

$$|\langle J_1 e_1, e_2 \rangle| = 1 = |\langle J_1 e_3, e_4 \rangle|.$$

Substituting (6.23) in (6.17) and (6.19), we obtain the system

$$\begin{cases} \alpha_1 \, w_{11} \langle J_1 e_1, e_2 \rangle - \alpha_2 \, w_{33} \langle J_1 e_3, e_4 \rangle = -\dfrac{2\lambda(\lambda + \nu^2)}{\tau \sqrt{\kappa}}, \\ \alpha_1^3 \, w_{11} \langle J_1 e_1, e_2 \rangle - \alpha_2^3 \, w_{33} \langle J_1 e_3, e_4 \rangle = -L, \end{cases}$$

a solution of which is given by

$$\langle J_1 e_1, e_2 \rangle = \frac{2\alpha_2^2(1 + \lambda \nu^2) - \tau L \sqrt{\kappa}}{\tau \sqrt{\kappa} \, w_{11} \, \alpha_1 (\alpha_1^2 - \alpha_2^2)}, \qquad \langle J_1 e_3, e_4 \rangle = \frac{2\alpha_1^2(1 + \lambda \nu^2) + \tau L \sqrt{\kappa}}{\tau \sqrt{\kappa} \, w_{33} \, \alpha_2 (\alpha_1^2 - \alpha_2^2)}.$$

Consequently, $\langle J_1 e_1, e_2 \rangle = \langle J_1 e_3, e_4 \rangle = -\lambda$ and $J_1 e_1 = -\lambda e_2$, $J_1 e_3 = \lambda e_4$. Then, if we fix the orthonormal basis of $\mathbb{R}_2^4$ given by

$$\tilde{E}_1 = (1, 0, 0, 0), \quad \tilde{E}_2 = (0, -\lambda, 0, 0), \quad \tilde{E}_3 = (0, 0, 1, 0), \quad \tilde{E}_4 = (0, 0, 0, \lambda),$$

there must exists a 1-parameter family of 4×4 pseudo-orthogonal matrices $A(y) \in O_2(4)$, with $J_1 A(y) = A(y) J_1$, such that $e_i(y) = A(y) \tilde{E}_i$ for all indices $i = 1, \ldots, 4$.. Replacing $e_i(y) = A(y) \tilde{E}_i$ in (6.16) we obtain

$$F(x, y) = A(y) \gamma(x),$$

where

$$\gamma(x) = (\sqrt{w_{11}} \cos(\alpha_1 x), -\lambda \sqrt{w_{11}} \sin(\alpha_1 x), \sqrt{-w_{33}} \cos(\alpha_2 x), \lambda \sqrt{-w_{33}} \sin(\alpha_2 x)),$$

is a twisted geodesic of the Lorentzian torus $\mathbb{S}^1(\sqrt{w_{11}}) \times \mathbb{S}^1(\sqrt{-w_{33}}) \subset \mathbb{H}_{1,\tau}^3$.

Let now examine the 1-parameter family $A(y)$ that, according to (3.7), depends on four functions $\xi_1(y), \xi_2(y), \xi_3(y)$ and $\xi(y)$. From (5.2), it follows that $\langle F_y, F_y \rangle = (\lambda + \nu^2) = $ constant. The latter implies that

$$\frac{\partial}{\partial x} \langle F_y, F_y \rangle_{|x=0} = 0. \tag{6.24}$$

Now, if we denote by $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ the four colons of $A(y)$, (6.24) implies that

$$\begin{cases} \langle \mathbf{c}_2', \mathbf{c}_3' \rangle = 0 \\ \langle \mathbf{c}_2', \mathbf{c}_4' \rangle = 0, \end{cases} \tag{6.25}$$

where by $'$ we mean the derivative with respect to y. Replacing in (6.25) the expressions of the $\mathbf{c}_i$'s as functions of $\xi_1(y), \xi_2(y), \xi_3(y)$ and $\xi(y)$, we obtain

$$\begin{cases} \xi' h_1(y) = 0 \\ \xi' k_1(y) = 0, \end{cases} \tag{6.26}$$

where $h_1(y)$ and $k_1(y)$ are two functions such that

$$h_1^2 + k_1^2 = 4(\xi_1')^2 + \sinh^2(2\xi_1)(-\xi' + \xi_2' + \xi_3')^2.$$

From (6.26) we have two possibilities:

