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Kernel Bounds for Parabolic Operators Having First-Order Degeneracy at the Boundary

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Received: 29 April 2024 | **Revised:** 16 September 2025 | **Accepted:** 30 June 2026

Keywords: boundary degeneracy | degenerate elliptic operators | kernel estimates

ABSTRACT

We study kernel estimates for parabolic problems governed by singular elliptic operators

$$\sum_{i,j=1}^{N+1} q_{ij}D_{ij} + c\frac{D_y}{y}, \qquad \frac{c}{\gamma} + 1 > 0, \quad \gamma = q_{N+1,N+1},$$

in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$ under Neumann boundary conditions at $y = 0$.

MSC(2010): 35K08, 35K67, 47D07, 35J70, 35J75

1 | Introduction

In this paper, we study upper estimates for the parabolic heat kernel of the degenerate operator

$$\mathcal{L} = \sum_{i,j=1}^{N+1} q_{ij}D_{ij} + c\frac{D_y}{y} \tag{1}$$

in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$, under Neumann boundary conditions at $y = 0$. Here, $Q = (q_{ij})_{i,j=1,\dots,N+1} \in \mathbb{R}^{N+1,N+1}$ is a constant and positive-definite matrix and $c \in \mathbb{R}$ satisfies $\frac{c}{\gamma} + 1 > 0$ where $\gamma = q_{N+1,N+1}$. In the special case,

$$\mathcal{L} = \Delta_x + B_y, \qquad B_y = D_{yy} + \frac{c}{y}D_y,$$

that is when the mixed second-order derivatives do not appear in the operator, the one-dimensional Bessel operator B_y and the Laplacian Δ_x commute and sharp kernel estimates can be proved after an explicit description of the kernel of

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the Bessel operator (see [1]). These operators play a major role in the investigation of the fractional powers $(-\Delta_x)^s$ and $(D_t - \Delta_x)^s$, $s = (1-c)/2$, through the "extension procedure" of Caffarelli and Silvestre, see [2]. The strategy above based on the commutativity of Δ_x and B_y clearly fails for the general operator (1) due to the presence of the mixed derivatives D_{xy}. On the other hand, adding this term is crucial for treating degenerate operators in domains, through a localization procedure and freezing of the coefficients as explained in [3, Section 1].

For this reason we investigate, in this work, the validity of upper kernel estimate for the general operator $\mathcal{L}$ defined in (1) by revisiting the classical method based upon the equivalence between Gagliardo–Nirenberg type inequalities and ultracontractivity estimates.

Before describing the strategy here employed, we remark that elliptic and parabolic solvability of the associated problems in weighted L^p spaces have been already investigated in [1, 3–5] where we proved that, under suitable assumptions on m and p and for $c/\gamma > -1$, $\mathcal{L}$, endowed with Neumann boundary conditions, generates an analytic semigroup in $L^p_m = L^p(\mathbb{R}^{N+1}_+; y^m dxdy)$. In addition, we characterized its domain as a weighted Sobolev space and showed that it has the maximal regularity. In addition, we also refer to [6–9] where the authors studied operators of this form, even for variable coefficients, using tools form linear PDE and Muckhenoupt weights. In [10], among other properties, kernel estimates for some second-order degenerate operators in the divergence form are considered. In our knowledge, kernel estimates for the operators in the general form (1) are new.

Our main results are Theorems 4.18 and 5.2 where, if $\frac{c}{\gamma} + 1 > 0$, we prove that $\mathcal{L}$ generates a contractive analytic semigroup in $L^2_{\frac{c}{\gamma}}$ and its heat kernel $p_{\mathcal{L}}$, written with respect the measure $y^{\frac{c}{\gamma}} dz$, satisfies the upper Gaussian estimate

$$0 \le p_{\mathcal{L}}(t, z_1, z_2) \le C t^{-\frac{N+1}{2}} y_1^{-\frac{c}{2\gamma}} \left(1 \wedge \frac{y_1}{\sqrt{t}}\right)^{\frac{c}{2\gamma}} y_2^{-\frac{c}{2\gamma}} \left(1 \wedge \frac{y_2}{\sqrt{t}}\right)^{\frac{c}{2\gamma}} \exp\left(-\frac{|z_1 - z_2|^2}{kt}\right),$$

where $t > 0$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \mathbb{R}^{N+1}_+$; we also extend the estimate to complex times. As consequence of our kernel estimates, we deduce, after a linear transformation, upper kernel estimates also for the operator

$$\mathcal{L} = \sum_{i,j=1}^{N+1} q_{ij} D_{ij} + \frac{b \cdot \nabla_x}{y} + \frac{c D_y}{y}, \qquad b \in \mathbb{R}^N, \quad c \neq 0, \tag{2}$$

under an oblique boundary conditions at $y = 0$.

We now describe the strategy of proof employed. A suitable linear change of variables, allows to deal with the equivalent degenerate operator

$$\mathcal{L} := \Delta_x u + 2a \cdot \nabla_x D_y u + B_y u, \qquad a = (a_1, \dots, a_N) \in \mathbb{R}^N, \ |a| < 1.$$

Note that the condition $|a| < 1$ is equivalent to the ellipticity of the top order coefficients.

We start by constructing, in Section 2, using form methods, the semigroup generated by $\mathcal{L}$ in a weighted L^2-space, where the weighted measure $y^c\, dx\, dy$ takes into account the degeneracy of the operator. We prove continuity, accretivity and sectoriality for the form associated with $\mathcal{L}$ and its adjoint.

Then, we try to use the standard machinery, relying on Gagliardo–Nirenberg type inequalities, to deduce the ultracontractivity of the semigroup. To this aim, we investigate in Section 3 the validity of some anisotropic weighted Sobolev inequalities. Some difficulties appear since the semigroup $e^{t\mathcal{L}}$ is not ultracontractive when $c < 0$: this is due to the fact that the Gagliardo–Nirenberg inequalities are valid only locally due to the singularity of the operator at the boundary $y = 0$.

To overcome this obstruction, we introduce, in Section 4, a further weight ϕ which allow us to take into account the degeneracy of the operator at the boundary. This introduction is essential for $c < 0$ since allows to prove Gagliardo–Nirenberg inequalities in the whole half-space for any $c > -1$. When $c \ge 0$, this approach, although not mandatory, is needed in order to deduce precise heat kernel estimates (see Remark 4.16). Another obstruction to the application of the standard machinery is the non-symmetry of the operator $\mathcal{L}$: this force to study at the same time also the adjoint operator $\mathcal{L}^*$, proving $L^2 - L^\infty$ estimate for $e^{t\mathcal{L}}$ and $e^{t\mathcal{L}^*}$. Then, we implement Davies' trick to add the Gaussian factor. Moreover, we extend the estimates to complex times. We do not prove lower bounds; however, in the case $\Delta_x + B_y$, our upper bounds coincide with those of [1] which are optimal.

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Finally, in Section 5 we deduce upper kernel estimates for the operator in (2) having the oblique boundary condition.

Notation. For $N \geq 0$, $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$. We write $\nabla u, D^2 u$ for the gradient and the Hessian matrix of a function u with respect to all x, y variables and $\nabla_x u, D_y u, D_{x_i x_j} u, D_{x_i y} u$ and so on, to distinguish the role of x and y.

For $m \in \mathbb{R}$, we consider the measure $y^m dxdy$ in $\mathbb{R}^{N+1}_+$ and we write $L^p_m(\mathbb{R}^{N+1}_+)$, and often only L^p_m when $\mathbb{R}^{N+1}_+$ is understood, for $L^p(\mathbb{R}^{N+1}_+; y^m dxdy)$.

$\mathbb{C}^+$ stands for $\{\lambda \in \mathbb{C} : Re\,\lambda > 0\}$ and, for $|\theta| \leq \pi$, we denote by Σ_θ the open sector $\{\lambda \in \mathbb{C} : \lambda \neq 0,\ |Arg(\lambda)| < \theta\}$.

Given a and $b \in \mathbb{R}$, $a \wedge b$, $a \vee b$ denote their minimum and maximum. We write $f(x) \simeq g(x)$ for x in a set I and positive f, g, if for some $C_1, C_2 > 0$

$$C_1\, g(x) \leq f(x) \leq C_2\, g(x), \quad x \in I.$$

2 | The operator $\mathcal{L} = \Delta_x + 2a \cdot \nabla_x D_y + B_y u$ in L^2_c

Let $c \in \mathbb{R}$, $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ such that $c + 1 > 0$, $|a| < 1$. In this section, we study the L^2 theory related to the degenerate operator

$$\mathcal{L} := \Delta_x + 2\sum_{i=1}^N a_i D_{x_i y} + D_{yy} + \frac{c}{y} D_y = \Delta_x u + 2a \cdot \nabla_x D_y u + B_y u \tag{3}$$

in L^2_c equipped with the Neumann boundary condition. The requirement $|a| < 1$ is equivalent to the ellipticity of the top order coefficients.

We use the Sobolev space $H^1_c := \{u \in L^2_c : \nabla u \in L^2_c\}$ equipped with the inner product

$$\langle u, v\rangle_{H^1_c} := \langle u, v\rangle_{L^2_c} + \langle \nabla u, \nabla v\rangle_{L^2_c}.$$

The condition $c + 1 > 0$ assures, by [11, Theorem 4.9], that the set

$$C := \left\{u \in C^\infty_c\left(\mathbb{R}^N \times [0,\infty)\right),\ D_y u(x,y) = 0 \text{ for } y \leq \delta \text{ and some } \delta > 0\right\}, \tag{4}$$

is dense in H^1_c. Moreover, by [11, Remark 4.14], if a function $u \in H^1_c$ has support in $\mathbb{R}^N \times [0,b]$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \in C$ such that supp $u_n \subseteq \mathbb{R}^N \times [0,b]$ and $u_n \to u$ in H^1_c.

We consider the form in L^2_c

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v}\rangle\, y^c dx\, dy + 2\int_{\mathbb{R}^{N+1}_+} D_y u\, a \cdot \nabla_x \overline{v}\, y^c dx\, dy, \quad D(\mathfrak{a}) = H^1_c$$

and its adjoint $\mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)}$

$$\mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)} := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v}\rangle\, y^c dx\, dy + 2\int_{\mathbb{R}^{N+1}_+} a \cdot \nabla_x u\, D_y \overline{v}\, y^c dx\, dy.$$

Proposition 2.1. *The forms $\mathfrak{a}$, $\mathfrak{a}^*$ are continuous, accretive, and sectorial.*

Proof. We consider only the form $\mathfrak{a}$ since the adjoint form can be handled similarly. If $u \in H^1_c$

$$Re\,\mathfrak{a}(u,u) \geq \|\nabla_x u\|^2_{L^2_c} + \|D_y u\|^2_{L^2_c} - 2|a|\|\nabla_x u\|_{L^2_c}\|D_y u\|_{L^2_c} \geq (1-|a|)(\|\nabla_x u\|^2_{L^2_c} + \|D_y u\|^2_{L^2_c}).$$

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By the ellipticity assumption $|a|<1$, the accretivity follows. Moreover,

$$|Im\,\mathfrak{a}(u,u)| \le 2|a|\|\nabla_x u\|_{L^2_c}\|D_y u\|_{L^2_c} \le |a|(\|\nabla_x u\|^2_{L^2_c}+\|D_y u\|^2_{L^2_c}) \le \frac{|a|}{(1-|a|)} Re\,\mathfrak{a}(u,u).$$

This proves the sectoriality and then the continuity of the form. □

We define the operators $\mathcal{L}$ and $\mathcal{L}^*$ associated respectively to the forms $\mathfrak{a}$ and $\mathfrak{a}^*$ by

$$D(\mathcal{L}) = \{u \in H^1_c \,:\, \exists f \in L^2_c \text{ such that } \mathfrak{a}(u,v) = \int_{\mathbb{R}^{N+1}_+} f\overline{v}y^c\,dz \text{ for every } v \in H^1_c\},$$

$$\mathcal{L}u = -f; \tag{5}$$

$$D(\mathcal{L}^*) = \{u \in H^1_c \,:\, \exists f \in L^2_c \text{ such that } \mathfrak{a}^*(u,v) = \int_{\mathbb{R}^{N+1}_+} f\overline{v}y^c\,dz \text{ for every } v \in H^1_c\},$$

$$\mathcal{L}^*u = -f. \tag{6}$$

If u, v are smooth function with compact support in the closure of $\mathbb{R}^{N+1}_+$ (so that they do not need to vanish on the boundary), it is easy to see integrating by parts that

$$-\mathfrak{a}(u,v) = \langle \Delta_x u + 2a\cdot\nabla_x D_y u + B_y u, \overline{v}\rangle_{L^2_c}$$

if $\lim_{y\to 0} y^c D_y u(x,y) = 0$. This means that $\mathcal{L}$ is the operator $\Delta_x + 2a\cdot\nabla_x D_y + B_y$ with Neumann boundary conditions at $y=0$. On the other hand,

$$-\mathfrak{a}^*(u,v) = \left\langle \Delta_x u + 2a\cdot\nabla_x D_y u + 2c\frac{a\cdot\nabla_x u}{y} + B_y u, \overline{v}\right\rangle_{L^2_c}$$

if $\lim_{y\to 0} y^c\left(D_y u(x,y) + 2a\cdot\nabla_x u(x,y)\right) = 0$ and therefore $\mathcal{L}^*$ is the operator $\Delta_x + 2a\cdot\nabla_x D_y + 2c\frac{a\cdot\nabla_x u}{y} + B_y$ with the above oblique condition at $y=0$. We refer the reader to [11] for further details about the Sobolev spaces H^1_c and their boundary conditions at $y=0$.

