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Leading large N giant graviton correction to Schur correlators in large representations

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A R T I C L E I N F O

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A B S T R A C T

We consider 4d $\mathcal{N}=4$ $U(N)$ SYM and the leading giant graviton correction at large N to the Schur defect 2-point functions of $\frac{1}{2}$-BPS Wilson lines in rank-k symmetric and antisymmetric representations. We study in particular the large k limit for the symmetric case and the regime $1 \ll k \ll N$ in the antisymmetric one. In both cases we present exact results for the correction. Wilson lines in symmetric/antisymmetric representations admit a description in terms of $D3_k$ and $D5_k$ brane probes representing a collection of k fundamental strings. In this picture, giant graviton corrections come from fluctuations of brane probes in the presence of a wrapped D3 brane giant graviton. In the antisymmetric case, our leading correction matches the half-index of the 4d $\mathcal{N}=4$ Maxwell theory living on the disk which is a part of the giant graviton divided out by the $D5_k$ probe, as recently proposed in arXiv:2404.08302. For the symmetric case at large k, we derive an explicit exact residue formula for the leading large N correction.

Contents

1. Introduction and results . . . 2
2. Schur line correlators in S_k or A_k representation . . . 5
   2.1. Finite N matrix integral data . . . 6
   2.2. $N \to \infty$ limit at fixed k . . . 6
      2.2.1. Taking $k \to \infty$ . . . 7
      2.2.2. Special limits . . . 8
   2.3. Finite N Schur line correlators from the HS functional . . . 8
3. Single giant graviton correction: symmetric representation . . . 9
   3.1. Explicit ratios $R^{U(N)}_{S_k}$ at fixed k and special limits . . . 9
   3.2. Large k limit and $R^{U(N)}_{S}$ . . . 10
      3.2.1. Unflavored limit . . . 12
      3.2.2. $\frac{1}{2}$-BPS limit . . . 12
4. Single giant graviton correction: antisymmetric representation . . . 13
   4.1. Special limits . . . 14
Declaration of competing interest . . . 14
Data availability . . . 14
Acknowledgements . . . 14

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Appendix A. Conventions for q-functions . . . 14
Appendix B. A useful differentiation identity . . . 15
Appendix C. Full details of the main calculation . . . 15
C.1. Plethystic representation of $\Phi_k^S(\boldsymbol{g};\zeta)$ . . . 17
References . . . 18

1. Introduction and results

The superconformal index [1–3] of 4d $\mathcal{N}=4$ $U(N)$ SYM at large N and fixed charge is the generating function of BPS single trace states and has finite N corrections associated with $U(N)$ trace relations in multi-trace states.[1] In the dual IIB superstring theory in $AdS_5\times S^5$, the large N limit reproduces the contributions to the index from BPS supergravity states. Finite N corrections are encoded in so-called giant graviton expansions [6,7] and in many instances take the schematic form

$$\mathrm{I}_N(q)=\mathrm{I}_\infty(q)\left[1+\sum_{n\geqslant 1}q^{nN}\,\widehat{I}_n(q)\right], \tag{1.1}$$

where $\mathrm{I}_N(q)$ is the superconformal index and q is its universal fugacity (coupled to the Hamiltonian and other charges). Other fugacities may be present but are omitted for the purpose of illustration. A key property of the functions $\widehat{I}_n(q)$ is that they do not depend on N at generic points in fugacity space. The gravity origin of the terms in (1.1) has been elucidated in [8] in the case of $\mathcal{N}=4$ $U(N)$ SYM and associated with D3 branes in the dual string theory on $AdS_5\times S^5$ wrapped on contractible cycles in S^5 with charge of order N. The prefactor q^{nN} is the classical Euclidean action of such a brane configuration with total winding n. The remaining factor $\widehat{I}_n(q)$ represents the superconformal index of fluctuations of the brane. The brane expansion (1.1) is usually called a giant graviton expansion being a generalization of the configurations studied in [9,10].[2] Apart from its string interpretation, the expansion (1.1) may be addressed at the level of the gauge theory where it requires to deal with a non-trivial matrix integral which is the main subject of this paper.

The Schur specialization [13–15] of the superconformal index is defined as the following trace[3]

$$\mathrm{I}^{U(N)}(\eta;q)=\mathrm{Tr}_{\mathrm{BPS}}[(-1)^{\mathrm{F}}\,q^{H+J+\bar{J}}\eta^R]. \tag{1.2}$$

It depends on two fugacities coupled to Cartan generators of the superconformal symmetry group $PSU(2,2|4)$ commuting with the supercharges entering the BPS condition. In more details, q is the fugacity coupled to the Hamiltonian H and the two left and right spins $J,\bar{J}$, while η is a flavor fugacity coupled to an R-charge generator. The Schur index has an explicit expression as the singlet projection of many-particle contributions built out of the Schur single-particle index $f(\eta;q)$ by usual plethystic[4]

$$\mathrm{I}^{U(N)}(\eta;q)=\int_{U(N)}DU\;\;\mathrm{PE}[f(\eta;q)\,\mathrm{Tr}\,U\;\mathrm{Tr}\,U^{-1}],\qquad f(\eta;q)=\frac{(\eta+\eta^{-1})\,q-2q^2}{1-q^2}. \tag{1.3}$$