(i) $\xi = $ constant;

or

(ii) $4(\xi_1')^2 + \sinh^2(2\xi_1)(-\xi' + \xi_2' + \xi_3')^2 = 0.$

We will show that case (ii) cannot occurs. Indeed, if (ii) occurs, then the parametrization $F(x, y) = A(y)\gamma(x)$ defines a Hopf tube, that is, the hyperbolic Hopf vector field E_1 is tangent to the surface. To this end, we write the unit normal vector field N to the parametrization $F(x, y)$ as

$$N = \frac{N_1 E_1 + N_2 E_2 + N_3 E_3}{\sqrt{|-N_1^2 + N_2^2 + N_3^2|}},$$

where

$$\begin{cases} N_1 = g_\tau(F_x, E_3)g_\tau(F_y, E_2) - g_\tau(F_x, E_2)g_\tau(F_y, E_3), \\ N_2 = g_\tau(F_x, E_1)g_\tau(F_y, E_3) - g_\tau(F_x, E_3)g_\tau(F_y, E_1), \\ N_3 = g_\tau(F_x, E_2)g_\tau(F_y, E_1) - g_\tau(F_x, E_1)g_\tau(F_y, E_2). \end{cases}$$

A long computation then gives

$$N_1 = 1/2(\alpha_1 + \alpha_2)\sqrt{w_{11}}\sqrt{-w_{33}}\,[-2\xi_1' \cos(\alpha_1 x + \alpha_2 x + \xi_2 - \xi_3) \\ + \sinh(2\xi_1)\sin(\alpha_1 x + \alpha_2 x + \xi_2 - \xi_3)(-\xi' + \xi_2' + \xi_3')].$$

Now, case (ii) occurs if and only if either $\xi_1 = $ constant $= 0$ or $\xi_1 = $ constant $\neq 0$ and $-\xi' + \xi_2' + \xi_3' = 0$. In both cases, we conclude that $N_1 = 0$ and this implies that

$$g_\tau(N, J_1 F) = \frac{2\tau}{\sqrt{\kappa}}g_\tau(N, E_1) = 0,$$

i.e. the Hopf vector field is tangent to the surface.

Thus, we are left with the case where $\xi = $ constant. In this case, (6.3) is equivalent to

$$\cosh^2(\xi_1(y))\,\xi_2'(y) + \sinh^2(\xi_1(y))\,\xi_3'(y) = 0$$

whence we conclude that condition (6.15) is satisfied.

The converse of the theorem follows immediately from Proposition 6.1, since a direct calculation shows that $g_\tau(F_x, F_x) = g_\tau(E_1, F_x) = -\lambda(\lambda + \nu^2)$ (and so, (6.2) holds), while (6.15) is equivalent to (6.3). $\qquad\square$

Corollary 6.1 *Let M be a helix spacelike (respectively, timelike) surface in $\mathbb{H}_{1,\tau}^3 \subset \mathbb{R}_2^4$ with constant angle function ν such that $B > 0$. Then, there exist local coordinates on M such that the position vector of M in $\mathbb{R}_2^4$ is given by*

$$F(s, y) = A(y)\,\gamma(s),$$

where

$$\gamma(s) = \frac{2}{\sqrt{\kappa}}\frac{1}{\sqrt{d^2 - 1}}\left(\cos\left(\frac{\sqrt{\kappa}}{2}\,ds\right), -\lambda \sin\left(\frac{\sqrt{\kappa}}{2}\,ds\right), d \cos\left(\frac{\sqrt{\kappa}}{2}\frac{s}{d}\right), \lambda d \sin\left(\frac{\sqrt{\kappa}}{2}\frac{s}{d}\right)\right)$$

is a twisted geodesic in the Lorentzian torus $\mathbb{S}^1\left(\frac{2}{\sqrt{\kappa}}\frac{1}{\sqrt{d^2-1}}\right) \times \mathbb{S}^1\left(\frac{2}{\sqrt{\kappa}}\frac{d}{\sqrt{d^2-1}}\right) \subset \mathbb{H}^3_{1,\tau}$
parametrized by arc length, whose slope is given by

$$d = \frac{\sqrt{B} + \tau\,|\nu|}{\sqrt{\lambda + \nu^2}}.$$

In addition, $A(y) = A(\xi, \xi_1, \xi_2, \xi_3)(y)$ *is a 1-parameter family of* 4×4 *pseudo-orthogonal matrices commuting with* J_1, *as described in* (3.7), *with* $\xi = $ constant *and*

$$\cosh^2(\xi_1(y))\xi_2'(y) + \sinh^2(\xi_1(y))\xi_3'(y) = 0.$$

Conversely, a parametrization $F(s, y) = A(y)\,\gamma(s)$, *with* $\gamma(s)$ *and* $A(y)$ *as above, defines a helix surface in* $\mathbb{H}^3_{1,\tau}$.