Proposition 2.2. *$\mathcal{L}$ and $\mathcal{L}^*$ generate contractive analytic semigroups $e^{t\mathcal{L}}$, $e^{t\mathcal{L}^*}$, $t \in \Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{1-|a|}}$, in L^2_c. Moreover the semigroups $(e^{t\mathcal{L}})_{t\ge 0}, (e^{t\mathcal{L}^*})_{t\ge 0}$ are positive and L^p_c-contractive for $1 \le p \le \infty$.*

Proof. We argue only for $\mathcal{L}$.

The generation result immediately follows from Proposition 2.1 and [12, Theorem 1.52]. The positivity follows by [12, Theorem 2.6] after observing that, if $u \in H^1_c$, u real, then $u^+ \in H^1_c$ and

$$\mathfrak{a}(u^+,u^-) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u^+, \nabla u^-\rangle\, y^c dx\,dy + 2\int_{\mathbb{R}^{N+1}_+} D_y u^+ a\cdot\nabla_x u^- y^c dx\,dy = 0.$$

Finally, the L^∞-contractivity follows by [12, Corolary 2.17] after observing that if $0 \le u \in H^1_c$, then $1\wedge u, (u-1)^+ \in H^1_c$ and, since $\nabla(1\wedge u) = \chi_{\{u<1\}}\nabla u$ and $\nabla(u-1)^+ = \chi_{\{u>1\}}\nabla u$, one has

$$\mathfrak{a}(1\wedge u, (u-1)^+) = 0.$$

□

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The previous proposition can be generalized by showing that the above semigroup extrapolates to the weighted spaces $L^p\left(\mathbb{R}^{N+1}_+, y^m dxdy\right)$, $m \in \mathbb{R}$, when $0 < \frac{m+1}{p} < c+1$. We refer the reader to [3, 5] where a characterization of the domain of the generators is given in terms of suitable anisotropic weighted Sobolev spaces (see [11]) and where it is also proved that the generated semigroups satisfy parabolic maximal regularity.

Finally, we collect in the following proposition the scaling properties of $\mathcal{L}$ and $e^{t\mathcal{L}}$, whose proofs follow after an easy straightforward computation.

Proposition 2.3. *The scale homogeneity of $\mathcal{L}$ is 2:*

$$s^2\mathcal{L} = I_s^{-1}\mathcal{L}I_s, \qquad I_s u(x,y) = u(sx,sy), \qquad s>0. \tag{7}$$

Moreover, the semigroup $e^{t\mathcal{L}}$ generated by $\mathcal{L}$ in L^2_c satisfies

$$e^{s^2 t\mathcal{L}} = I_s^{-1} e^{t\mathcal{L}} I_s, \qquad t \in \Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{1-|a|}}, \quad s>0. \tag{8}$$

The same results hold for the adjoint operator $\mathcal{L}^*$ and its generated semigroup.

3 | Weighted Sobolev Inequalities

In this section, we investigate the validity of anisotropic weighted Sobolev inequalities for the space $H^1_c(\mathbb{R}^{N+1}_+)$. An essential role is played by the sign of c. Indeed assuming that for some $p>2$ one has

$$H^1_c(\mathbb{R}^{N+1}_+) \hookrightarrow L^p_c\left(\mathbb{R}^{N+1}_+\right), \qquad \|u\|_{L^p_c\left(\mathbb{R}^{N+1}_+\right)} \le C\|\nabla u\|_{L^2_c\left(\mathbb{R}^{N+1}_+\right)}.$$

then a simple scaling argument implies that the right expression of the Sobolev exponent p is

$$2^*_c \overset{def}{=} 2\frac{N+1+c}{N+c-1}, \qquad \text{i.e.,} \qquad \frac{1}{2^*_c} = \frac{1}{2} - \frac{1}{N+1+c}.$$

Then, a necessary condition for the validity of the latter inequalities is $c \ge 0$: this assumption guarantees that 2^*_c is not greater than the Sobolev exponent $2^* = 2\frac{N+1}{N-1}$ for the Lebesgue measure since otherwise this would imply better Sobolev embedding on sets far from the singularity $y=0$, which is impossible.

For this reason, we distinguish between the cases $c \ge 0$ and $c<0$. When $c \ge 0$ we prove that the embedding $H^1_c(\mathbb{R}^{N+1}_+) \hookrightarrow L^q_c(\mathbb{R}^{N+1}_+)$ holds for any $2 \le q < 2^*_c$. On the other hand, when $c<0$, we prove that the Sobolev immersion holds only locally and with the same Sobolev exponent 2^* as for the Lebesgue measure.

We start by the case $c \ge 0$. We need some preparation. We treat first the case $c=0$ where the latter embedding is just the standard Sobolev inequality which we state below in the form we use.

Lemma 3.1. *Let $N \in \mathbb{N}$. Then for every $q \in (2,\infty)$ satisfying $\frac{1}{q} \ge \frac{1}{2} - \frac{1}{N+1}$, there exists $\widetilde{C}_q > 0$ such that for every $u \in H^1(\mathbb{R}^{N+1}_+)$,*

$$\|u\|_{L^q(\mathbb{R}^{N+1}_+)} \le \widetilde{C}_q \|\nabla v\|^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2(\mathbb{R}^{N+1}_+)} \|v\|^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2(\mathbb{R}^{N+1}_+)}.$$

We state now the following lemmas which can be found in [13].

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Lemma 3.2. *If $\alpha \in \mathbb{R}$ and $q \geq 1$ satisfies*

$$1 \leq q \leq (N+1)' = \frac{N+1}{N}, \qquad \beta = \alpha - 1 + \frac{q-1}{q}(N+1) > -\frac{1}{q}$$

(in particular $\alpha + 1 > 0$) then there exists a positive constant C such that

$$\|y^{\beta} u\|_{L^q(\mathbb{R}^{N+1}_+)} \leq C \|y^{\alpha} \nabla u\|_{L^1(\mathbb{R}^{N+1}_+)}, \quad \forall u \in C_c^{\infty}\left(\overline{\mathbb{R}^{N+1}_+}\right). \tag{9}$$

Proof. See [13, Corollary 1, Section 2.1.7, p. 138]. □

Lemma 3.3. *Let $q,\ \beta,\ \alpha \in \mathbb{R}$, satisfying the following assumptions:*

(i) $2 \leq q \leq 2^* = 2\frac{N+1}{N-1}$ *($2 \leq q$ when $N = 1$);*

(ii) $\beta + \frac{1}{q} > 0$;

(iii) $\beta = \alpha - 1 + (N+1)\left(\frac{1}{2} - \frac{1}{q}\right)$, *i.e.,* $\frac{1}{q} = \frac{1}{2} - \frac{\beta - \alpha + 1}{N+1}$.

Then, one has for some positive constant C

$$\|y^{\beta} u\|_{L^q\left(\mathbb{R}^{N+1}_+\right)} \leq C \|y^{\alpha} \nabla u\|_{L^2\left(\mathbb{R}^{N+1}_+\right)}, \qquad \text{for all} \quad u \in C_c^{\infty}\left(\overline{\mathbb{R}^{N+1}_+}\right).$$

Proof. The proof follows by applying Lemma 3.2 with q, α, β replaced respectively by $\frac{2q}{q+2}, \alpha + \frac{q\beta}{2}, \left(\frac{q}{2} + 1\right)\beta$ to the function $|u|^{\frac{q}{2}+1}$ which yields

$$\|y^{\beta} u\|^{\frac{q}{2}+1}_{L^q(\mathbb{R}^{N+1}_+)} \leq C \frac{q+2}{2} \|y^{\alpha} |\nabla u| y^{\frac{q\beta}{2}} |u|^{\frac{q}{2}}\|_{L^1(\mathbb{R}^{N+1}_+)}.$$

Applying then the Cauchy–Schwarz inequality to the right-hand side yields

$$\|y^{\beta} u\|^{\frac{q}{2}+1}_{L^q(\mathbb{R}^{N+1}_+)} \leq C \frac{q+2}{2} \|y^{\alpha} \nabla u\|_{L^2(\mathbb{R}^{N+1}_+)} \|y^{\beta} u\|^{\frac{q}{2}}_{L^q(\mathbb{R}^{N+1}_+)}$$

which is the required claim. □

We can now prove the Sobolev immersion $H^1_c(\mathbb{R}^{N+1}_+) \hookrightarrow L_c^{2_c^*}\left(\mathbb{R}^{N+1}_+\right)$ assuming

$$2_c^* = 2\frac{N+1+c}{N+c-1}, \qquad c \geq 0.$$

Proposition 3.4. *Let $c > 0$ if $N = 1$, $c \geq 0$ otherwise. Then for some positive constant $C = C(c, N)$ one has*

$$\|u\|_{L_c^{2_c^*}\left(\mathbb{R}^{N+1}_+\right)} \leq C \|\nabla u\|_{L_c^2\left(\mathbb{R}^{N+1}_+\right)}, \qquad \text{for all} \quad u \in H_c^1\left(\mathbb{R}^{N+1}_+\right).$$

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Proof. Since the set C defined in (4) is dense in H^1_c, we can assume without any loss of generality $u \in C$. The proof then follows using Lemma 3.3 with $\alpha = \frac{c}{2}$, $\beta = \frac{c}{q}$. □

The following Gagliardo–Nirenberg type inequalities easily follow.

Proposition 3.5. *Assume $c \geq 0$ and that $q \in (2, \infty)$ satisfies $\frac{1}{q} \geq \frac{1}{2} - \frac{1}{N+1+c}$ i.e.*

(i) $2 < q \leq 2^*_c$ *if $N > 1$ and $c \geq 0$, or $N = 1$ and $c > 0$;*

(ii) $q > 2$ *if $N = 1$ and $c = 0$.*

Then, there exists a positive constant C, which may depends on c, N (and also on q in case (ii)), such that

$$\|u\|_{L^q_c(\mathbb{R}^{N+1}_+)} \leq C \|\nabla u\|^{(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_c(\mathbb{R}^{N+1}_+)} \|u\|^{1-(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_c(\mathbb{R}^{N+1}_+)}, \qquad \text{for all} \quad u \in H^1_c\left(\mathbb{R}^{N+1}_+\right).$$

Proof. Case (ii) is just Lemma 3.1. To prove (i) let us fix $u \in H^1_c\left(\mathbb{R}^{N+1}_+\right)$. Then, by Holder's inequality and Proposition 3.4, one has for some $C(c, N)$

$$\begin{aligned}\|u\|_{L^q_c(\mathbb{R}^{N+1}_+)} &\leq \|u\|^{(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^{2^*_c}_c(\mathbb{R}^{N+1}_+)} \|u\|^{1-(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_c(\mathbb{R}^{N+1}_+)} \\ &\leq \left[C\|\nabla u\|_{L^2_c(\mathbb{R}^{N+1}_+)}\right]^{(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^{1-(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_c(\mathbb{R}^{N+1}_+)}\end{aligned}$$

which is the required estimates. □

When $c < 0$, as explained before, the immersion $H^1_c(\mathbb{R}^{N+1}_+) \hookrightarrow L^{2^*_c}_c\left(\mathbb{R}^{N+1}_+\right)$ fails since otherwise would imply better Sobolev embedding on sets far from the singularity $y = 0$. In the next lemma, we show that, in this case, the Sobolev immersion still holds locally (note that in this case the Sobolev exponent for $|y|^c\, dz$ is the same as for the Lebesgue measure).