This matrix integral was computed exactly at finite N in [17–19] for $\eta=0$ and in [20] for $\eta\neq 0$, and for other gauge groups in [21]. The structure of its large and finite N corrections was first studied in [6,7] (and later developed in [22] as a double sum over D3 brane windings) and takes the form, *cf.* (1.1),

$$\mathrm{I}^{U(N)}(\eta;q)=\mathrm{I}^{U(\infty)}(\eta;q)\left[1+\left(\eta^N\,\widehat{\mathrm{I}}^{+}(\eta;q)+\eta^{-N}\,\widehat{\mathrm{I}}^{-}(\eta;q)\right)q^N+\mathcal{O}(q^{2N})\right], \tag{1.4}$$

where $\mathrm{I}^{U(\infty)}(\eta;q)$ is the large N supergravity contribution, while terms $\widehat{\mathrm{I}}^{\pm}(\eta;q)$ come from wrapped $S^1\times S^3$ D3 branes in $AdS_5\times S^5$ with $S^1\subset AdS_5$ and $S^3\subset S^5$.[5] They may be given by analytic continuation of the $\mathcal{N}=4$ $U(n)$ SYM index [22], following the approach in [29–35] and were computed exactly in [36].

We remark that (1.4) is a specific choice where η is kept fixed and one expands at small q. Other limits may be considered (like expansions in $x=\eta q$ or $y=\eta^{-1}q$) whose physical meaning was elucidated in [6,7]. A motivation for our choice is the analysis in [25] where the index was computed by D3 brane fluctuations in a twisted $AdS_5\times S^5$ background. In that case η was related to a fixed rotation angle mixing AdS time and two angles in $S^3\subset S^5$.

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1 For the relation between large N limit of the full index and black hole physics, see [4,5].

2 Alternatively, giant graviton contributions can also be examined in terms of supergravity bubbling solutions [11,12].

3 Superconformal indices do not depend on the Yang-Mills gauge coupling because of supersymmetry. Indeed, they may be regarded as the Witten index [16] in radial quantization supplemented by suitable chemical potential coupled to charges commuting with the supercharge used to define the index.

4 For a set of fugacities, group matrices, *etc.*, collectively denoted by $\boldsymbol{x}$ one has $\mathrm{PE}[F(\boldsymbol{x})]=\exp\sum_{n\geqslant 1}\frac{1}{n}F(\boldsymbol{x}^n)$.

5 For a direct D3 brane fluctuation analysis reproducing the leading correction in (1.4), see for instance [23–26]. Higher giant-graviton contributions for the $\frac{1}{2}$-BPS index were analyzed by localization on the brane world volume [27,28].

Schur line correlators and string fluctuations The Schur index is a supersymmetric partition function on $S^1\times S^3$ and it is possible to consider additional defect 't Hooft or Wilson lines along S^1 and at positions on a great circle of S^3 [37–40]. The resulting correlation functions are actually topological, *i.e.* do not depend on the line positions. Wilson lines in general representations $\mathsf{R}_1,\mathsf{R}_2,\dots$ correspond to the following generalization of (1.3)[6]

$$\mathrm{I}^{U(N)}_{\mathsf{R}_1,\mathsf{R}_2,\dots}(\eta;q)=\int_{U(N)} DU\prod_{n\geqslant 1}\mathrm{Tr}_{\mathsf{R}_n}(U)\,\mathrm{PE}[f(\eta;q)\,\mathrm{Tr}\,U\ \mathrm{Tr}\,U^{-1}],\tag{1.5}$$

and exact results were obtained in [41,39,42–46] at fixed or infinite N.

The giant graviton expansion of Schur line correlators (1.5) was recently considered in [47–49]. In the simplest case of the defect 2-point function with one line in the fundamental and one in the antifundamental $\mathrm{I}^{U(N)}_{\mathrm{F}}(\eta;q)\equiv\mathrm{I}^{U(N)}_{\square,\overline{\square}}(\eta;q)$, the large N limit has the simple factorized form [38]

$$\mathrm{I}^{U(\infty)}_{\mathrm{F}}(\eta;q)=\mathrm{I}_{\mathrm{F1}}(\eta;q)\ \mathrm{I}^{U(\infty)}(\eta;q),\tag{1.6}$$

where the additional factor $\mathrm{I}_{\mathrm{F1}}(\eta;q)$ reads

$$\mathrm{I}_{\mathrm{F1}}(\eta;q)=\frac{1}{1-f(\eta;q)}=\mathrm{PE}[f_{\mathrm{F1}}(\eta;q)],\qquad f_{\mathrm{F1}}(\eta;q)=-q^2+(\eta+\eta^{-1})\,q\,.\tag{1.7}$$

Its gravity interpretation is expected to involve fluctuations of a fundamental string stretched along AdS_2 inside AdS_5 [50,51]. They fill a short multiplet of $OSp(4^*|4)$ with 8_B+8_F states and the three terms in f_{F1} agree with the contributions to the single particle index from three BPS modes [38]. At finite N, the leading single giant graviton correction to $\mathrm{I}^{U(N)}_{\mathrm{F}}(\eta;q)$ is due to fluctuations of a world-sheet bounded by two semi-infinite strings attached to the Wilson lines and ending on the giant graviton, *i.e.* a wrapped D3 brane [47]. Explicitly, at leading order, one has[7]

$$\frac{\mathrm{I}^{U(N)}_{\mathrm{F}}-I_{\mathrm{F1}}\,\mathrm{I}^{U(N)}}{\mathrm{I}^{U(\infty)}}=1+\Big(\mathsf{G}^{+}_{\mathrm{F}}(\eta;q)\,\eta^N+\mathsf{G}^{-}_{\mathrm{F}}(\eta;q)\,\eta^{-N}\Big)\,q^N+\cdots,\quad \mathsf{G}^{-}_{\mathrm{F}}(\eta;q)=\mathsf{G}^{+}_{\mathrm{F}}(\eta^{-1};q).\tag{1.8}$$