Proof We consider the curve $\gamma(x)$ given in the Theorem 6.1. Since $\langle \gamma'(x), \gamma'(x) \rangle = \frac{4}{\kappa}\alpha_1\,\alpha_2$, considering

$$d := \sqrt{\frac{\alpha_1}{\alpha_2}} = \frac{\sqrt{B} + \tau\,|\nu|}{\sqrt{\lambda + \nu^2}},$$

from Eq. (6.4) and taking into account the equation (6.5) with $w_{13} = 0$, we get

$$w_{11} = \frac{4}{\kappa}\frac{1}{d^2 - 1}, \qquad w_{33} = -\frac{4}{\kappa}\frac{d^2}{d^2 - 1}.$$

Observe that $d > 1$. Therefore, we can consider the arc length reparametrization of the curve γ given by

$$\gamma(s) = \frac{2}{\sqrt{\kappa}}\frac{1}{\sqrt{d^2-1}}\left(\cos\left(\frac{\sqrt{\kappa}}{2}\,d\,s\right), -\lambda\sin\left(\frac{\sqrt{\kappa}}{2}\,d\,s\right), d\,\cos\left(\frac{\sqrt{\kappa}}{2}\frac{s}{d}\right), \lambda\,d\,\sin\left(\frac{\sqrt{\kappa}}{2}\frac{s}{d}\right)\right).$$

Finally, we observe that d represents the slope of the geodesic γ. $\qquad\square$

6.2 Helix surfaces of $\mathbb{H}^3_{1,\tau}$ in the case $B = 0$

Integrating (5.4) and taking into account $\partial_x F = T$, we prove at once the following.

Proposition 6.3 *Let M be a helix surface in* $\mathbb{H}^3_{1,\tau} \subset \mathbb{R}^4_2$ *with constant angle function* ν *such that* $B = 0$. *Then, with respect to the local coordinates* (x, y) *defined in* (5.2), *the position vector F of M in* $\mathbb{R}^4_2$ *is given by*

$$F(x, y) = T(y)x + w(y),$$

where $w(y)$ *is a timelike unit vector field in* $\mathbb{R}^4_2$, *depending only on* y.

We are now ready to prove the following result.

Theorem 6.2 (Of characterization for $B = 0$) *Let M be a helix surface in* $\mathbb{H}^3_{1,\tau} \subset \mathbb{R}^4_2$ *with constant angle function* ν *such that* $B = 0$. *Then, with respect to the local coordinates* (x, y) *defined in* (5.2), *the position vector F of M in* $\mathbb{R}^4_2$ *is given by*

$$F(x, y) = A(y)\gamma(x),$$

where

$$\gamma(x) = \left(v^2\tau x, 0, \frac{2}{\sqrt{\kappa}}, v^2\tau\lambda x\right)$$

is a straight line of $\mathbb{H}^3_{1,\tau}$ *(contained in the plane* $x_2 = x_3 - \frac{2}{\sqrt{\kappa}} = 0$*) and* $A(y) = A(\xi, \xi_1, \xi_2, \xi_3)(y)$ *is a 1-parameter family of* 4×4 *pseudo-orthogonal matrices commuting with* J_1 *as described in* (3.7)*, with*

$$
\begin{aligned}
[\xi_2'(y) + \xi_3'(y) - \xi'(y)] & \sin(\xi_2(y) - \xi_3(y)) \sinh(2\xi_1(y)) \\
& - 2\lambda(\xi'(y) - \xi_2'(y)) \cosh^2\xi_1(y) \\
& + 2[\xi_1'(y) \cos(\xi_2(y) - \xi_3(y)) + \lambda\xi_3'(y) \sinh^2\xi_1(y)] = 0.
\end{aligned}
\tag{6.27}
$$

Conversely, a parametrization

$$F(x, y) = A(y)\left(v^2\tau x, 0, \frac{2}{\sqrt{\kappa}}, v^2\tau\lambda x\right),$$

with $A(y)$ *as above, defines a helix surface in the anti-De Sitter space* $\mathbb{H}^3_{1,\tau}$ *with constant angle function* v.