Lemma 3.6. *Let $c + 1 > 0$. The following properties hold.*

(i) If $N > 1$, then one has for some positive constant C

$$\|u\|_{L^{2^*_c}_c(\mathbb{R}^{N+1}_+)} \leq C \, \|y^{-\frac{c}{N+1}} \nabla u\|_{L^2_c(\mathbb{R}^{N+1}_+)}, \qquad \text{for all} \quad u \in C^\infty_c\left(\overline{\mathbb{R}^{N+1}_+}\right).$$

(ii) If $N = 1$, then for every $q \geq 2$ there exists a positive constant C such that

$$\|u\|_{L^q_c(\mathbb{R}^{N+1}_+)} \leq C \, \|y^{\frac{c+2}{q}-\frac{c}{2}} \nabla u\|_{L^2_c(\mathbb{R}^{N+1}_+)}, \qquad \text{for all} \quad u \in C^\infty_c\left(\overline{\mathbb{R}^{N+1}_+}\right).$$

Proof. The proof follows by using Lemma 3.3 with $\beta = \frac{c}{q}$ and, for claim (i), by imposing also $\frac{1}{q} = \frac{1}{2^*}$. □

Proposition 3.7. *Let $-1 < c < 0$ and $r > 0$. The following properties hold.*

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(i) If $N > 1$ then one has for some positive constant $C = C(r)$

$$\|u\|_{L_c^{2^*}(\mathbb{R}_+^{N+1})} \le C\,\|\nabla u\|_{L_c^2(\mathbb{R}_+^{N+1})}, \qquad \text{for all} \quad u \in H_c^1\left(\mathbb{R}_+^{N+1}\right), \quad \text{supp}\, u \subseteq \mathbb{R}^N \times [0,r].$$

(ii) If $N = 1$ then for every $q \ge 2$ there exists a positive constant $C = C(r)$ such that

$$\|u\|_{L_c^{q}(\mathbb{R}_+^{N+1})} \le C\,\|\nabla u\|_{L_c^2(\mathbb{R}_+^{N+1})}, \qquad \text{for all} \quad u \in H_c^1\left(\mathbb{R}_+^{N+1}\right), \quad \text{supp}\, u \subseteq \mathbb{R}^N \times [0,r].$$

Proof. For simplicity, we can assume $r = 1$. Using the density of C, we can assume without any loss of generality $u \in C$. The proof then follows by (i) and (ii) of Lemma 3.6 after noticing that if $y \le 1$ then in (i) one has $y^{-\frac{c}{N+1}} \le 1$ and in (ii), $y^{\frac{c+2}{q}-\frac{c}{2}} \le 1$. □

We can prove some local Gagliardo–Nirenberg inequalities.

Proposition 3.8. *Let $-1 < c < 0$, $r > 0$ and $q \in (2,\infty)$ such that $\frac{1}{q} \ge \frac{1}{2} - \frac{1}{N+1}$. Then for any $u \in H_c^1\left(\mathbb{R}_+^{N+1}\right)$ such that $\text{supp}\, u \subseteq \mathbb{R}^N \times [0,r]$ one has*

$$\|u\|_{L_c^{q}(\mathbb{R}_+^{N+1})} \le C\,\|\nabla u\|_{L_c^2(\mathbb{R}_+^{N+1})}^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|_{L_c^2(\mathbb{R}_+^{N+1})}^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}.$$

Proof. For simplicity, we can assume $r = 1$. The claim for $N > 1$ follows from (i) of Proposition 3.7 using Holder's inequality as in Proposition 3.5.

Let us prove the claim for $N = 1$. Using Lemma 3.2 with $q = 2$, $\beta = \alpha = \frac{c}{2}$, we obtain

$$\|u\|_{L_c^2(\mathbb{R}_+^{N+1})} \le C\|y^{-\frac{c}{2}}\nabla u\|_{L_c^1(\mathbb{R}_+^{N+1})}, \quad \forall u \in C_c^\infty\left(\overline{\mathbb{R}_+^{N+1}}\right).$$

In particular, since $c < 0$, we get

$$\|u\|_{L_c^2(\mathbb{R}_+^{N+1})} \le C\|\nabla u\|_{L_c^1(\mathbb{R}_+^{N+1})}, \quad \forall u \in C_c^\infty\left(\overline{\mathbb{R}_+^{N+1}}\right), \quad \text{supp}\, u \subseteq \mathbb{R}^N \times [0,1].$$

Let $m \in \mathbb{N}$; applying the latter inequality to $|u|^{m-1}u$ instead of u and using Holder's inequality we obtain

$$\|u\|_{L_c^{2m}(\mathbb{R}_+^{N+1})}^m \le Cm\|\nabla u|u|^{m-1}\|_{L_c^1(\mathbb{R}_+^{N+1})} \le Cm\|\nabla u\|_{L_c^2(\mathbb{R}_+^{N+1})}\|u\|_{L_c^{2m-2}(\mathbb{R}_+^{N+1})}^{m-1}$$

that is,

$$\|u\|_{L_c^{2m}(\mathbb{R}_+^{N+1})} \le (Cm)^{\frac{1}{m}}\|\nabla u\|_{L_c^2(\mathbb{R}_+^{N+1})}^{\frac{1}{m}}\|u\|_{L_c^{2(m-1)}(\mathbb{R}_+^{N+1})}^{1-\frac{1}{m}} \tag{10}$$

which is valid for any $u \in C_c^\infty\left(\overline{\mathbb{R}_+^{N+1}}\right)$ with $\text{supp}\, u \subseteq \mathbb{R}^N \times [0,1]$. The inequality above and an easy inductive argument then proves that, for some positive constant $C = C(m) > 0$, one has

$$\|u\|_{L_c^{2m}(\mathbb{R}_+^{N+1})} \le C\|\nabla u\|_{L_c^2(\mathbb{R}_+^{N+1})}^{1-\frac{1}{m}}\|u\|_{L_c^2(\mathbb{R}_+^{N+1})}^{\frac{1}{m}}, \quad \forall u \in C_c^\infty\left(\overline{\mathbb{R}_+^{N+1}}\right), \quad \text{supp}\, u \subseteq \mathbb{R}^N \times [0,1] \tag{11}$$

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which is the required claim for $q = 2m$. Indeed for $m = 1$ the latter inequality is simply $\|u\|_{L^2_c(\mathbb{R}^{N+1}_+)} \le C\|u\|_{L^2_c(\mathbb{R}^{N+1}_+)}$ which is obviously true. Let us suppose (11) true for $m = k$; then applying (10) we get for any $u \in C^\infty_c\left(\overline{\mathbb{R}^{N+1}_+}\right)$ with $\operatorname{supp} u \subseteq \mathbb{R}^N \times [0,1]$ and for some positive constant C depending on k

$$\begin{aligned}\|u\|_{L^{2(k+1)}_c(\mathbb{R}^{N+1}_+)} &\le C\|\nabla u\|^{\frac{1}{k+1}}_{L^2_c(\mathbb{R}^{N+1}_+)}\|u\|^{1-\frac{1}{k+1}}_{L^{2k}_c(\mathbb{R}^{N+1}_+)}\\ &\le C\|\nabla u\|^{\frac{1}{k+1}}_{L^2_c(\mathbb{R}^{N+1}_+)}\left(C\|\nabla u\|^{1-\frac{1}{k}}_{L^2_c(\mathbb{R}^{N+1}_+)}\|u\|^{\frac{1}{k}}_{L^2_c(\mathbb{R}^{N+1}_+)}\right)^{1-\frac{1}{k+1}}\\ &= C\|\nabla u\|^{1-\frac{1}{k+1}}_{L^2_c(\mathbb{R}^{N+1}_+)}\|u\|^{\frac{1}{k+1}}_{L^2_c(\mathbb{R}^{N+1}_+)}\end{aligned}$$

which is the claim for $m = k + 1$.

The claim for a general $q > 2$ then follows by interpolating $L^q_c(\mathbb{R}^{N+1}_+)$ between $L^{2m}_c(\mathbb{R}^{N+1}_+)$ and $L^{2m+2}_c(\mathbb{R}^{N+1}_+)$ for a suitable $m \in \mathbb{N}$. □

4 | Kernel Estimates

In this section, we deduce kernel estimates for the degenerate operator

$$\mathcal{L} = \Delta_x u + 2a \cdot \nabla_x D_y u + B_y u$$

of Section 2. We use the standard machinery of [12, Chapter 6], relying on Gagliardo–Nirenberg type inequalities for deducing the ultracontractivity of the semigroup, which gives bounds on the diagonal, and on Davies' method to add the Gaussian term. Some difficulties appear since the semigroup $e^{t\mathcal{L}}$ is not ultracontractive when $c < 0$: this is due to the fact that the Sobolev immersions of Proposition 3.5 fail to hold, in the whole space, for $c < 0$ due to the singularity of the operator at the boundary $y = 0$. To overcome this obstruction, we introduce then a further weight ϕ and an equivalent operator $\tilde{\mathcal{L}}$ which allow us to take into account the degeneracy of the operator at the boundary. This introduction is essential for $c < 0$ since allows to prove the Gagliardo–Nirenberg inequalities of Proposition 4.11 which hold in the whole half-space and for any $c > -1$; when $c \ge 0$, this approach, although not mandatory, is needed in order to deduce better heat kernel estimates (see Remark 4.16).

Definition 4.1. Let $\eta \in C^\infty[0,\infty)$ satisfying $0 \le \eta \le 1$, $\eta = 1$ on $[0,1/2]$ and $\eta = 0$ on $[2,\infty)$ and let us define the function $\phi \in C^\infty[0,+\infty)$ by

$$\phi(y) := \eta(y) + (1 - \eta(y))y^{-\frac{c}{2}}. \tag{12}$$

By elementary computations, ϕ satisfies for some positive constant $C_0 > 0$, see also [14, Lemma 3.3]:

$$\phi(y) \simeq y^{-\frac{c}{2}}(1 \wedge y)^{\frac{c}{2}}, \qquad \left|D_y\phi\right| \le C_0 y^{-1}\chi_{\{y\ge 1\}}\phi, \qquad |B\phi| \le C_0 y^{-2}\chi_{\{y\ge 1\}}\phi. \tag{13}$$

In particular, $\left|D_y\phi\right| \le C_0\,\phi$, $|B\phi| \le C_0\,\phi$.

Definition 4.2. We introduce the Hilbert space $L^2_\nu := L^2(\mathbb{R}^{N+1}_+, d\nu)$, and consider the isometry

$$\Phi : L^2(\mathbb{R}^{N+1}_+, d\nu) \to L^2_c\left(\mathbb{R}^{N+1}_+\right), \qquad f \mapsto \phi f \tag{14}$$

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where by (13)

$$d\nu = \phi^2\, y^c dx\, dy \simeq (y \wedge 1)^c\, dxdy. \tag{15}$$

The map Φ defines a similar operator $\tilde{\mathcal{L}} = \Phi^{-1}\mathcal{L}\Phi$ which acts on $L^2(\mathbb{R}^{N+1}_+, d\nu)$.