The functions $\mathsf{G}^{\pm}$ factorize and read [48,47] (see Appendix A for special function conventions)

$$\mathsf{G}^{\pm}_{\mathrm{F}}(\eta;q)=\frac{1}{\eta q}\ \mathrm{PE}[f_{\mathrm{F}}(\eta;q)]\ G^{\pm}_{\mathrm{D3}}(\eta;q),\qquad G^{+}_{\mathrm{D3}}(\eta;q)=-\eta^2 q\,\frac{(\frac{q}{\eta})^3_\infty}{\vartheta(\eta^2,\frac{q}{\eta})},\tag{1.9}$$

where $G^{\pm}_{\mathrm{D3}}(\eta;q)$ is the leading giant graviton contribution to the Schur index without insertions computed from wrapped D3 branes in [22,36], and

$$f_{\mathrm{F}}(\eta;q)=2\eta^{-1}q-2q^2,\tag{1.10}$$

is the single particle index of fluctuations of the system of two semi-infinite strings. Up to a factor 2 accounting for the two strings, it selects two terms out of the three in f_{F1} since one BPS state is missing due to the boundary condition that they should end on the giant graviton [47]. At the moment, the meaning of the prefactor $1/(\eta q)$ in (1.9) is unclear. Similar prefactors appear also in higher order giant graviton contributions and in more general Schur correlations functions [47,48].

New results for large representations In this paper, we study the Schur 2-point function when the two Wilson lines are in the rank-k symmetric (k) or antisymmetric $[k]$ representation. Although the BPS Wilson line in fundamental representation may be represented in terms of a fundamental string, an alternative description is by a D3 brane carrying electric flux and pinching off at the boundary of AdS_5 along the Wilson line [53,54,50,55].

This D-brane probe picture is convenient when the fundamental representation is replaced by large higher representations. In particular, a Wilson line in rank-k symmetric representation corresponds to k fundamental strings emerging as spikes on an extra D3-brane wrapping $AdS_2\times S^2\subset AdS_5$, denoted D3$_k$ [55–59]. The case of rank-k antisymmetric representation corresponds instead to k fundamental strings attached to a D5-brane wrapping $AdS_2\times S^4\subset AdS_5\times S^5$, denoted D5$_k$ [60,56–58,61].[8]

For the 2-point function in the symmetric or antisymmetric cases, we examine the leading giant graviton correction to line indices, *cf.* (1.5),

$$\mathrm{I}^{U(N)}_{S_k}(\eta;q)=\mathrm{I}^{U(N)}_{(k),\overline{(k)}}(\eta;q),\qquad \mathrm{I}^{U(N)}_{A_k}(\eta;q)=\mathrm{I}^{U(N)}_{[k],\overline{[k]}}(\eta;q).\tag{1.11}$$

[6] In a basis where $U=\mathrm{diag}(e^{iz_1},\dots,e^{iz_n})$, the trace $\mathrm{Tr}\,U$ is the character of the fundamental representation $\mathrm{Tr}\,U=\chi_{\square}(z)$ and $\mathrm{Tr}_{\mathsf{R}}\,U$ stands similarly for $\chi_{\mathsf{R}}(z)$.

[7] Notice that the giant graviton expansion of the finite N index suffers ambiguities at higher orders in the q^N expansion. This may be interpreted as an operator mixing issue in the gauge theory, see [28] and [52]. This problem does not affect the leading correction discussed which is isolated in large N limit.

[8] More general representations and their D-brane probe description are discussed in [56].

We find that the ratios[9]

$$R_{\mathsf{R}}^{U(N)}(\eta;q)=\frac{\mathrm{I}_{\mathsf{R}}^{U(N)}(\eta;q)}{\mathrm{I}_{\mathsf{R}}^{U(\infty)}(\eta;q)},\qquad \mathsf{R}=S_k,A_k\,, \tag{1.12}$$

may be written at large N and fixed k as

$$\begin{aligned} R_{S_k}^{U(N)}(\eta;q)&=1+[\eta^N\,G^+_{S_k}(\eta;q)+\eta^{-N}\,G^-_{S_k}(\eta;q)]\,q^N+\cdots,\\ R_{A_k}^{U(N)}(\eta;q)&=1+[\eta^{N-k}\,G^+_{A_k}(\eta;q)+\eta^{-(N-k)}\,G^-_{A_k}(\eta;q)]\,q^{N-k}+\cdots, \end{aligned} \tag{1.13}$$

where again $G^-_{\mathsf{R}}(\eta;q)=G^+_{\mathsf{R}}(\eta^{-1};q)$ and, at small q, $G^+_{S_k}(\eta;q)=\mathcal{O}(1)$ and $G^+_{A_k}(\eta;q)=\mathcal{O}(q)$. Dots in (1.13) include in both cases higher giant graviton contributions of the form $q^{2N+\delta}$ where δ is a k-dependent integer. These are beyond our scope and will not be discussed. Here, we present explicit formulas for the functions $G^{\pm}_{\mathsf{R}}(\eta;q)$.