Proof From Proposition 6.3 we say that

$$F(x, y) = T(y)x + w(y), \tag{6.28}$$

where $w(y)$ is a vector field in $\mathbb{R}^4_2$, depending only on y. Using (6.28) and evaluating the first three equations of (5.9) and the second equation of (6.1) at $(0, y)$, we get the following identities:

$$
\begin{aligned}
\langle F, F\rangle = \langle w(y), w(y)\rangle = -\frac{4}{\kappa}, \quad \langle F_x, F_x\rangle = \langle T(y), T(y)\rangle = 0, \\
\langle F, F_x\rangle = \langle w(y), T(y)\rangle = 0, \quad \langle J_1 w, T\rangle = -\frac{2\lambda(\lambda + v^2)}{\tau\sqrt{\kappa}}.
\end{aligned}
\tag{6.29}
$$

Moreover, evaluating (5.6) in $(0, y)$, setting

$$G(0, y) = -v\cos\varphi \, E_{2|F(0,y)} - v\sin\varphi \, E_{3|F(0,y)}.$$

and using (6.29), we have

$$\langle J_1 w, G(0, y)\rangle = 0, \quad \langle G(0, y), G(0, y)\rangle = v^2.$$

In particular, setting

$$g^1(y) = \frac{1}{|v|}G(0, y), \quad g^3(y) = \frac{\sqrt{\kappa}}{2}w(y),$$

we have that $\{g^1(y), J_1 g^1(y), g^3(y), J_1 g^3(y)\}$ is an orthonormal basis of $\mathbb{R}^4_2$. Consequently, if we fix the orthonormal basis $\{\hat{E}_i\}_{i=1}^4$ of $\mathbb{R}^4_2$ given by

$$\hat{E}_1 = (1, 0, 0, 0), \quad \hat{E}_2 = (0, 1, 0, 0), \quad \hat{E}_3 = (0, 0, 1, 0), \quad \hat{E}_4 = (0, 0, 0, 1),$$

there exists a 1-parameter family of matrices $A(y) \in O_2(4)$, with $J_1 A(y) = A(y)J_1$ such that

$$g^1(y) = A(y)\hat{E}_1, \quad J_1 g^1(y) = A(y)\hat{E}_2, \quad g^3(y) = A(y)\hat{E}_3, \quad J_1 g^3(y) = A(y)\hat{E}_4.$$

Then, (6.28) becomes

$$F(x, y) = \frac{2}{\sqrt{\kappa}} g^3(y) + v^2 \tau x \left(g^1(y) + \lambda J_1 g^3(y)\right) = A(y) \left(v^2 \tau x, 0, \frac{2}{\sqrt{\kappa}}, v^2 \tau \lambda x\right).$$

Finally, according to (3.7), the 1-parameter family $A(y)$ depends on four functions $\xi_1(y)$, $\xi_2(y)$, $\xi_3(y)$ and $\xi(y)$ and, in this case, condition (6.3) reduces to $\langle F_u, F_v \rangle = 0$ which is equivalent to (6.27).

In order to prove the converse, let

$$F(x, y) = A(y) \left(v^2 \tau x, 0, \frac{2}{\sqrt{\kappa}}, v^2 \tau \lambda x\right)$$

be a parametrization, where $A(y) = A(\xi(y), \xi_1(y), \xi_2(y), \xi_3(y))$ is a 1-parameter family of pseudo-orthogonal matrices with functions $\xi(y), \xi_1(y), \xi_2(y), \xi_3(y)$ satisfying (6.27). Since $A(y)$ satisfies (6.27), F satisfies (6.3). Thus, in virtue of Proposition 6.1, we only have to show that (6.2) is satisfied. We put

$$\gamma(x) = \left(v^2 \tau x, 0, \frac{2}{\sqrt{\kappa}}, v^2 \tau \lambda x\right).$$

Now, using (3.1) and taking into account the fact that $A(y)$ commutes with J_1, we get

$$g_\tau(F_x, F_x) = (1 - \tau^2) v^4 \tau^2 = -(1 + \lambda v^2)$$

and similarly,

$$g_\tau(E_1, F_x) = \tau \langle X_1, F_x \rangle = -\lambda v^2 \tau^2 = -(1 + \lambda v^2),$$

which ends the proof. □

6.3 Helix surfaces of $\mathbb{H}_{1,\tau}^3$ in the case $B < 0$

In this case, we start from (5.5) and prove the following result.

Proposition 6.4 *Let M be a helix surface in $\mathbb{H}_{1,\tau}^3$ with constant angle function v such that $B < 0$. Then, with respect to the local coordinates (x, y) defined above, the position vector F of M in $\mathbb{R}_2^4$ is given by*

$$\begin{aligned}F(x, y) = &\cos(\alpha x) \left[\cosh(\beta x) w^1(y) + \sinh(\beta x) w^3(y)\right] \\ &+ \sin(\alpha x) \left[\cosh(\beta x) w^2(y) + \sinh(\beta x) w^4(y)\right],\end{aligned}$$

where

$$\alpha = -\frac{\sqrt{\kappa}}{2} \frac{B}{\lambda \tau}, \qquad \beta = |v| \frac{\sqrt{-\kappa B}}{2},$$

are real constants and $w^i(y)$, $i = 1, 2, 3, 4$, are linearly independent vector fields in $\mathbb{R}_2^4$, depending only on y, such that:

$$w_{11} = w_{22} = -w_{33} = -w_{44} = -\frac{4}{\kappa}, \qquad w_{14} = -w_{23} = \frac{4\lambda |v| \tau}{\kappa \sqrt{-B}}. \tag{6.30}$$