In what follows we prove that the semigroup generated by $\tilde{\mathcal{L}}$ is ultracontractive. We need some preparation. Let us consider the symmetric form on L^2_ν induced by Φ and defined by

$$\mathfrak{b}(u,v) := \mathfrak{a}(\phi u, \phi v) \qquad D(\mathfrak{b}) = \Phi^{-1}(D(\mathfrak{a})) = \frac{1}{\phi} H^1_c.$$

$\mathfrak{b}$ inherits, by construction, all the properties of $\mathfrak{a}$ stated in Propositions 2.1 and 2.2. In particular, since $\mathcal{C}$ is preserved by Φ, $\mathcal{C}$ is dense in $D(\mathfrak{b})$. Moreover, $\mathfrak{b}$ is formally associated with $-\tilde{\mathcal{L}}$. By (13), one can recognize that $D(\mathfrak{b})$ coincides with the Hilbert space

$$H^1_\nu \overset{\text{def}}{=} \overline{\mathcal{C}}^{\|\cdot\|_{H^1_\nu}}, \qquad \langle u, v\rangle_{H^1_\nu} := \langle u, v\rangle_{L^2_\nu} + \langle \nabla u, \nabla v\rangle_{L^2_\nu}$$

where the closure is taken with respect to the norm

$$\|u\|_{H^1_\nu} := \sqrt{\|\nabla u\|^2_{L^2_\nu} + \|u\|^2_{L^2_\nu}}.$$

We clarify the latter result in the following propositions.

Lemma 4.3. *For any $u, v \in D(\mathfrak{b})$ one has*

$$\mathfrak{b}(u,v) = \langle \nabla u, \nabla v\rangle_{L^2_\nu} + 2\big\langle D_y u, a\cdot\nabla_x v\big\rangle_{L^2_\nu} - \left\langle \frac{B\phi}{\phi}\, u, v\right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}\, u, a\cdot\nabla_x v\right\rangle_{L^2_\nu} \tag{16}$$

Proof. By density we can assume $u, v \in \mathcal{C}$; then by the definition of $\mathfrak{a}$ we have

$$\begin{aligned}
\mathfrak{a}(\phi u, \phi v) &= \int_{\mathbb{R}^{N+1}_+} \langle \nabla(\phi u), \nabla(\phi \overline{v})\rangle\, y^c dx\, dy + 2\int_{\mathbb{R}^{N+1}_+} D_y(\phi u)\, a\cdot\nabla_x \overline{v}\phi\, y^c dx\, dy \\
&= \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v}\rangle\, \phi^2 y^c dx\, dy + 2\int_{\mathbb{R}^{N+1}_+} D_y u\, a\cdot\nabla_x \overline{v}\phi^2\, y^c dx\, dy \\
&\quad + \int_{\mathbb{R}^{N+1}_+} \left[D_y\phi D_y u\overline{v}\phi + D_y\phi D_y\overline{v}u\phi + D_y\phi D_y\phi u\overline{v}\right] y^c dx\, dy \\
&\quad + 2\int_{\mathbb{R}^{N+1}_+} D_y\phi u\, a\cdot\nabla_x\overline{v}\phi\, y^c dx\, dy
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\mathfrak{a}(\phi u, \phi v) &= \langle \nabla u, \nabla v\rangle_{L^2_\nu} + 2\big\langle D_y u, a\cdot\nabla_x v\big\rangle_{L^2_\nu} \\
&\quad + \int_{\mathbb{R}^{N+1}_+} D_y\phi D_y\big(u\overline{v}\phi\big)\, y^c dx\, dy + 2\left\langle \frac{D_y\phi}{\phi}\, u, a\cdot\nabla_x v\right\rangle_{L^2_\nu}.
\end{aligned}$$

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The claim then follows after observing that using (13) and integrating by parts we get

$$\int_{\mathbb{R}^{N+1}_+} D_y\phi D_y\left(u\overline{v}\phi\right) y^c dx\, dy = -\int_{\mathbb{R}^{N+1}_+} (B\phi)u\overline{v}\,\phi\, y^c dx\, dy + \left[D_y\phi\,\phi\, u\overline{v}\, y^c\right]_0^{+\infty}$$

$$= -\int_{\mathbb{R}^{N+1}_+} (B\phi)u\overline{v}\,\phi\, y^c dx\, dy.$$

□

Proposition 4.4. *The following two norms are equivalent on C:*

$$\|u\|_{\mathfrak{b}} = \|\phi u\|_{\mathfrak{a}} = \sqrt{\mathrm{Re}\,\mathfrak{a}(\phi u, \phi u) + \|\phi u\|^2_{L^2_c}}, \qquad \|u\|_{H^1_\nu} = \sqrt{\|\nabla u\|^2_{L^2_\nu} + \|u\|^2_{L^2_\nu}}$$

In particular, $D(\mathfrak{b}) = H^1_\nu$.

Proof. We start by proving that for some positive constant C one has $\|u\|_{\mathfrak{b}} \le C\|u\|_{H^1_\nu}$ for any $u \in C$. Since $\|\phi u\|_{L^2_c} = \|u\|_{L^2_\nu}$, it is enough to prove that

$$Re\,\mathfrak{b}(u,u) = Re\,\mathfrak{a}(\phi u, \phi u) \le C\|u\|^2_{H^1_\nu}, \qquad u \in C.$$

Let C_0 be the constant in (13). Then using Lemma 4.3, the property of ϕ in (13) and Young's inequality, we have

$$Re\,\mathfrak{b}(u,u) = \langle \nabla u, \nabla u\rangle_{L^2_\nu} + 2\left\langle D_y u, a\cdot\nabla_x u\right\rangle_{L^2_\nu} - \left\langle \frac{B\phi}{\phi}u, u\right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}u, a\cdot\nabla_x u\right\rangle_{L^2_\nu}$$

$$\le \|\nabla u\|^2_{L^2_\nu} + |a|\|\nabla u\|^2_{L^2_\nu} + C_0\|u\|^2_{L^2_\nu} + C_0|a|\left(\|u\|^2_{L^2_\nu} + \|\nabla_x u\|^2_{L^2_\nu}\right)$$

$$\le (1+|a|(1+C_0))\|\nabla u\|^2_{L^2_\nu} + C_0(1+|a|)\,\|u\|^2_{L^2_\nu}$$

which is the required claim.

On the other hand, with C_0 as in (13) and using Lemma 4.3, the property of ϕ in (13) and Young's inequality again we have also, for every $\varepsilon > 0$,

$$Re\,\mathfrak{b}(u,u) = \langle \nabla u, \nabla u\rangle_{L^2_\nu} + 2\left\langle D_y u, a\cdot\nabla_x u\right\rangle_{L^2_\nu} - \left\langle \frac{B\phi}{\phi}u, u\right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}u, a\cdot\nabla_x u\right\rangle_{L^2_\nu}$$

$$\ge (1-|a|)\|\nabla u\|^2_{L^2_\nu} - C_0\|u\|^2_{L^2_\nu} - \frac{C_0}{\varepsilon}|a|\|u\|^2_{L^2_\nu} - C_0|a|\varepsilon\|\nabla_x u\|^2_{L^2_\nu}.$$

Choosing ε small enough such that $(1-|a|) - C_0|a|\epsilon > 0$, it follows that

$$\|\nabla u\|^2_{L^2_\nu} \le C\left(Re\,\mathfrak{b}(u,u) + \|u\|^2_{L^2_\nu}\right)$$

for some positive constant C. □

By similarity and by Proposition 2.2, $\tilde{\mathcal{L}}$ generates a contractive analytic semigroup of angle $\frac{\pi}{2} - \arctan\frac{|a|}{1-|a|}$ in L^2_ν. However, in order to deduce pointwise kernel estimates by the Davies' method, we need to introduce perturbed forms as follows.

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Definition 4.5. Let

$$\Psi := \left\{ \psi \in C^\infty(\overline{\mathbb{R}^{N+1}_+}) \cap L^\infty(\mathbb{R}^{N+1}_+) \, ; |\nabla \psi| \le 1; \quad |D^2\psi| \le 1 \right\}. \tag{17}$$

We introduce the quadratic forms $\{\mathfrak{b}_{\alpha\psi}\}_{\alpha\in\mathbb{R},\psi\in\Psi}$ in L^2_ν defined by

$$\mathfrak{b}_{\alpha\psi}(u,v) = \mathfrak{a}(e^{\alpha\psi}\phi u, e^{-\alpha\psi}\phi v), \qquad D(\mathfrak{b}_{\alpha\psi}) = D(\mathfrak{b}).$$

Remark 4.6. By standard computation, one can recognize that

$$d(z_1,z_2) \overset{def}{=} \sup_{\psi\in\Psi}\{\psi(z_1)-\psi(z_2)\}, \qquad \forall z_1, z_2 \in \mathbb{R}^{N+1}_+$$

defines a distance in $\mathbb{R}^{N+1}$ which is actually equivalent to the Euclidean one, i.e.,

$$|z_1 - z_2| \simeq \sup_{\psi\in\Psi}\{\psi(z_1)-\psi(z_2)\}, \qquad \forall z_1, z_2 \in \mathbb{R}^{N+1}_+.$$

We refer the reader to [15, pp. 200–203] or to [16, Corollary 3.2.8] for the proof of this classic result.

Since Φ is an isometry from L^2_ν onto L^2_c which maps $D(\mathfrak{b})$ onto $D(\mathfrak{a})$ and since $v \in L^2_\nu \to \mathrm{e}^{\pm\alpha\psi}v \in L^2_\nu$ are bounded operators preserving $D(\mathfrak{b})$, the following result easily follows.

Proposition 4.7. *For every $\alpha \in \mathbb{R}$ and $\psi \in \Psi$ the operator*

$$\mathcal{B}_{\alpha,\psi}u = \phi^{-1}e^{-\alpha\psi}\mathcal{L}(\phi e^{\alpha\psi}u)$$

with domain $D(\mathcal{B}_{\alpha\psi}) = \phi^{-1}e^{-\alpha\psi}(D(\mathcal{L}))$ is associated with the form $\mathfrak{b}_{\alpha\psi}$. Moreover, $\mathcal{B}_{\alpha,\psi}$ generates a bounded analytic semigroups $e^{z\mathcal{B}_{\alpha,\psi}}$, $z \in \Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{1-|a|}}$ in L^2_ν which satisfies

$$e^{z\mathcal{B}_{\alpha\psi}}f = \phi^{-1}e^{-\alpha\psi}e^{z\mathcal{L}}\left(\phi e^{\alpha\psi}f\right), \quad f \in L^2_\nu. \tag{18}$$

Proposition 4.8. *For every $\alpha \in \mathbb{R}$ and $\psi \in \Psi$, $\mathfrak{b}_{\alpha\psi}$ is quasi-accretive in L^2_ν and for any $0 < \epsilon < 1$ there exists some positive constant $C_\epsilon > 0$ dependent only on ϵ, a, c such that*

$$\mathrm{Re}\,\mathfrak{b}_{\alpha\psi}(u,u) \ge \epsilon(1-|a|)\|\nabla u\|^2_{L^2_\nu} - C_\epsilon\left(\alpha^2+1\right)\|u\|^2_{L^2_\nu}.$$

Moreover, $e^{t\mathcal{B}_{\alpha\psi}}$ is positive and one has

$$\|e^{t\mathcal{B}_{\alpha\psi}}f\|_\infty \le e^{C_\epsilon(1+\alpha^2)t}\|f\|_\infty, \quad f \in L^2_\nu \cap L^\infty(\mathbb{R}^{N+1}_+).$$