From the finite k results, we may examine the structure of the correction for large k. In the symmetric case, when the functions $G^{\pm}_{S_k}(\eta;q)$ are expanded in powers of q, k-dependent corrections appear taking the form of powers q^{k-n} with positive n. This implies that when $k\gg 1$, and irrespectively of its possible scaling with N, one has a definite $k\to\infty$ limit of the form

$$\lim_{k\to\infty}R_{S_k}^{U(N)}(\eta;q)\equiv R_S^{U(N)}(\eta;q)=1+[\eta^N\,G^+_S(\eta;q)+\eta^{-N}\,G^-_S(\eta;q)]\,q^N+\cdots. \tag{1.14}$$

We prove that the asymptotic functions $G^{\pm}_S(\eta;q)$ are given by the exact residue formula

$$G^+_S(\eta;q)=\frac{1}{\eta^2 q}\,\underset{\varepsilon=0}{\mathrm{Res}}\left[\frac{1}{\varepsilon^3}\,\frac{(\varepsilon;\frac{q}{\eta})^2_\infty}{(\varepsilon q^2;\frac{q}{\eta})^2_\infty}\,\frac{\vartheta(\varepsilon\eta^2;\frac{q}{\eta})}{\vartheta(\varepsilon;\frac{q}{\eta})}\right]G^+_{\mathrm{D3}}(\eta;q), \tag{1.15}$$

that evaluates to

$$G^+_S(\eta;q)=\left[\frac{1}{\eta q}+1+\frac{2}{\eta^2}+\frac{1}{\eta^4}+\left(\frac{2}{\eta^5}+\frac{1}{\eta^3}-\frac{2}{\eta}\right)q+\left(\frac{5}{\eta^6}-\frac{2}{\eta^4}-\frac{3}{\eta^2}\right)q^2+\cdots\right]G^+_{\mathrm{D3}}(\eta;q). \tag{1.16}$$

Expression (1.15) is the main result of our analysis and has been purposely written with explicit factorization of the undecorated index, similar to (1.9). About its physical interpretation, it is expected to represent fluctuations of the probe D3$_k$ brane in the presence of the D3 giant graviton in the same spirit as [47] for the case of a pair of semi-infinite fundamental strings. This statement clearly deserves explicit brane fluctuation calculations that are unavailable at the moment. It would be very interesting to compare our result (1.15) with finite N effects from fluctuations of the D3$_k$ probe in the presence of the giant graviton also to clarify the puzzling different structure of fluctuations even at infinite N for the D3$_k$ and D5$_k$ probes [62,63], despite equality of the symmetric and antisymmetric Schur 2-point functions in this limit.

In the antisymmetric case, $G^{\pm}_{A_k}(\eta;q)$ has again powers of q and k-dependent corrections as well. The latter takes now the form q^{-k+n} with positive n, and are dominant in the strict regime $1\ll k\ll N$. They determine the correction

$$R_{A_k}^{U(N)}(\eta;q)\overset{1\ll k\ll N}{\equiv}R_A^{U(N)}(\eta;q)=1+[\eta^{N-k}\,G^+_A(\eta;q)+\eta^{-(N-k)}\,G^-_A(\eta;q)]\,q^{N-k}+\cdots, \tag{1.17}$$

where the functions $G^{\pm}_A(\eta;q)$ are k-independent. Our methods show that they are given exactly by

$$G^+_A(\eta;q)=-\frac{\eta q}{1-\eta q}\,\frac{(\frac{q}{\eta})_\infty}{(\frac{1}{\eta^2};\frac{q}{\eta})_\infty}, \tag{1.18}$$

that we may write like in (1.16) as

$$G^+_A(\eta;q)=\frac{1}{1-\eta q}\,\frac{(\eta q;\frac{q}{\eta})_\infty}{(\frac{q}{\eta})_\infty}\,G^+_{\mathrm{D3}}(\eta;q)=\left[1+\frac{1}{\eta}\,q+\left(\frac{2}{\eta^2}-1\right)q^2+\left(\frac{3}{\eta^3}-\frac{2}{\eta}\right)q^3+\cdots\right]G^+_{\mathrm{D3}}(\eta;q). \tag{1.19}$$

The exact result (1.18) agrees with the recent analysis by Imamura and Inoue who conjectured a quadruple-sum giant graviton expansion of the form [49]

$$\frac{\mathrm{I}_{A_k}^{U(N)}(\eta;q)}{I_{A_\infty}^{U(\infty)}(\eta;q)}=\sum_{m,m',n,n'\geqslant 0}(\eta q)^{km+(N-k)m'}(\eta^{-1}q)^{kn+(N-k)n'}\,\mathcal{F}_{m,n,m',n'}(\eta;q), \tag{1.20}$$

[9] Notice that in (1.12), one has same k in numerator and denominator. So $R_{\mathsf{R}}^{U(N)}(\eta;q)\to 1$ for $N\to\infty$ with corrections vanishing for fixed k and large N. A different ratio was considered in [49] where k was taken infinite in the denominator. In this case, the associated ratio admitted a quadruple sum expansion with summation indices coupled to two independent charges k and $N-k$.

where some of the functions $\mathcal{F}$ are computed in analytic form and other are extracted from the q-series of the ratio in the l.h.s. For the leading term in $1 \ll k \ll N$ regime, their analysis implies the relation and physical interpretation

$$G^{+}_{A}(\eta; q) = \mathrm{PE}\left[\frac{\frac{1}{\eta q} - \eta^{-1} q}{1 - \eta^{-1} q}\right], \tag{1.21}$$

where the plethystic exponential in the r.h.s. is the analytic continuation of the half-index [37] of 4d $\mathcal{N} = 4$ $U(1)$ Maxwell theory living on a disk which is a part of the giant graviton separated out by the probe D5$_k$ brane. It is readily seen that (1.21) agrees with (1.18), that we obtained by an independent matrix model analysis of the Schur correlation function. This confirms the construction in [49].