Proof As $B < 0$, we have $\tilde{b}^2 + 4\tilde{a} < 0$. Integrating Eq. (5.5), we obtain

$$F(x, y) = \cos(\alpha x)[\cosh(\beta x) w^1(y) + \sinh(\beta x) w^3(y)]$$
$$+ \sin(\alpha x)[\cosh(\beta x) w^2(y) + \sinh(\beta x) w^4(y)],$$

where

$$\alpha = \frac{\tilde{b}}{2}, \qquad \beta = \frac{1}{2}\sqrt{-(\tilde{b}^2 + 4\tilde{a})}$$

are real constants and $w^i(y)$, $i = 1, 2, 3, 4$, are vector fields in $\mathbb{R}_2^4$, depending only on y. Moreover, using the definition of $\tilde{a}$ and $\tilde{b}$, we get

$$\alpha = -\frac{\sqrt{\kappa}}{2}\frac{B}{\lambda\tau}, \qquad \beta = |\nu|\frac{\sqrt{-\kappa B}}{2}.$$

Defining $w_{ij} = \langle w^i(y), w^j(y)\rangle$, for all indices i, j and evaluating the relations (5.9) in $(0, y)$, we find:

$$w_{11} = -\frac{4}{\kappa}, \tag{6.31}$$

$$\alpha^2 w_{22} + \beta^2 w_{33} + 2\alpha \beta w_{23} = \frac{4}{\kappa}\tilde{a}, \tag{6.32}$$

$$\alpha w_{12} + \beta w_{13} = 0, \tag{6.33}$$

$$\alpha\left(\beta^2 - \alpha^2\right) w_{12} + 2\alpha \beta^2 w_{34} + 2\alpha^2 \beta w_{24} + \beta\left(\beta^2 - \alpha^2\right) w_{13} = 0, \tag{6.34}$$

$$\left(\beta^2 - \alpha^2\right)^2 w_{11} + 4\alpha^2\beta^2 w_{44} + 4\alpha \beta\left(\beta^2 - \alpha^2\right) w_{14} = D, \tag{6.35}$$

$$\left(\beta^2 - \alpha^2\right) w_{11} + 2\alpha \beta w_{14} = -\frac{4}{\kappa}\tilde{a}, \tag{6.36}$$

$$\alpha^2\left(3\beta^2 - \alpha^2\right) w_{22} + \beta^2\left(\beta^2 - 3\alpha^2\right) w_{33} + 4\alpha \beta(\beta^2 - \alpha^2) w_{23} = -D, \tag{6.37}$$

$$\alpha\left(3\beta^2 - \alpha^2\right)\left(\beta^2 - \alpha^2\right) w_{12} + 2\alpha \beta^2\left(\beta^2 - 3\alpha^2\right) w_{34}$$

$$+ \beta\left(\beta^2 - 3\alpha^2\right)\left(\beta^2 - \alpha^2\right) w_{13} + 2\alpha^2 \beta\left(3\beta^2 - \alpha^2\right) w_{24} = 0, \tag{6.38}$$

$$\alpha\left(3\beta^2 - \alpha^2\right) w_{12} + \beta\left(\beta^2 - 3\alpha^2\right) w_{13} = 0, \tag{6.39}$$

$$\alpha^2\left(3\beta^2 - \alpha^2\right)^2 w_{22} + \beta^2\left(\beta^2 - 3\alpha^2\right)^2 w_{33} + 2\alpha \beta\left(3\beta^2 - \alpha^2\right)\left(\beta^2 - 3\alpha^2\right) w_{23} = E. \tag{6.40}$$

From (6.31), (6.35) and (6.36), it follows that

$$w_{11} = -w_{44} = -\frac{4}{\kappa}, \qquad w_{14} = \frac{4\beta}{\kappa\alpha} = \frac{4\lambda|\nu|\tau}{\kappa\sqrt{-B}}.$$

Also, from (6.33) and (6.39), we obtain

$$w_{12} = w_{13} = 0$$

and, therefore, from (6.34) and (6.38),

$$w_{24} = w_{34} = 0.$$

Moreover, using (6.32), (6.37) and (6.40), we get

$$w_{22} = -w_{33} = -\frac{4}{\kappa}, \qquad w_{23} = -\frac{4\beta}{\kappa\alpha} = -\frac{4\lambda|\nu|\tau}{\kappa\sqrt{-B}}.$$

$\square$

We now prove the following.