Proof. Since the form is real (i.e., $\mathfrak{b}_{\alpha\psi}(u,v) \in \mathbb{R}$ if $u,\ v \in \mathbb{R}$), one has $Re\,\mathfrak{b}_{\alpha\psi}(u,u) = \mathfrak{b}_{\alpha\psi}(Re\,u, Re\,u) + \mathfrak{b}_{\alpha\psi}(Im\,u, Im\,u)$ so we can restrict the analysis, without any loss of generality, to real $u \in D(\mathfrak{b})$. Using Lemma 4.3 and a straightforward calculation, we get

$$\mathfrak{b}_{\alpha\psi}(u,u) = \left\langle \nabla(\mathrm{e}^{\alpha\psi}u), \nabla(\mathrm{e}^{-\alpha\psi}u)\right\rangle_{L^2_\nu} + 2\left\langle D_y(\mathrm{e}^{\alpha\psi}u), a\cdot\nabla_x(\mathrm{e}^{-\alpha\psi}u)\right\rangle_{L^2_\nu}$$

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$$
\begin{aligned}
&-\left\langle \frac{B\phi}{\phi}\,\mathrm{e}^{\alpha\psi}u, \mathrm{e}^{-\alpha\psi}u\right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}\,\mathrm{e}^{\alpha\psi}u, a\cdot\nabla_x(\mathrm{e}^{-\alpha\psi}u)\right\rangle_{L^2_\nu}\\
=&\|\nabla u\|^2_{L^2_\nu} - \alpha^2\langle\nabla\psi u, \nabla\psi u\rangle_{L^2_\nu} - 2\alpha^2\left\langle D_y\psi u, a\cdot\nabla_x\psi u\right\rangle_{L^2_\nu} + 2\left\langle D_y u, a\cdot\nabla_x u\right\rangle_{L^2_\nu}\\
&-2\alpha\left\langle D_y u, a\cdot\nabla_x\psi u\right\rangle_{L^2_\nu} + 2\alpha\left\langle D_y\psi u, a\cdot\nabla_x u\right\rangle_{L^2_\nu}\\
&-\left\langle \frac{B\phi}{\phi}u, u\right\rangle_{L^2_\nu} - 2\alpha\left\langle \frac{D_y\phi}{\phi}u, a\cdot\nabla_x\psi u\right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}u, a\cdot\nabla_x u\right\rangle_{L^2_\nu}.
\end{aligned}
$$

Using the properties of ϕ in (13), we obtain

$$
\begin{aligned}
\mathfrak{b}_{\alpha\psi}(u,u) \geq&\|\nabla u\|^2_{L^2_\nu} - \alpha^2\|u\|^2_{L^2_\nu} - 2\alpha^2|a|\|u\|^2_{L^2_\nu} - 2|a|\|D_y u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu}\\
&-2|\alpha||a|\|u\|_{L^2_\nu}\|D_y u\|_{L^2_\nu} - 2|\alpha||a|\|u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu}\\
&-C_0\|u\|^2_{L^2_\nu} - 2|\alpha||a|C_0\|u\|^2_{L^2_\nu} - 2C_0|a|\|u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu}.
\end{aligned}
\tag{19}
$$

Using the Cauchy–Schwarz inequality, we get for any $\epsilon_1 > 0$

$$
2|\alpha|\|u\|_{L^2_\nu}\|D_y u\|_{L^2_\nu} = 2\left(\frac{|\alpha|}{\sqrt{\epsilon_1}}\|u\|_{L^2_\nu}\right)\left(\sqrt{\epsilon_1}\|D_y u\|_{L^2_\nu}\right) \leq \frac{\alpha^2}{\epsilon_1}\|u\|^2_{L^2_\nu} + \epsilon_1\|D_y u\|^2_{L^2_\nu}
$$

and similarly $2|\alpha|\|u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu} \leq \frac{\alpha^2}{\epsilon_1}\|u\|^2_{L^2_\nu} + \epsilon_1\|\nabla_x u\|^2_{L^2_\nu}$ which implies

$$
2|\alpha|\|u\|_{L^2_\nu}\|D_y u\|_{L^2_\nu} + 2|\alpha|\|u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu} \leq 2\frac{\alpha^2}{\epsilon_1}\|u\|^2_{L^2_\nu} + \epsilon_1\|\nabla u\|^2_{L^2_\nu}.
$$

Analogously, we get $2|a|\|D_y u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu} \leq |a|\|\nabla u\|^2_{L^2_\nu}$ and for any $\epsilon_2 > 0$

$$
2\|u\|_{L^2_\nu}\|\nabla_x u\|_{L^2_\nu} \leq \frac{1}{\epsilon_2}\|u\|^2_{L^2_\nu} + \epsilon_2\|\nabla_x u\|^2_{L^2_\nu}
$$

Inserting the latter estimates into (19) yields

$$
\begin{aligned}
\mathfrak{b}_{\alpha\psi}(u,u) \geq&[1 - |a|(1+\epsilon_1 + C_0\epsilon_2)]\|\nabla u\|^2_{L^2_\nu}\\
&-\left[\alpha^2\left(1 + 2|a| + \frac{2}{\epsilon_1}|a| + \frac{C_0}{\epsilon_2}\right) + 2|\alpha||a|C_0 + C_0 + \frac{C_0|a|}{\epsilon_2}\right]\|u\|^2_{L^2_\nu}.
\end{aligned}
$$

Let us fix now $0 < \epsilon < 1$. Since $|a| < 1$ we can choose ϵ_1, ϵ_2 sufficiently small such that

$$
1 - |a|(1+\epsilon_1 + C_0\epsilon_2) \geq \epsilon(1-|a|).
$$

Consequently, using the elementary inequality $\alpha^2 + c_1|\alpha| + c_2 \leq 2\alpha^2 + c_3$ for a suitable $c_3 > 0$, we get for some positive constant $C_1 > 0$ dependent only on ϵ, a

$$
\mathfrak{b}_{\alpha\psi}(u,u) \geq \epsilon(1-|a|)\|\nabla u\|^2_{L^2_\nu} - C_1\left(\alpha^2+1\right)\|u\|^2_{L^2_\nu}.
$$

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The proof of the remaining claim follows similarly as in the proof of Proposition 2.2. Indeed the positivity follows by [12, Theorem 2.6] after observing that, if $u \in H^1_\nu$, u real, then $u^+ \in H^1_\nu$ and, by Proposition 2.2,

$$\mathfrak{b}_{\alpha\psi}(u^+, u^-) = \mathfrak{a}(e^{\alpha\psi}\phi u^+, e^{-\alpha\psi}\phi u^-) = \mathfrak{a}((e^{\alpha\psi}\phi u)^+, (e^{-\alpha\psi}\phi u)^-) = 0.$$

Finally, the quasi-L^∞-contractivity follows by [12, Corollary 2.17] after proving that if $0 \le u \in H^1_\nu$ then, observing that $1 \wedge u, (u-1)^+ \in H^1_\nu$, one has

$$\mathfrak{b}_{\alpha\psi}(1 \wedge u, (u-1)^+) + C_2(1 \wedge u, (u-1)^+) \ge 0$$

for some positive constant $C_2 > 0$ independent of α, ψ. After this the required claim of the proposition follows after taking $C_\epsilon = \max\{C_1, C_2\}$. Indeed, let $0 \le u \in H^1_\nu$ and, using the density of C in (4), let us assume, without any loss of generality, $u \in C$. Then

$$\begin{aligned}
&\mathfrak{b}_{\alpha\psi}(1 \wedge u, (u-1)^+)\\
&= \big\langle \nabla(e^{\alpha\psi}(1 \wedge u)), \nabla(e^{-\alpha\psi}(u-1)^+)\big\rangle_{L^2_\nu} + 2\big\langle D_y(e^{\alpha\psi}(1 \wedge u)), a \cdot \nabla_x(e^{-\alpha\psi}(u-1)^+)\big\rangle_{L^2_\nu}\\
&\quad - \left\langle \frac{B\phi}{\phi} e^{\alpha\psi}(1 \wedge u), e^{-\alpha\psi}(u-1)^+ \right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi} e^{\alpha\psi}(1 \wedge u), a \cdot \nabla_x(e^{-\alpha\psi}(u-1)^+) \right\rangle_{L^2_\nu}\\
&= \big\langle \nabla(1 \wedge u), \nabla(u-1)^+\big\rangle_{L^2_\nu} - \alpha^2\big\langle \nabla\psi(1 \wedge u), \nabla\psi(u-1)^+\big\rangle_{L^2_\nu}\\
&\quad + \alpha\big\langle \nabla\psi(1 \wedge u), \nabla(u-1)^+\big\rangle_{L^2_\nu} - \alpha\big\langle \nabla(1 \wedge u), \nabla\psi(u-1)^+\big\rangle_{L^2_\nu}\\
&\quad - 2\alpha^2\big\langle D_y\psi(1 \wedge u), a \cdot \nabla_x\psi(u-1)^+\big\rangle_{L^2_\nu} + 2\big\langle D_y(1 \wedge u), a \cdot \nabla_x(u-1)^+\big\rangle_{L^2_\nu}\\
&\quad - 2\alpha\big\langle D_y(1 \wedge u), a \cdot \nabla_x\psi(u-1)^+\big\rangle_{L^2_\nu} + 2\alpha\big\langle D_y\psi(1 \wedge u), a \cdot \nabla_x(1-u)^+\big\rangle_{L^2_\nu}\\
&\quad - \left\langle \frac{B\phi}{\phi}(1 \wedge u), (u-1)^+ \right\rangle_{L^2_\nu} - 2\alpha\left\langle \frac{D_y\phi}{\phi}(1 \wedge u), a \cdot \nabla_x\psi u \right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}(1 \wedge u), a \cdot \nabla_x(1-u)^+ \right\rangle_{L^2_\nu}.
\end{aligned}$$

Since $\nabla(1 \wedge u) = \chi_{\{u<1\}}\nabla u$ and $\nabla(u-1)^+ = \chi_{\{u>1\}}\nabla u$, one has

$$\begin{aligned}
\mathfrak{b}_{\alpha\psi}(1 \wedge u, (u-1)^+) = &- \alpha^2\big\langle \nabla\psi, \nabla\psi(u-1)^+\big\rangle_{L^2_\nu} + \alpha\big\langle \nabla\psi, \nabla(u-1)^+\big\rangle_{L^2_\nu}\\
&- 2\alpha^2\big\langle D_y\psi, a \cdot \nabla_x\psi(u-1)^+\big\rangle_{L^2_\nu} + 2\alpha\big\langle D_y\psi, a \cdot \nabla_x(u-1)^+\big\rangle_{L^2_\nu}\\
&- \left\langle \frac{B\phi}{\phi}, (u-1)^+ \right\rangle_{L^2_\nu} - 2\alpha\left\langle \frac{D_y\phi}{\phi}, a \cdot \nabla_x\psi(u-1)^+ \right\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}, a \cdot \nabla_x(u-1)^+ \right\rangle_{L^2_\nu}.
\end{aligned}$$

Using the properties of ψ in (17) and of ϕ in (13) we get

$$\begin{aligned}
\mathfrak{b}_{\alpha\psi}(1 \wedge u, (u-1)^+) \ge &\big(-\alpha^2 - 2|a|\alpha^2 - C_0 - 2|\alpha|C_0|a|\big)\int_{\mathbb{R}^{N+1}_+}(u-1)^+ d\nu\\
&+ \alpha\big\langle \nabla\psi, \nabla(u-1)^+\big\rangle_{L^2_\nu} + 2\alpha\big\langle D_y\psi, a \cdot \nabla_x(u-1)^+\big\rangle_{L^2_\nu} + 2\left\langle \frac{D_y\phi}{\phi}, a \cdot \nabla_x(u-1)^+ \right\rangle_{L^2_\nu}.
\end{aligned}$$

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To estimate the second term, we integrate by parts and we use again the properties of ψ and of ϕ (which implies in particular the estimate $\frac{|D_y(\phi^2 y^c)|}{\phi^2 y^c} \le C$, for some $C > 0$) obtaining

$$\begin{aligned}\left|\alpha\left\langle \nabla\psi, \nabla(u-1)^+\right\rangle_{L^2_\nu}\right| &= \left|\alpha\int_{\mathbb{R}^{N+1}_+} \nabla\psi\cdot\nabla(u-1)^+ d\nu\right| \\ &= \left|-\alpha\int_{\mathbb{R}^{N+1}_+}\left(\Delta\psi + \frac{D_y\psi D_y(\phi^2 y^c)}{\phi^2 y^c}\right)(u-1)^+ d\nu\right| \\ &\le |\alpha|(N+1+C)\int_{\mathbb{R}^{N+1}_+}(u-1)^+ d\nu.\end{aligned}$$