Plan of the paper In Section 2, we present the explicit definitions of the Schur 2-point functions in symmetric and antisymmetric representation and discuss explicit finite N, k data from their matrix model representation. In Sections 2.2, we discuss their $N \to \infty$ limit and the form it takes when k is also large. Section 2.3 uses the Hubbard-Stratonovich transformation to give a representation of the 2-point functions suitable for the analysis of their finite N corrections. This is worked out in Section 3 in the case of the symmetric representation. In particular, in Section 3.2, we discuss the large k form of the leading giant graviton correction (1.14). Section 4 treats the antisymmetric case. The detailed derivations are presented in technical Appendices.

2. Schur line correlators in S_k or A_k representation

The matrix integral (1.5) can be expressed as a contour integral over holonomies $\boldsymbol{z} = (z_1, \dots, z_N)$

$$\begin{aligned} &\mathrm{I}^{U(N)}_{\mathrm{R}_1,\mathrm{R}_2,\dots}(\eta; q) = \oint_{|z|=1} D^N \boldsymbol{z} \prod_{n\geqslant 1} \chi_{\mathrm{R}_n}(\boldsymbol{z})\, \mathrm{PE}[f(\eta; q)\chi_{\Box}(\boldsymbol{z})\chi_{\Box}(\boldsymbol{z}^{-1})], \\ &D^N \boldsymbol{z} = \frac{1}{N!}\prod_{n=1}^{N} \frac{dz_n}{2\pi i\, z_n} \prod_{n\neq m}\left(1 - \frac{z_n}{z_m}\right). \end{aligned} \tag{2.1}$$

For the symmetric representation (k) and antisymmetric $[k]$, the characters are

$$\chi_{(k)}(\boldsymbol{z}) = \sum_{1\leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant N} z_{i_1} z_{i_2} \cdots z_{i_k}, \qquad \chi_{[k]}(\boldsymbol{z}) = \sum_{1\leqslant i_1 < i_2 < \dots < i_k \leqslant N} z_{i_1} z_{i_2} \cdots z_{i_k}. \tag{2.2}$$

In the following we will make extensive use of Young tableaux expansions. To set notation, we recall that a partition λ of the positive integer k, denoted $\lambda \vdash k$, can be represented as a Young tableau with λ_n blocks in its n-th row, *i.e.* $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ or in frequency representation $1^{r_1} 2^{r_2} \cdots$. The number of parts of the partition λ is $\ell(\lambda) = \sum_n r_n$ (the number of non-zero λ_i) and is the number of rows in the associated Young tableau. The weight of the partition λ is $|\lambda| = \sum_n \lambda_n = \sum_n n r_n$, the number of blocks in the Young tableau, and $\lambda \vdash k$ is same as $|\lambda| = k$. With this notation, we also define

$$\rho_\lambda = \prod_{n=1}^{\infty} r_n!\, n^{r_n}. \tag{2.3}$$

The characters (2.2) may be given as sums over partitions $\lambda \vdash k$

$$\chi_{(k)}(\boldsymbol{z}) = \sum_{\lambda\vdash k} \frac{1}{\rho_\lambda} \chi_{\Box}(\boldsymbol{z})^{\lambda}, \qquad \chi_{[k]}(\boldsymbol{z}) = \sum_{\lambda\vdash k} \frac{(-1)^{k-\ell(\lambda)}}{\rho_\lambda} \chi_{\Box}(\boldsymbol{z})^{\lambda}, \tag{2.4}$$

where we denoted, for a generic function $F(\boldsymbol{z})$

$$F(\boldsymbol{z})^{\lambda} = \prod_{n\geqslant 1} F(\boldsymbol{z}^{\lambda_n}), \qquad \boldsymbol{z}^m = (z_1^m, \dots, z_N^m). \tag{2.5}$$

Given the set $\boldsymbol{A} = (A_1, A_2, \dots)$, the relation between plethystic exponentiation and Young tableau expansion is

$$\mathrm{PE}[\boldsymbol{A}] = \exp \sum_{n\geqslant 1} \frac{1}{n} A_n = \sum_{\lambda} \frac{1}{\rho_\lambda} \boldsymbol{A}^{\lambda}, \qquad \boldsymbol{A}^{\lambda} = \prod_{n\geqslant 1} A_n^{\lambda_n}. \tag{2.6}$$

It follows from this relation that

$$\mathrm{PE}[-\boldsymbol{A}] = \mathrm{PE}[\boldsymbol{A}]^{-1} = \sum_{\lambda} \frac{(-1)^{\ell(\lambda)}}{\rho_\lambda} \boldsymbol{A}^{\lambda}. \tag{2.7}$$

Comparing with (2.4) gives the well-known generating functions of symmetric and antisymmetric characters in the form

$$S(x;z)=\sum_{k\geqslant 0}x^k\,\chi_{(k)}(z)=\mathrm{PE}[x\,\chi_{\square}(z)]=\prod_{n=1}^{N}\frac{1}{1-xz_n},$$
$$A(x;z)=\sum_{k\geqslant 0}(-1)^k\,x^k\,\chi_{[k]}(z)=\mathrm{PE}[-x\,\chi_{\square}(z)]=\prod_{n=1}^{N}(1-xz_n)\,. \tag{2.8}$$