Theorem 6.3 (Of characterization for $B < 0$) *Let M be a helix surface in $\mathbb{H}^3_{1,\tau}$ with constant angle function ν so that $B < 0$. Then, locally, the position vector of M in $\mathbb{R}^4_2$, with respect to the local coordinates (x, y) on M defined in (5.2), is given by*

$$F(x, y) = A(y)\,\gamma(x),$$

where the curve $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x), \gamma_4(x))$ has components

$$\begin{cases} \gamma_1(x) = \dfrac{2\sqrt{\lambda + \nu^2}}{\sqrt{-\kappa B}}\,\cos(\alpha x)\,\sinh(\beta x), \\[2mm] \gamma_2(x) = \dfrac{2\sqrt{\lambda + \nu^2}}{\sqrt{-\kappa B}}\,\sin(\alpha x)\,\sinh(\beta x), \\[2mm] \gamma_3(x) = \dfrac{2}{\sqrt{\kappa}}\cos(\alpha x)\,\cosh(\beta x) - \dfrac{2\lambda\tau|\nu|}{\sqrt{-\kappa B}}\,\sin(\alpha x)\,\sinh(\beta x), \\[2mm] \gamma_4(x) = \dfrac{2}{\sqrt{\kappa}}\sin(\alpha x)\,\cosh(\beta x) + \dfrac{2\lambda\tau|\nu|}{\sqrt{-\kappa B}}\,\cos(\alpha x)\,\sinh(\beta x), \end{cases}$$

with

$$\alpha = -\frac{\sqrt{\kappa}}{2}\frac{B}{\lambda\tau}, \qquad \beta = |\nu|\frac{\sqrt{-\kappa B}}{2},$$

and $A(y) = A(\xi, \xi_1, \xi_2, \xi_3)(y)$ is a 1-parameter family of 4×4 pseudo-orthogonal matrices commuting with J_1, as described in (3.6), where ξ is a constant and

$$|\nu|\sqrt{\lambda + \nu^2}\,[2\cos(\xi_2(y) - \xi_3(y))\,\xi_1'(y) + (\xi_2'(y) + \xi_3'(y))\sin(\xi_2(y) - \xi_3(y))\,\sinh(2\xi_1(y))]$$
$$+ 2\lambda\tau\nu^2\,[\cosh^2(\xi_1(y))\,\xi_2'(y) + \sinh^2(\xi_1(y))\,\xi_3'(y)] = 0. \tag{6.41}$$

Conversely, a parametrization $F(x, y) = A(y)\,\gamma(x)$, with $\gamma(x)$ and $A(y)$ as above, defines a helix surface in $\mathbb{H}^3_{1,\tau}$ with constant angle function $\nu \neq 0$.

Proof From (6.30), we can define the following orthonormal basis in $\mathbb{R}^4_2$:

$$\begin{cases} e_1(y) = \dfrac{\sqrt{\kappa}}{2\sqrt{\lambda + \nu^2}}[\sqrt{-B}\,w^3(y) - \lambda\tau|\nu|\,w^2(y)], \\[2mm] e_2(y) = \dfrac{\sqrt{\kappa}}{2\sqrt{\lambda + \nu^2}}[\sqrt{-B}\,w^4(y) + \lambda\tau|\nu|\,w^1(y)], \\[2mm] e_3(y) = \dfrac{\sqrt{\kappa}}{2}w^1(y), \\[2mm] e_4(y) = \dfrac{\sqrt{\kappa}}{2}w^2(y), \end{cases}$$

with $\langle e_1, e_1 \rangle = 1 = \langle e_2, e_2 \rangle$ and $\langle e_3, e_3 \rangle = -1 = \langle e_4, e_4 \rangle$. Evaluating the identities (6.1) in $(0, y)$, and taking into account that:

$$F(0, y) = w^1(y),$$
$$F_x(0, y) = \alpha \, w^2(y) + \beta \, w^3(y),$$
$$F_{xx}(0, y) = \left(\beta^2 - \alpha^2\right) w^1(y) + 2\alpha \beta \, w^4(y),$$
$$F_{xxx}(0, y) = \alpha \left(3\beta^2 - \alpha^2\right) w^2(y) + \beta \left(\beta^2 - 3\alpha^2\right) w^3(y),$$
$$F_{xxxx}(0, y) = \left(\beta^4 - 6\alpha^2 \beta^2 + \alpha^4\right) w^1(y) + 4\alpha \beta \left(\beta^2 - \alpha^2\right) w^4(y),$$

we conclude that

$$\langle J_1 w^1, w^2 \rangle = -\langle J_1 w^3, w^4 \rangle = -\frac{4}{\kappa},$$
$$\langle J_1 w^3, w^2 \rangle = \langle J_1 w^1, w^4 \rangle = 0,$$
$$\langle J_1 w^2, w^4 \rangle = \langle J_1 w^1, w^3 \rangle = -\frac{4\lambda\tau \, |\nu|}{\kappa \sqrt{-B}}.$$