Similarly, we get

$$\left|2\alpha\left\langle D_y\psi, a\cdot\nabla_x(u-1)^+\right\rangle_{L^2_\nu}\right| \le 2|\alpha||a|\int_{\mathbb{R}^{N+1}_+}(u-1)^+ d\nu$$

$$2\left\langle\frac{D_y\phi}{\phi}, a\cdot\nabla_x(u-1)^+\right\rangle_{L^2_\nu} = 0.$$

Putting together all the above estimates and using again the elementary inequality $\alpha^2 + c_1|\alpha| + c_2 \le 2\alpha^2 + c_3$, we finally get, for some positive constant $C > 0$,

$$\mathfrak{b}_{\alpha\psi}(1\wedge u, (u-1)^+) \ge -C\left(\alpha^2+1\right)\int_{\mathbb{R}^{N+1}_+}(u-1)^+ d\nu.$$

□

By analogous computations, we can get similar results for the adjoint form $\{\mathfrak{b}^*_{\alpha\psi}\}_{\alpha\in\mathbb{R},\psi\in\Psi}$ in L^2_ν defined by

$$\mathfrak{b}^*_{\alpha\psi}(u,v) = \overline{\mathfrak{b}^*_{\alpha\psi}(v,u)} = \mathfrak{a}^*(e^{-\alpha\psi}\phi u, e^{\alpha\psi}\phi v), \qquad D(\mathfrak{b}^*{}_{\alpha\psi}) = D(\mathfrak{b})$$

whose associated operator is the adjoint operator of $\mathcal{B}_{\alpha,\psi}$ in L^2_ν which we denote by $\mathcal{B}^*_{\alpha,\psi}$.

Proposition 4.9. *For every $\alpha\in\mathbb{R}$ and $\psi\in\Psi$ the operator*

$$\mathcal{B}^*_{\alpha,\psi}u = \phi^{-1}e^{\alpha\psi}\mathcal{L}^*(\phi e^{-\alpha\psi}u)$$

with domain $D(\mathcal{B}^*_{\alpha\psi}) = \phi^{-1}e^{\alpha\psi}(D(\mathcal{L}^*))$ is associated with the form $\mathfrak{b}^*_{\alpha\psi}$. Moreover, $\mathcal{B}^*_{\alpha,\psi}$ generates a bounded analytic semigroups $e^{z\mathcal{B}^*_{\alpha,\psi}}$, $z\in\Sigma_{\frac{\pi}{2}-\arctan\frac{|a|}{1-|a|}}$ in L^2_ν which satisfies

$$e^{z\mathcal{B}^*_{\alpha\psi}}f = \phi^{-1}e^{\alpha\psi}e^{z\mathcal{L}^*}\left(\phi e^{-\alpha\psi}f\right), \quad f\in L^2_\nu. \tag{20}$$

As before, we can also prove that

Proposition 4.10. *For every $\alpha\in\mathbb{R}$ and $\psi\in\Psi$, $\mathfrak{b}^*{}_{\alpha\psi}$ is quasi-accretive in L^2_ν and for any $0<\epsilon<1$ there exist some positive constant $C_\epsilon > 0$ dependent only on ϵ, a such that*

$$\operatorname{Re}\mathfrak{b}^*_{\alpha\psi}(u,u) \ge \epsilon(1-|a|)\|\nabla u\|^2_{L^2_\nu} - C_\epsilon\left(\alpha^2+1\right)\|u\|^2_{L^2_\nu}.$$

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Moreover, $e^{t\mathcal{B}^*_{\alpha\psi}}$ is positive and one has

$$\|e^{t\mathcal{B}^*_{\alpha\psi}} f\|_\infty \le e^{C_\epsilon(1+\alpha^2)t}\|f\|_\infty, \quad f \in L^2_\nu \cap L^\infty(\mathbb{R}^{N+1}_+).$$

4.1 | Gagliardo–Nirenberg Type Inequalities for $\mathfrak{b}_{\alpha\psi}$ and $\mathfrak{b}^*_{\alpha\psi}$

In this section, we use the results of Section 3 to derive some Gagliardo–Nirenberg inequalities for the space H^1_ν which are the main technical tools to prove upper bounds for the heat kernel of $\mathcal{L}$.

Proposition 4.11. *Assume $c+1>0$. Then for every $q \in (2,\infty)$ such that $\frac{1}{q} \ge \frac{1}{2} - \frac{1}{N+1+c^+}$ one has*

$$\|u\|_{L^q_\nu} \le C_q \|u\|^\theta_{H^1_\nu} \|u\|^{1-\theta}_\nu, \qquad \text{for all} \quad u \in H^1_\nu.$$

where C_q is a positive constant independent of u and $\theta = \left(N+1+c^+\right)\left(\frac{1}{2}-\frac{1}{q}\right) \in (0,1]$ ($\theta<1$ if $N=1$ and $c\le 0$).

Proof. Let $\eta \in C^\infty[0,\infty)$ satisfying $\eta = 1$ on $[0,1/2]$ and $\eta = 0$ on $[2,\infty)$ and ϕ as defined in (12). We note preliminarily that by (15) we have

$$d\nu \simeq y^c dxdy, \quad \text{if } y \in \operatorname{supp}\eta, \qquad d\nu \simeq dxdy, \quad \text{if } y \in \operatorname{supp}(1-\eta).$$

We treat separately the case $c<0$ and $c \ge 0$.

We start by the case $c<0$. From Lemmas 3.7 and 3.1, respectively, we deduce, for some positive constant $C = C_q > 0$ which may depend on q and which may vary in each occurrence, that

$$\begin{aligned}\|\eta u\|_{L^q_\nu} &\le C\|\eta u\|_{L^q_c} \le C_q \|\nabla(\eta u)\|^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_c} \|\eta u\|^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_c} \\ &\le C_q \|\eta\|^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{W^{1,\infty}} \left(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu}\right)^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_\nu}\end{aligned}$$

and

$$\begin{aligned}\|(1-\eta)u\|_{L^q_\nu} &\le C\|(1-\eta)u\|_q \\ &\le C_q \|\nabla((1-\eta)u)\|^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_2 \|(1-\eta)u\|^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_2 \\ &\le C_q \|\eta\|^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{W^{1,\infty}} \left(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_{d\nu}}\right)^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_\nu}.\end{aligned}$$

Combining the above estimates, we deduce

$$\|u\|_{L^q_\nu} \le C\left(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu}\right)^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)}_{L^2_\nu}$$

which is the required claim.

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We treat now, similarly, the case $c \geq 0$. Observing that $\frac{1}{q} > \frac{1}{2} - \frac{1}{N+1+c} > \frac{1}{2} - \frac{1}{N+1}$, and applying Lemmas 3.7 and 3.1, respectively, we get proceeding as before that

$$\|\eta u\|_{L^q_\nu} \leq C_q \Big(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu} \Big)^{(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|_{L^2_\nu}^{1-(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}$$

and

$$\begin{aligned} \|(1-\eta)u\|_{L^q_\nu} &\leq C_q \Big(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu} \Big)^{(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|_{L^2_\nu}^{1-(N+1)\left(\frac{1}{2}-\frac{1}{q}\right)} \\ &\leq C_q \Big(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu} \Big)^{(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|_{L^2_\nu}^{1-(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \end{aligned}$$

where in the last inequality we used the fact that $(A+B)^r A^{1-r}$ is increasing in $r \in [0,1]$, for $A, B \geq 0$. Combining the above estimates, we deduce

$$\|u\|_{L^q_\nu} \leq C_q \Big(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu} \Big)^{(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|_{L^2_\nu}^{1-(N+1+c)\left(\frac{1}{2}-\frac{1}{q}\right)}$$

which is the required claim. □

Using Propositions 4.8 and 4.10, we can immediately reformulate the latter result in terms of $\mathfrak{b}_{\alpha\psi}$ and $\mathfrak{b}^*_{\alpha\psi}$.

Corollary 4.12. *Assume $c+1 > 0$. Then for every $q \in (2,\infty)$ such that $\frac{1}{q} \geq \frac{1}{2} - \frac{1}{N+1+c^+}$ one has for all $u \in H^1_\nu = D(\mathfrak{b}_{\alpha\psi}) = D(\mathfrak{b}^*_{\alpha\psi})$*

$$\|u\|_{L^q_\nu} \leq C\big((1+\alpha^2)\|u\|_\nu^2 + \operatorname{Re}\mathfrak{b}_{\alpha\psi}(u,u)\big)^{\frac{\theta}{2}} \|u\|_\nu^{1-\theta}$$

and

$$\|u\|_{L^q_\nu} \leq C\Big((1+\alpha^2)\|u\|_\nu^2 + \operatorname{Re}\mathfrak{b}^*_{\alpha\psi}(u,u)\Big)^{\frac{\theta}{2}} \|u\|_\nu^{1-\theta}$$

where $\theta = \left(N+1+c^+\right)\left(\frac{1}{2}-\frac{1}{q}\right) \in (0,1]$ ($\theta < 1$ if $N=1$ and $c \leq 0$) and $C > 0$ is a positive constant which may depend only on q, a, c.

4.2 | Derivation of Kernel Estimates for $\mathrm{e}^{z\mathcal{L}}$ and $\mathrm{e}^{z\mathcal{L}^*}$

We obtain kernel estimates for $t > 0$ via Davies' method. For $x \in \mathbb{R}^N$, $y \in \mathbb{R}_+$, we write $z = (x,y)$ to denote by a generic point in $\mathbb{R}^{N+1}_+$. We consider preliminarily the operator $\tilde{\mathcal{L}}$ on $L^2(\mathbb{R}^{N+1}_+, d\nu)$ of Definition 4.2; we also consider the perturbed operators $\mathcal{B}_{\alpha,\psi}$ of Proposition 4.7 and, for clarity sake, we recall that

$$\tilde{\mathcal{L}} = \mathcal{B}_{0,\psi}, \qquad \mathcal{B}_{\alpha,\psi} u = \mathrm{e}^{-\alpha\psi}\tilde{\mathcal{L}}(e^{\alpha\psi}u), \qquad \mathrm{e}^{t\mathcal{B}_{\alpha,\psi}} f = \mathrm{e}^{-\alpha\psi}\mathrm{e}^{t\tilde{\mathcal{L}}}(\mathrm{e}^{\alpha\psi} f). \tag{21}$$

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Lemma 4.13. *Let $c+1>0$, $|a|<1$. The semigroup $(e^{t\tilde{\mathcal{L}}})_{t>0}$ consists of integral operators, i.e., there exists $p_{\tilde{\mathcal{L}}}(t,\cdot,\cdot)\in L^\infty(\mathbb{R}^{N+1}_+\times\mathbb{R}^{N+1}_+)$ such that for $t>0$, $z_1,\ z_2\in\mathbb{R}^{N+1}_+$*

$$e^{t\tilde{\mathcal{L}}}f(z_1)=\int_{\mathbb{R}^{N+1}_+}p_{\tilde{\mathcal{L}}}(t,z_1,z_2)f(z_2)\,d\nu,\quad f\in L^2_\nu.$$

Its heat kernel $p_{\tilde{\mathcal{L}}}$, written with respect the measure $d\nu$, satisfies for some $C,\varkappa>0$,

$$0\le p_{\tilde{\mathcal{L}}}(t,z_1,z_2)\le Ct^{-\frac{N+1+c^+}{2}}\exp\left(Ct-\frac{|z_1-z_2|^2}{kt}\right).\tag{22}$$