2.1. Finite N matrix integral data

Explicit expressions for ratios (1.12) with fixed N and increasing k are obtained by evaluation of (2.1). For $U(2)$ gauge group we get

$$R^{U(2)}_{S_2}(\eta;q)=1-(1+\eta^{-2}+\eta^2)\,q^2-2\,(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^3-(\eta^{-4}-5\eta^{-2}-8-5\eta^2+\eta^4)\,q^4+\cdots,$$
$$R^{U(2)}_{S_3}(\eta;q)=1-(1+\eta^{-2}+\eta^2)\,q^2-2\,(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^3-(2\eta^{-4}-3\eta^{-2}-5-3\eta^2+2\eta^4)\,q^4+\cdots, \tag{2.9}$$

and same for S_k with $k>3$ at this order in q. For $U(3)$ gauge group

$$\begin{aligned}R^{U(3)}_{S_2}(\eta;q)&=1-(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^3-2(\eta^{-4}+\eta^{-2}+2+\eta^2+\eta^4)\,q^4\\&\quad-(\eta^{-5}-3\eta^{-3}-8\eta^{-1}-8\eta-3\eta^3+\eta^5)\,q^5-(\eta^{-6}-3\eta^{-4}-5\eta^{-2}-5-5\eta^2-3\eta^4+\eta^6)\,q^6+\cdots,\\R^{U(3)}_{S_3}(\eta;q)&=1-(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^3-2(\eta^{-4}+\eta^{-2}+2+\eta^2+\eta^4)\,q^4\\&\quad-(2\eta^{-5}-\eta^{-3}-4\eta^{-1}-4\eta-\eta^3+2\eta^5)\,q^5-(\eta^{-6}-4\eta^{-4}-8\eta^{-2}-13-8\eta^2-4\eta^4+\eta^6)\,q^6+\cdots,\\R^{U(3)}_{S_4}(\eta;q)&=1-(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^3-2(\eta^{-4}+\eta^{-2}+2+\eta^2+\eta^4)\,q^4\\&\quad-(2\eta^{-5}-\eta^{-3}-4\eta^{-1}-4\eta-\eta^3+2\eta^5)\,q^5-2\,(\eta^{-6}-\eta^{-4}-2\eta^{-2}-4-2\eta^2-\eta^4+\eta^6)\,q^6+\cdots,\end{aligned} \tag{2.10}$$

and same for S_k with $k>4$ at this order in q. Continuing at higher N shows that there is a well-defined limit

$$R^{U(N)}_{S}(\eta;q)=\lim_{k\to\infty}R^{U(N)}_{S_k}(\eta;q). \tag{2.11}$$

The same analysis for the antisymmetric representation makes no sense since we cannot take k large at fixed N. In this case the first correction is $\sim q^{N-k+1}$. For instance taking $N=5$ and $k=2,3,4,5$ one finds

$$R^{U(5)}_{A_2}(\eta;q)=1-(\eta^{-4}+\eta^{-2}+1+\eta^2+\eta^4)\,q^4-(\eta^{-5}-\eta^{-1}-\eta-\eta^5)\,q^5-(\eta^{-6}-1+\eta^6)\,q^6+\cdots, \tag{2.12}$$

$$\begin{aligned}R^{U(5)}_{A_3}(\eta;q)&=1-(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^3-(\eta^{-4}+1+\eta^4)\,q^4-(\eta^{-5}-\eta^{-1}-\eta+\eta^5)\,q^5\\&\quad-(\eta^{-6}-\eta^{-4}-\eta^{-2}-1-\eta^2-\eta^4+\eta^6)\,q^6+\cdots,\end{aligned} \tag{2.13}$$

$$\begin{aligned}R^{U(5)}_{A_4}(\eta;q)&=1-(\eta^{-2}+1+\eta^2)\,q^2-(\eta^{-3}+\eta^3)\,q^3-(\eta^{-4}-\eta^{-2}-1-\eta^2+\eta^4)\,q^4\\&\quad+(\eta^{-3}+\eta^{-1}+\eta+\eta^3)\,q^5+(\eta^{-4}+\eta^{-2}+\eta^2+\eta^4)\,q^6+\cdots,\end{aligned} \tag{2.14}$$

$$R^{U(5)}_{A_5}(\eta;q)=1-(\eta^{-1}+\eta)\,q-(\eta^{-2}-1+\eta^2)\,q^2+(\eta^{-1}+\eta)\,q^3+q^4+(\eta^{-5}+\eta^5)\,q^5-(\eta^{-4}+\eta^4)\,q^6+\cdots. \tag{2.15}$$

In the following, we will derive exact expressions for these expansions, up to terms of order $q^{2N+\cdots}$ where higher order giant graviton effects are to be taken into account.

In particular, we will examine the dependence on the rank k. At single giant graviton level, k-dependent corrections in symmetric case will contribute terms $q^{N+k+\cdots}$ (dots being fixed numerical integers) and will thus be neglibile if k is large, consistent with the limit (1.14). Instead, in antisymmetric case, k-dependent terms will be of the form $q^{N-k+\cdots}$ and will mix with $q^{N+\cdots}$ terms. They are disentangled in the limit $1\ll k\ll N$ and will be shown to be captured by (1.17).

2.2. $N\to\infty$ limit at fixed k

Let us begin with the Schur 2-point function in the symmetric representation. According to (2.1), we need

$$I^{U(N)}_{S_k}(\eta;q)=\oint_{|z|=1}D^N z\sum_{\lambda,\lambda'\vdash k}\frac{1}{\rho_\lambda\rho_{\lambda'}}\chi_\square(z)^\lambda\chi_\square(z^{-1})^{\lambda'}\,\mathrm{PE}[f(\eta;q)\chi_\square(z)\chi_\square(z^{-1})]. \tag{2.16}$$