We point out that to obtain the previous identities, we divided by

$$\alpha^2 - \beta^2 = \frac{\kappa}{4} \frac{B}{\tau^2} (\lambda + \nu^2)$$

which is always different from zero. Then,

$$\langle J_1 e_1, e_2 \rangle = -\langle J_1 e_3, e_4 \rangle = 1,$$
$$\langle J_1 e_1, e_4 \rangle = \langle J_1 e_1, e_3 \rangle = \langle J_1 e_2, e_3 \rangle = \langle J_1 e_2, e_4 \rangle = 0.$$

Therefore, we have

$$J_1 e_1 = e_2, \qquad J_1 e_3 = e_4.$$

Consequently, if we consider the orthonormal basis $\{\hat{E}_i\}_{i=1}^4$ of $\mathbb{R}_2^4$ given by

$$\hat{E}_1 = (1, 0, 0, 0), \quad \hat{E}_2 = (0, 1, 0, 0), \quad \hat{E}_3 = (0, 0, 1, 0), \quad \hat{E}_4 = (0, 0, 0, 1),$$

there must exists a 1-parameter family of matrices $A(y) \in O_2(4)$, with $J_1 A(y) = A(y) J_1$, such that $e_i(y) = A(y)\hat{E}_i$ for all indices $i = 1, \ldots, 4$. As

$$F = \langle F, e_1 \rangle e_1 + \langle F, e_2 \rangle e_2 - \langle F, e_3 \rangle e_3 - \langle F, e_4 \rangle e_4,$$

computing $\langle F, e_i \rangle$ and substituting $e_i(y) = A(y)\hat{E}_i$, we obtain that $F(x, y) = A(y)\gamma(x)$, where $\gamma(x)$ is the curve of $\mathbb{H}_{1,\tau}^3$ described in the statement. Proceeding as in the proof of Theorem 6.1, we now examine the 1-parameter family $A(y)$ that, according to (3.7), depends on four functions $\xi_1(y), \xi_2(y), \xi_3(y)$ and $\xi(y)$. from $\langle F_y, F_y \rangle = \lambda + \nu^2 = $ constant we have

$$\frac{\partial}{\partial x} \langle F_y, F_y \rangle|_{x=0} = 0. \tag{6.42}$$

If we denote by c_1, c_2, c_3, c_4 the four columns of $A(y)$, Eq. (6.42) implies that

$$\begin{cases} \langle c_1', c_3' \rangle = 0, \\ 2\tau \, |\nu| \, \langle c_2', c_3' \rangle + \lambda \sqrt{\lambda + \nu^2} \left[\langle c_2', c_2' \rangle + \langle c_3', c_3' \rangle \right] = 0, \end{cases} \tag{6.43}$$

where $'$ denotes the derivative with respect to y. Replacing in (6.43) the expressions of the $\mathbf{c}_i$'s as functions of $\xi_1(y)$, $\xi_2(y)$, $\xi_3(y)$ and $\xi(y)$, we obtain

$$\begin{cases} \xi' \, h_2(y) = 0, \\ \xi' \, k_2(y) = 0, \end{cases} \tag{6.44}$$

where $h_2(y)$ and $k_2(y)$ are given by

$$\begin{cases} h_2(y) = 2\sin(\xi_2 - \xi_3)\,\xi_1' + (\xi' - \xi_2' - \xi_3')\cos(\xi_2 - \xi_3)\,\sinh(2\xi_1), \\ k_2(y) = \tau\,|\nu|\,[(\xi' - \xi_2' - \xi_3')\sin(\xi_2 - \xi_3)\,\sinh(2\xi_1) - 2\cos(\xi_2 - \xi_3)\,\xi_1'] \\ \qquad\quad + \lambda\,(\lambda + \nu^2)\,[2\cosh^2(\xi_1)\xi_2' + 2\sinh^2(\xi_1)\,\xi_3' - \xi'\cosh(2\xi_1)]. \end{cases}$$

From (6.44) we have two possibilities:

(i) $\xi =$ constant;

   or

(ii) $h_2 = k_2 = 0$.