Proof. We use Propositions 4.8 and 4.10 with $\epsilon=\frac{1}{2}$ and correspondingly we consider the forms

$$\mathfrak{b}_{\alpha,\psi}+C(1+\alpha^2)\langle\ ,\ \rangle_{L^2_\nu},\quad \mathfrak{b}^*_{\alpha,\psi}+C(1+\alpha^2)\langle\ ,\ \rangle_{L^2_\nu}$$

which are accretive in L^2_ν and L^∞-contractive. By Corollary 4.12, we can apply [12, Theorem 6.2] to get that

$$\|\mathrm{e}^{t\mathcal{B}_{\alpha,\psi}}\|_{L^2_\nu\to L^\infty}\le Ct^{-\frac{N+1+c^+}{4}}\mathrm{e}^{C(1+\alpha^2)t},\quad \|\mathrm{e}^{t\mathcal{B}^*_{\alpha,\psi}}\|_{L^2_\nu\to L^\infty}\le Ct^{-\frac{N+1+c^+}{4}}\mathrm{e}^{C(1+\alpha^2)t},$$

(note that [12, Theorem 6.2] apply also to non-symmetric forms, see [12, Remark (ii), p. 159]). By duality the latter inequality imply also

$$\|\mathrm{e}^{t\mathcal{B}_{\alpha,\psi}}\|_{L^1_\nu\to L^2_\nu}\le Ct^{-\frac{N+1+c^+}{4}}\mathrm{e}^{C(1+\alpha^2)t},\quad \|\mathrm{e}^{t\mathcal{B}^*_{\alpha,\psi}}\|_{L^1_\nu\to L^2_\nu}\le Ct^{-\frac{N+1+c^+}{4}}\mathrm{e}^{C(1+\alpha^2)t}.$$

Then combining the latter estimates we obtain, for some possibly different constants $C>0$,

$$\|\mathrm{e}^{t\mathcal{B}_{\alpha,\psi}}\|_{L^1_\nu\to L^\infty}\le Ct^{-\frac{N+1+c^+}{2}}\mathrm{e}^{C(1+\alpha^2)t},\quad \|\mathrm{e}^{t\mathcal{B}^*_{\alpha,\psi}}\|_{L^1_\nu\to L^\infty}\le Ct^{-\frac{N+1+c^+}{2}}\mathrm{e}^{C(1+\alpha^2)t}.$$

The Dunford–Pettis criterion yields then the existence of a positive kernel $p_{\alpha,\psi,\nu}$ such that

$$\mathrm{e}^{t\mathcal{B}_{\alpha\psi}}f(z_1)=\int_{\mathbb{R}^{N+1}_+}p_{\alpha,\psi,\nu}(t,z_1,z_2)f(z_2)\,d\nu,\quad f\in L^2_\nu$$

and $0\le p_{\alpha,\psi,\nu}(t,z_1,z_2)\le Ct^{-\frac{N+1+c^+}{2}}\mathrm{e}^{C(1+\alpha^2)t}$ with $C>0$ independent of α,ψ. Using the equivalence $\mathrm{e}^{t\mathcal{B}_{\alpha,\psi}}=\mathrm{e}^{-\alpha\psi}\mathrm{e}^{t\tilde{\mathcal{L}}}\mathrm{e}^{\alpha\psi}$ of Equation (21), the heat kernel $p_{\tilde{\mathcal{L}}}$ of $\tilde{\mathcal{L}}$ satisfies

$$0\le\mathrm{e}^{-\alpha\psi(z_1)}p_{\tilde{\mathcal{L}}}(t,z_1,z_2)\mathrm{e}^{\alpha\psi(z_2)}=p_{\alpha,\psi,\nu}(t,z_1,z_2)\le Ct^{-\frac{N+1+c^+}{2}}\mathrm{e}^{C(1+\alpha^2)t}.$$

Optimizing on $\alpha\in\mathbb{R}$ we get

$$0\le p_{\tilde{\mathcal{L}}}(t,z_1,z_2)\le Ct^{-\frac{N+1+c^+}{2}}\mathrm{e}^{Ct}\exp\left(-\frac{(\psi(z_1)-\psi(z_2))^2}{4Ct}\right).$$

Optimizing finally on $\psi\in\Psi$ and recalling Remark 4.6 we get the required assertion. □

We can now prove heat kernel estimates for the degenerate operator

$$\mathcal{L}=\Delta_x+2a\cdot\nabla_xD_y+B_y$$

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in L^2_c with Neumann boundary conditions at $y = 0$: $\lim_{y\to 0} y^c D_y u = 0$. We recall that from Definition 4.2 we have $\mathcal{L} = \phi\tilde{\mathcal{L}}\phi^{-1}$ and then for $z_1 = (x_1, y_1), z_1 = (x_2, y_2) \in \mathbb{R}^{N+1}_+$

$$e^{t\mathcal{L}} f(z_1) = \phi(y_1)\left[e^{t\tilde{\mathcal{L}}}(\phi^{-1} f)\right](z_1) = \int_{\mathbb{R}^{N+1}_+} \phi(y_1) p_{\tilde{\mathcal{L}}}(t, z_1, z_2)\phi(y_2)^{-1} f(z_2)\, d\nu(z_2)$$

$$:= \int_{\mathbb{R}^{N+1}_+} p_{\mathcal{L}}(t, z_1, z_2) f(z_2)\, y_2^c dz_2$$

where

$$p_{\mathcal{L}}(t, z_1, z_2) = \phi(y_1) p_{\tilde{\mathcal{L}}}(t, z_1, z_2)\phi(y_2). \tag{23}$$

Using the scaling property of $e^{t\mathcal{L}}$, see (8), we obtain the following kernel estimates which constitute the main result of this section. Note that the distinction between the cases $c < 0$ and $c \geq 0$ disappears.

Theorem 4.14. *Let $c + 1 > 0$, $|a| < 1$. The semigroup $(e^{t\mathcal{L}})_{t>0}$ consists of integral operators, i.e., there exists $p_{\mathcal{L}}(t, \cdot, \cdot) \in L^\infty(\mathbb{R}^{N+1}_+ \times \mathbb{R}^{N+1}_+)$ such that for $t > 0$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \mathbb{R}^{N+1}_+$*

$$e^{t\mathcal{L}} f(z_1) = \int_{\mathbb{R}^{N+1}_+} p_{\mathcal{L}}(t, z_1, z_2) f(z_2)\, y_2^c\, dz_2, \quad f \in L^2_c.$$

Its heat kernel $p_{\mathcal{L}}$, written with respect the measure $y^c dz$, satisfies for some $C, k > 0$

$$0 \leq p_{\mathcal{L}}(t, z_1, z_2) \leq C t^{-\frac{N+1}{2}} y_1^{-\frac{c}{2}} \left(1 \wedge \frac{y_1}{\sqrt{t}}\right)^{\frac{c}{2}} y_2^{-\frac{c}{2}} \left(1 \wedge \frac{y_2}{\sqrt{t}}\right)^{\frac{c}{2}} \exp\left(-\frac{|z_1 - z_2|^2}{kt}\right).$$

The last estimate can be written equivalently as

$$0 \leq p_{\mathcal{L}}(t, z_1, z_2) \leq C t^{-\frac{N+1}{2}} y_2^{-c} \left(1 \wedge \frac{y_2}{\sqrt{t}}\right)^{c} \exp\left(-\frac{|z_1 - z_2|^2}{kt}\right).$$

Proof. The existence of the heat kernel follows from Lemma 4.13 and (26). To prove the required estimate, we use (22) for $t = 1$ and (13) and (26) to get

$$p_{\mathcal{L}}(1, z_1, z_2) \leq C y_1^{-\frac{c}{2}} (1 \wedge y_1)^{\frac{c}{2}} y_2^{-\frac{c}{2}} (1 \wedge y_2)^{\frac{c}{2}} y_2^c \exp\left(\frac{-|z_1 - z_2|^2}{k}\right)$$

for any $(z_1, z_2) \in \mathbb{R}^{N+1}_+ \times \mathbb{R}^{N+1}_+$. On the other hand using the scaling property (8) of Proposition 2.3, we get

$$p_{\mathcal{L}}(s^2 t, z_1, z_2) = s^{-(N+1+c)} p_{\mathcal{L}}(t, s^{-1} z_1, s^{-1} z_2), \qquad s, t > 0, \quad z_1, z_2 \in \mathbb{R}^{N+1}_+.$$

In particular,

$$p_{\mathcal{L}}(t, z_1, z_2) = t^{-\frac{N+1+c}{2}} p_{\mathcal{L}}\left(1, \frac{z_1}{\sqrt{t}}, \frac{z_2}{\sqrt{t}}\right)$$

which combined with the above inequality for $t = 1$ gives the first required estimate.

The second equivalent estimate, up to a possibly different constants C, k, follows by applying [1, Lemma 10.2]. □

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Remark 4.15. We remark that the above estimates seem to be optimal since in the case $N = 0$ and $a = 0$ coincide with the optimal estimate proved in [17]. Moreover, in the case $\mathcal{L} = \Delta_x + B_y$, our upper bounds coincide with those of [1] which are optimal.

Remark 4.16. When $c \geq 0$, with the same proof of Lemma 4.13, one could have obtained directly kernel estimates for $e^{t\mathcal{L}}$ (without relying on the weight ϕ and on the equivalent operator $\tilde{\mathcal{L}}$) by using the Gagliardo–Nirenberg inequalities of Proposition 3.8 in place of Corollary 4.12. This approach leads to the Gaussian estimate

$$0 \leq p_{\mathcal{L}}(t, z_1, z_2) \leq Ct^{-\frac{N+1+c}{2}} \exp\left(-\frac{|z_1 - z_2|^2}{kt}\right)$$

which is weaker than the estimate proved in Theorem 4.14.

With the same proofs of Lemma 4.13 and Theorem 4.14, we can deduce heat kernel estimates for the degenerate operator

$$\mathcal{L}^* = \Delta_x + 2a \cdot \nabla_x D_y + 2c\frac{a \cdot \nabla_x u}{y} + B_y$$

in L^2_c with the oblique condition at $y = 0$: $\lim_{y\to 0} y^c \left(D_y u + 2a \cdot \nabla_x u\right) = 0$.

Theorem 4.17. *Let $c + 1 > 0$, $|a| < 1$. The semigroup $(e^{t\mathcal{L}^*})_{t>0}$ consists of integral operators, i.e., there exists $p_{\mathcal{L}^*}(t, \cdot, \cdot) \in L^\infty(\mathbb{R}^{N+1}_+ \times \mathbb{R}^{N+1}_+)$ such that for $t > 0$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \mathbb{R}^{N+1}_+$*

$$e^{t\mathcal{L}^*} f(z_1) = \int_{\mathbb{R}^{N+1}_+} p_{\mathcal{L}^*}(t, z_1, z_2) f(z_2)\, y_2^c\, dz_2, \quad f \in L^2_c.$$

Its heat kernel $p_{\mathcal{L}^*}$, written with respect the measure $y^c dz$, satisfies for some $C, k > 0$,

$$0 \leq p_{\mathcal{L}^*}(t, z_1, z_2) \leq Ct^{-\frac{N+1}{2}} y_1^{-\frac{c}{2}} \left(1 \wedge \frac{y_1}{\sqrt{t}}\right)^{\frac{c}{2}} y_2^{-\frac{c}{2}} \left(1 \wedge \frac{y_2}{\sqrt{t}}\right)^{\frac{c}{2}} \exp\left(-\frac{|z_1 - z_2|^2}{kt}\right).$$

The last estimate can be written equivalently as

$$0 \leq p_{\mathcal{L}^*}(t, z_1, z_2) \leq Ct^{-\frac{N+1}{2}} y_2^{-c} \left(1 \wedge \frac{y_2}{\sqrt{t}}\right)^{c} \exp\left(-\frac{|z_1 - z_2|^2}{kt}\right).$$

Once kernel estimates have obtained for positive t, based on the argument of [12, Theorem 6.16] relying on the Phragmen–Lindelöf theorem, we can extend them for complex times $t \in \Sigma_{\frac{\pi}{2} - \arctan\frac{|a|}{1-|a|}}$. We do not repeat the proof which is identical to that of [3, Section 5.3]. For simplicity, we state the result only for the operator $\mathcal{L}$.