In the limit $N\to\infty$, it has been conjectured in [43] that only diagonal terms survive in the sum over λ,λ'. We will assume this property without proof and exploit it to derive a novel representation of the index which will turn out to be useful in the following. Diagonal terms give

$$\mathrm{I}^{U(\infty)}_{S_k}(\eta;q)=\sum_{\lambda\vdash k}\frac{1}{\rho_\lambda^2}\,\mathrm{I}_\lambda(\eta;q), \tag{2.17}$$

with

$$\mathrm{I}_\lambda(\eta;q) = \lim_{N\to\infty} \oint_{|z|=1} D^N z\, \chi_\square(z)^\lambda \chi_\square(z^{-1})^\lambda \,\mathrm{PE}[f(\eta;q)\chi_\square(z)\chi_\square(z^{-1})]\,. \tag{2.18}$$

Same is obtained starting from the antisymmetric representation because $[(-1)^{k-\ell(\lambda)}]^2 = 1$, *i.e.*

$$\mathrm{I}^{U(\infty)}_{S_k}(\eta;q) = \mathrm{I}^{U(\infty)}_{A_k}(\eta;q)\,. \tag{2.19}$$

To compute (2.18) it is convenient to introduce a multi-coupling unitary matrix integral [64] with $\boldsymbol{g} = (g_1, g_2, \dots)$

$$Z_N(\boldsymbol{g}) = \int_{U(N)} dU\, \exp\left(\sum_{n=1}^\infty \frac{1}{n} g_n \,\mathrm{Tr}\, U^n\, \mathrm{Tr}\, U^{-n}\right) = \oint_{|z|=1} D^N z\, \mathrm{PE}[\boldsymbol{g}\,\chi_\square(z)\chi_\square(z^{-1})]\,. \tag{2.20}$$

It reduces to the Schur index by identification

$$g_n = f(\eta^n; q^n)\,. \tag{2.21}$$

For $N \to \infty$, one has [64]

$$Z_\infty(\boldsymbol{g}) = \prod_{n=1}^\infty \frac{1}{1-g_n}\,, \tag{2.22}$$

and thus the multi-coupling version of (2.18) reads

$$\mathrm{I}_\lambda(\boldsymbol{g}) = \lambda_1 \frac{\partial}{\partial g_{\lambda_1}} \cdots \lambda_{\ell(\lambda)} \frac{\partial}{\partial g_{\lambda_{\ell(\lambda)}}} \prod_{n\geqslant 1} \frac{1}{1-g_n} = \prod_{m\geqslant 1} m^{r_m} \partial^{r_m}_{g_m} \prod_{n\geqslant 1} \frac{1}{1-g_n} = \prod_{n\geqslant 1} \frac{n^{r_n} r_n!}{(1-g_n)^{r_n}} \times Z_\infty(\boldsymbol{g})\,. \tag{2.23}$$

Hence, the multi-coupling version of (2.17), denoted $Z^{S_k}_\infty(\boldsymbol{g})$, obeys

$$\begin{aligned} W_k(\boldsymbol{g}) \equiv \frac{Z^{S_k}_\infty(\boldsymbol{g})}{Z_\infty(\boldsymbol{g})} &= \sum_{\lambda\vdash k} \frac{1}{\rho^2_\lambda} \prod_{n\geqslant 1} \frac{r_n!}{(1-g_n)^{r_n}} = \sum_{\lambda\vdash k} \prod_{n\geqslant 1} \frac{1}{(r_n!)^2\, n^{2r_n}} \frac{n^{r_n} r_n!}{(1-g_n)^{r_n}} \\ &= \sum_{\lambda\vdash k} \prod_{n\geqslant 1} \frac{1}{r_n!\, n^{r_n}} \frac{1}{(1-g_n)^{r_n}} = \exp\left[\sum_{n=1}^\infty \frac{1}{n} \frac{\varepsilon^n}{1-g_n}\right]\Bigg|_{\varepsilon^k}. \end{aligned} \tag{2.24}$$

This gives the following expression for the ratio at $N = \infty$

$$W_k(\eta;q) \equiv \frac{\mathrm{I}^{U(\infty)}_{S_k}(\eta;q)}{\mathrm{I}^{U(\infty)}(\eta;q)} = \exp\left[\sum_{n=1}^\infty \frac{1}{n} \frac{\varepsilon^n}{1-f(\eta^n;q^n)}\right]\Bigg|_{\varepsilon^k} = \mathrm{PE}\left[\frac{\varepsilon}{1-f(\eta;q)}\right]\Bigg|_{\varepsilon^k}. \tag{2.25}$$

By working out its small q expansion, we checked at very high order that the result (2.25) agrees with the conjectured formula Eq. (5.13) in [43], *i.e.*

$$\mathrm{PE}\left[\sum_{n=1}^\infty \frac{1}{n} \frac{\varepsilon^n}{1-f(\eta^n;q^n)}\right]\Bigg|_{\varepsilon^k} = \sum_{n=0}^k \frac{1}{(\eta q;\eta q)_n (\eta^{-1}q;\eta^{-1}q)_{k-n}} - \sum_{n=0}^{k-1} \frac{1}{(\eta q;\eta q)_n (\eta^{-1}q;\eta^{-1}q)_{k-n-1}}\,. \tag{2.26}$$

It could be worth to prove rigorously the non-trivial identity (2.26).