As

$$N_1 = g_\tau(F_x, E_3)g_\tau(F_y, E_2) - g_\tau(F_x, E_2)g_\tau(F_y, E_3)$$

$$= \sqrt{\frac{\nu^2\,(\lambda + \nu^2)}{\kappa}}\left[\cosh(2\tilde{b}x)\,h_2(y) - \frac{\lambda\sinh(2\tilde{b}x)\,k_2(y)}{\sqrt{-B}}\right],$$

it results that if the case (ii) happens than the parametrization $F(x, y) = A(y)\gamma(x)$ defines a Hopf tube. Thus, we can assume that $\xi =$ constant and in this case (6.3) is equivalent to (6.41).

The converse easily follows from Proposition 6.1, since a direct calculation shows that $g_\tau(F_x, F_x) = g_\tau(E_1, F_x) = -\lambda(\lambda + \nu^2)$ (and so, (6.2) holds), while (6.41) is equivalent to (6.3). $\qquad\square$

7 Characterization of the helix surfaces of $\mathbb{H}^3_{1,\tau}$ by general helices

As a consequence of Proposition 6.1 and the characterization Theorems 6.1, 6.2 and 6.3, in the next result we will prove that the curves used to describe helix surfaces in $\mathbb{H}^3_{1,\tau}$, are general helices with axis the infinitesimal generator of the Hopf fibers. We recall that a *general helix* is a non-null curve α in a Lorentzian manifold (N, h), admitting a Killing vector field V of constant length along α, such that the function angle between V and α' is a non-zero constant (for the concept of general helix in a Lorentzian ambient space, we may also refer to [1]). We say that V is an axis of the general helix α. We now prove the following.

Proposition 7.1 *The curves* $\gamma : \mathbb{R} \to \mathbb{H}^3_{1,\tau}$ *used in the Theorems* 6.1, 6.2 *and* 6.3 *to character-ize a constant angle spacelike (respectively, timelike) surface* M, *are spacelike (respectively, timelike) general helices in* $\mathbb{H}^3_{1,\tau}$ *with axis* E_1, *so that they meet at constant angle the fibers of the Hopf fibration. This angle is the same in all the three cases.*

Proof We first observe that in the three cases the position vector of M has been expressed as

$$F(x, y) = A(y)\gamma(x),$$

where $A(y) = A(\xi, \xi_1, \xi_2, \xi_3)(y)$ is a 1-parameter family of 4×4 pseudo-orthogonal matrices commuting with J_1 and $\gamma(x)$ is a curve on $\mathbb{H}^3_{1,\tau}$. Therefore, as $F_x = A(y)\gamma'$, from (6.2) we get

$$g_\tau(\gamma', \gamma') = g_\tau(F_x, F_x) = -(1 + \lambda \nu^2),$$

thus we conclude that if M is a spacelike (respectively, timelike) surface, then γ is a spacelike (respectively, timelike) curve. In both cases, the above equation yields

$$\|\gamma'\|_\tau = \sqrt{\lambda + \nu^2}.$$

Moreover, as $J_1 A(y) = A(y) J_1$, we have

$$E_{1|F} = \frac{\sqrt{k}}{2\tau} J_1 F = \frac{\sqrt{k}}{2\tau} A(y) J_1 \gamma$$

and then, from (6.2), we obtain

$$g_\tau(\gamma', E_{1|\gamma}) = g_\tau\left(\gamma', \frac{\sqrt{k}}{2\tau} J_1 \gamma\right) = g_\tau\left(A(y)\gamma', \frac{\sqrt{k}}{2\tau} A(y) J_1 \gamma\right)$$

$$= g_\tau(F_x, E_{1|F}) = -\lambda(\lambda + \nu^2).$$

Therefore, the angle function that γ forms with the hyperbolic Hopf vector field is given by

$$\frac{g_\tau(\gamma', E_1)}{\|\gamma'\|_\tau} = -\lambda \sqrt{\lambda + \nu^2},$$

that is, in the three cases described in Theorems 6.1, 6.2 and 6.3, γ is a general helix, forming the same constant angle with its axis E_1. □

Remark 7.1 As we observed in Remark 5.2, when $\tau = 1$ we get flat helix surfaces in $\mathbb{H}^3_1(\kappa/4)$ equipped with its standard metric. The results we obtained are consistent with the ones deduced in [12], under the requirement of constant angle between N and E_1. In this case, $B = -\lambda$ and so:

- the case $B > 0$ corresponds to Lorentzian helix surfaces considered in [12], as $\lambda = -1$;
- the case $B = 0$ cannot occur;
- the case $B < 0$ corresponds to Riemannian helix surfaces considered in [12], as $\lambda = 1$.

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