Theorem 4.18. *Let $c + 1 > 0$, $|a| < 1$. The semigroup $(e^{t\mathcal{L}})$, $t \in \Sigma_{\frac{\pi}{2} - \arctan\frac{|a|}{1-|a|}}$ consists of integral operators, i.e., there exists $p_{\mathcal{L}}(t, \cdot, \cdot) \in L^\infty(\mathbb{R}^{N+1}_+ \times \mathbb{R}^{N+1}_+)$ such that for $t \in \Sigma_{\frac{\pi}{2} - \arctan\frac{|a|}{1-|a|}}$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \mathbb{R}^{N+1}_+$*

$$e^{t\mathcal{L}} f(z_1) = \int_{\mathbb{R}^{N+1}_+} p_{\mathcal{L}}(t, z_1, z_2) f(z_2)\, y_2^c\, dz_2, \quad f \in L^2_c.$$

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Moreover for every $\varepsilon > 0$ there exist constants $C_\varepsilon, k_\varepsilon > 0$ such that if $t \in \Sigma_{\frac{\pi}{2} - \arctan \frac{|a|}{1-|a|} - \varepsilon}$ then

$$|p_{\mathcal{L}}(t, z_1, z_2)| \le C_\varepsilon |t|^{-\frac{N+1}{2}} y_2^{-c} \left(1 \wedge \frac{y_2}{\sqrt{|t|}}\right)^c \exp\left(-\frac{|z_1 - z_2|^2}{k_\varepsilon |t|}\right).$$

5 | General Operators and Oblique Derivative

Kernel estimates for more general operators and boundary conditions follow by a linear change of variables, as we explain below. Let us first remove the assumption on the special form of $\mathcal{L} = \Delta_x + 2a \cdot \nabla_x D_y + B_y u$ by considering the general operator in $\mathbb{R}^{N+1}_+$

$$\mathcal{L} = \mathrm{Tr}\left(QD^2 u\right) + \frac{c}{y} D_y = \sum_{i,j=1}^{N} q_{ij} D_{x_i x_j} + 2 \sum_{i=1}^{N} q_i D_{x_i y} + \gamma D_{yy} + \frac{c}{y} D_y$$

with Neumann boundary condition $\lim_{y \to 0} y^{\frac{c}{\gamma}} D_y u = 0$. If Q_1 is the $N \times N$ matrix (q_{ij}), $\gamma = q_{N+1,N+1}$ and $q = (q_1, \dots, q_N)$, we assume that the quadratic form $Q(\xi, \eta) = Q_1(\xi, \xi) + \gamma \eta^2 + 2q \cdot \xi\, \eta$ is positive definite. Through a linear change of variables in the x variables, the term $\sum_{i,j=1}^{N} q_{ij} D_{x_i x_j}$ is transformed into $\gamma \Delta_x$ and all the results of the previous sections remain true. In particular, if $\frac{c}{\gamma} + 1 > 0$ then $\mathcal{L}$ generates a contractive analytic semigroups $\mathrm{e}^{z\mathcal{L}}$ in $L^2_{\frac{c}{\gamma}}$ and the kernel estimates of Theorems 4.14 and 4.18 remain true.

A further change of variables allows to deal with the operator

$$\mathcal{L} = \mathrm{Tr}\left(QD^2 u\right) + \frac{v \cdot \nabla}{y} = \sum_{i,j=1}^{N+1} q_{ij} D_{ij} + \frac{b \cdot \nabla_x}{y} + \frac{c D_y}{y}, \qquad c \neq 0,$$

where $v = (b, c) \in \mathbb{R}^{N+1}$ and is Q positive definite. We impose, in this case, an oblique derivative boundary condition

$$\lim_{y \to 0} y^{\frac{c}{\gamma}}\, v \cdot \nabla u = 0.$$

We transform $\mathcal{L}$ into a similar operator with $b = 0$ and Neumann boundary conditions by defining the following isometry of $L^2_{\frac{c}{\gamma}}$:

$$T\, u(x, y) := u\left(x - \frac{b}{c} y, y\right), \quad (x, y) \in \mathbb{R}^{N+1}_+. \tag{24}$$

Lemma 5.1. *Let $1 < p < \infty$, $v = (b, c) \in \mathbb{R}^{N+1}$, $c \neq 0$. Then for $u \in W^{2,1}_{loc}\left(\mathbb{R}^{N+1}_+\right)$*

(i) $T^{-1}\, (v \cdot \nabla)\, T\, u = c D_y u$;

(ii) $T^{-1}\left(Tr\, \left(QD^2\right) + \frac{v \cdot \nabla}{y}\right) T\, u = Tr\, \left(\tilde{Q} D^2 u\right) + \frac{c}{y} D_y u$.

Here, $\tilde{Q}$ is a uniformly elliptic symmetric matrix defined by

$$\tilde{Q} = \left(\begin{array}{c|c} Q_N - \frac{2}{c} b \otimes q + \frac{\gamma}{c^2} b \otimes b & q^t - \frac{\gamma}{c} b^t \\ \hline q - \frac{\gamma}{c} b & \gamma \end{array}\right)$$

and $\gamma = q_{N+1,N+1}$.

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Proof. The proof follows by a straightforward computation. □

The latter lemma shows that the operator $\mathcal{L}$, endowed with the oblique boundary condition $\lim_{y\to 0} y^{\frac{c}{\gamma}}\, v\cdot\nabla u = 0$ is unitarily equivalent to the operator

$$\tilde{\mathcal{L}} := T^{-1}\mathcal{L}T = \operatorname{Tr}\left(\tilde{Q}D^2\right) + \frac{c}{y}D_y \tag{25}$$

with the Neumann boundary condition $\lim_{y\to 0} y^{\frac{c}{\gamma}} D_y u = 0$. Then, by similarity and the previous results, if $\frac{c}{\gamma}+1>0$, $\mathcal{L}$ generates a contractive analytic semigroups $e^{z\mathcal{L}}$. In order to deduce kernel estimates we observe that, if $p_{\tilde{\mathcal{L}}}$ is the heat kernel of $\tilde{\mathcal{L}}$ with respect to the measure $y^{\frac{c}{\gamma}}dz$, then one has for $z_1=(x_1,y_1), z_2=(x_2,y_2)\in\mathbb{R}^{N+1}_+$

$$\begin{aligned} e^{t\mathcal{L}}f(z_1) &= T\left[e^{t\tilde{\mathcal{L}}}(T^{-1}f)\right](z_1) = \left[e^{t\tilde{\mathcal{L}}}(T^{-1}f)\right]\left(x_1-\frac{b}{c}y_1,y_1\right)\\ &= \int_{\mathbb{R}^{N+1}_+} p_{\tilde{\mathcal{L}}}\left(t,\left(x_1-\frac{b}{c}y_1,y_1\right),z_2\right) f\left(x_2+\frac{b}{c}y_2,y_2\right) y_2^{\frac{c}{\gamma}}\,dx_2dy_2\\ &= \int_{\mathbb{R}^{N+1}_+} p_{\tilde{\mathcal{L}}}\left(t,\left(x_1-\frac{b}{c}y_1,y_1\right),\left(x_2-\frac{b}{c}y_2,y_2\right)\right) f(z_2)\, y_2^{\frac{c}{\gamma}}\,dz_2. \end{aligned}$$

This shows that the heat kernel of $\mathcal{L}$ is given by

$$p_{\mathcal{L}}(t,z_1,z_2) := p_{\tilde{\mathcal{L}}}\left(t,\left(x_1-\frac{b}{c}y_1,y_1\right),\left(x_2-\frac{b}{c}y_2,y_2\right)\right). \tag{26}$$

By Theorem 4.14, we then obtain kernel estimates for $\mathcal{L}$. We state them for real t only for simplicity.

Theorem 5.2. *Let $v=(b,c)\in\mathbb{R}^{N+1}$ with $b=0$ if $c=0$, $\left(\begin{array}{c|c} Q & q^t \\ \hline q & \gamma \end{array}\right)$ an elliptic matrix. If $\frac{c}{\gamma}+1>0$ then the operator*

$$\mathcal{L} = Tr\left(QD^2u\right) + \frac{v\cdot\nabla}{y}$$

endowed with the boundary conditions

$$\lim_{y\to 0} y^{\frac{c}{\gamma}}\, v\cdot\nabla u = 0, \qquad \text{(if } c\neq 0\text{)}, \qquad\qquad \lim_{y\to 0} y^{\frac{c}{\gamma}}\, D_y u = 0, \qquad \text{(if } c = 0\text{)},$$

generates a contractive analytic semigroup in $L^2_{\frac{c}{\gamma}}$. The semigroup $(e^{t\mathcal{L}})_{t>0}$ consists of integral operators and its heat kernel $p_{\mathcal{L}}$, written with respect the measure $y^{\frac{c}{\gamma}}dz$, satisfies for some $C,k>0$,

$$0\le p_{\mathcal{L}}(t,z_1,z_2) \le Ct^{-\frac{N+1}{2}} y_1^{-\frac{c}{2\gamma}}\left(1\wedge\frac{y_1}{\sqrt{t}}\right)^{\frac{c}{2\gamma}} y_2^{-\frac{c}{2\gamma}}\left(1\wedge\frac{y_2}{\sqrt{t}}\right)^{\frac{c}{2\gamma}} \exp\left(-\frac{|z_1-z_2|^2}{kt}\right).$$

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Proof. The generation property follows by similarity using the equality (25). To prove the kernel estimates, we use (26) and Theorem 4.14 getting

$$0 \le p_{\mathcal{L}}(t, z_1, z_2) \le C t^{-\frac{N+1}{2}} y_1^{-\frac{c}{2\gamma}} \left(1 \wedge \frac{y_1}{\sqrt{t}}\right)^{\frac{c}{2\gamma}} y_2^{-\frac{c}{2\gamma}} \left(1 \wedge \frac{y_2}{\sqrt{t}}\right)^{\frac{c}{2\gamma}} \times \exp\left(-\frac{|x_1 - x_2 - \frac{b}{c}(y_1 - y_2)|^2 + |y_1 - y_2|^2}{kt}\right).$$

The required claim then follows by observing that for any $k \in \mathbb{R}^N \setminus \{0\}$ one has, for some positive constant $C_k > 0$,

$$|x - ky|^2 + y^2 \ge C_k\big(|x|^2 + y^2\big), \qquad \forall x \in \mathbb{R}^n,\ y \in \mathbb{R}.$$

Indeed, for $\epsilon > 0$, the Cauchy–Schwarz inequality gives

$$\begin{aligned} |x - ky|^2 + y^2 &= |x|^2 + (1 + |k|^2)y^2 - 2x \cdot ky \ge |x|^2 + (1 + |k|^2)y^2 - 2|x||k||y| \\ &\ge |x|^2 + (1 + |k|^2)y^2 - \frac{1}{\epsilon}|x|^2 - \epsilon|k|^2 y^2 \\ &= (1 - \frac{1}{\epsilon})|x|^2 + (1 + (1 - \epsilon)|k|^2)y^2. \end{aligned}$$

The claimed inequality then follows by choosing $1 < \epsilon < 1 + \frac{1}{|k|^2}$. □

Having at disposal good kernel estimates for $e^{t\mathcal{L}}$ implies various consequences about the regularity of the semigroup and its spectral properties; as an application one can study, for example, generation results in L^p space. We refer the reader to [3, 5] where it has been proved that the above semigroup extrapolates to the weighted spaces $L^p\left(\mathbb{R}^{N+1}_+, y^m dxdy\right), m \in \mathbb{R}$, when $0 < \frac{m+1}{p} < \frac{c}{\gamma} + 1$; furthermore, a characterization of the domain of the generators is given in terms of suitable anisotropic weighted Sobolev spaces.

Acknowledgments

The research is supported by GNAMPA ("Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni") of which all the authors are members.

Open access publishing facilitated by Universita del Salento, as part of the Wiley - CRUI-CARE agreement.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

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