2.2.1. Taking $k \to \infty$

When $g_n = f(q^n)$ with $f(q) = \mathcal{O}(q)$ it is interesting to look at the large k limit of (2.25) within the q-expansion. To this aim, we start from

$$\mathrm{PE}\left[\frac{\varepsilon}{1-f(\eta;q)}\right]\Bigg|_{\varepsilon^k} = \oint \frac{d\varepsilon}{2\pi i \varepsilon^{k+1}} \frac{1}{1-\varepsilon}\, \mathrm{PE}\left[\varepsilon \frac{f(\eta;q)}{1-f(\eta;q)}\right]. \tag{2.27}$$

If $k \to \infty$ at some order in q expansion, the only pole that is relevant is $\varepsilon = 1$ because other poles give suppressed contributions $\sim q^{k+\cdots}$. Thus, *cf.* (2.25),

$$\mathcal{W}(\eta;q) \equiv \lim_{k\to\infty} W_k(\eta;q) = \lim_{k\to\infty} \frac{\mathrm{I}^{U(\infty)}_{S_k}(\eta;q)}{\mathrm{I}^{U(\infty)}(\eta;q)} = \mathrm{PE}\left[\frac{f(\eta;q)}{1-f(\eta;q)}\right], \tag{2.28}$$

and we remind that same is obtained using A_k. The expression (2.28) agrees with the probe D3 brane computed in Eq. (6.12) of [38] since

$$\frac{f(\eta;q)}{1-f(\eta;q)} = \frac{\eta q}{1-\eta q} + \frac{\eta^{-1}q}{1-\eta^{-1}q} = \sum_{n=1}^\infty (\eta^n + \eta^{-n}) q^n\,. \tag{2.29}$$

This also gives

$$\begin{aligned}\lim_{k\to\infty} \mathrm{I}_{S_k}^{U(\infty)}(\eta;q) &= \mathrm{PE}\left[\frac{f(\eta;q)}{1-f(\eta;q)}\right]\prod_{n=1}^{\infty}\frac{1}{1-f(\eta^n;q^n)}\\ &= \prod_{n\geqslant 1}\mathrm{PE}\left[(\eta^n+\eta^{-n})q^n - q^{2n} + (\eta^n+\eta^{-n})q^n\right] = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-\eta^n q^n)^2(1-\eta^{-n}q^n)^2}\,, \end{aligned}\tag{2.30}$$

in agreement with Eq. (5.14) of [43].

2.2.2. Special limits

In the following we will consider two special limits. The first is simply the unrefined case

$$\text{unrefined limit:}\qquad \eta = 1\,. \tag{2.31}$$

The second is the $\frac{1}{2}$-BPS limit

$$\tfrac{1}{2}\text{-BPS limit:}\qquad \eta\, q = t,\qquad q\to 0\,. \tag{2.32}$$

For the key quantity $\mathcal{W}(\eta;q)$ we get

$$\mathcal{W}(1;q) = \mathrm{PE}\left[\frac{2q}{1-q}\right] = \prod_{n\geqslant 0}\mathrm{PE}[2q^{n+1}] = \prod_{n\geqslant 1}\frac{1}{(1-q^n)^2} = \frac{1}{(q)_\infty^2}, \tag{2.33}$$

$$\mathcal{W}(t) \equiv \lim_{q\to 0}\mathcal{W}(t/q;q) = \mathrm{PE}\left[\frac{t}{1-t}\right] = \prod_{n\geqslant 1}\frac{1}{1-t^n} = \frac{1}{(t)_\infty}\,. \tag{2.34}$$

2.3. Finite N Schur line correlators from the HS functional

At finite N, we need to keep non-diagonal terms in the double tableaux sum in (2.16), *i.e.* include contributions from $\lambda'\neq\lambda$. It is convenient to use the Hubbard-Stratonovich functional considered in [64]

$$\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-) = \int_{U(N)} dU\,\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(t_n^+\,\mathrm{Tr}\,U^n + t_n^-\,\mathrm{Tr}\,U^{-n})\right). \tag{2.35}$$

For a function $f(\boldsymbol{t}^+,\boldsymbol{t}^-)$, we define

$$\langle f\rangle_{\boldsymbol{g}} = \int\prod_{n=1}^{\infty}\frac{dt_n^+dt_n^-}{2\pi\, ng_n}e^{-\frac{1}{ng_n}t_n^+t_n^-}\,f(\boldsymbol{t}^+,\boldsymbol{t}^-),\qquad \int\frac{dt^+dt^-}{2\pi g}e^{-\frac{1}{g}t^+t^-}(t^+)^{n_+}(t^-)^{n_-} = n!\,g^n\,\delta_{n^+,n^-}\,. \tag{2.36}$$

Differentiating, we get

$$Z_N^{S_k}(\boldsymbol{g}) = \sum_{\lambda,\lambda'\vdash k}\frac{1}{\rho_\lambda\rho_{\lambda'}}\int\prod_{n\geqslant 1}\frac{dt_n^+dt_n^-}{2\pi\, ng_n}e^{-\frac{1}{ng_n}t_n^+t_n^-}n^{r_n+r'_n}\partial_{t_n^+}^{r_n}\partial_{t_n^-}^{r'_n}\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-)\,. \tag{2.37}$$

The previous result (2.24) for $N\to\infty$ is easily recovered as a special case. To see this, we use

$$\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}t_n^+t_n^-\right), \tag{2.38}$$

and relation (B.7) with $X=1/n$ gives

$$Z_\infty^{S_k}(\boldsymbol{g}) = \sum_{\lambda,\lambda'\vdash k}\frac{1}{\rho_\lambda\rho_{\lambda'}}\int\prod_{n\geqslant 1}\frac{dt_n^+dt_n^-}{2\pi\, ng_n}e^{-\frac{1}{ng_n}t_n^+t_n^-}n^{r_n+r'_n}e^{\frac{1}{n}t_n^+t_n^-}(t_n^+)^{r'_n}(t_n^-)^{r_n}n^{-r_n-r'_n}\sum_{p\geqslant 0}p!\binom{r_n}{p}\binom{r'_n}{p}\left(\frac{1}{n}t_n^+t_n^-\right)^{-p}. \tag{2.39}$$

Integrating over $\boldsymbol{t}^\pm$, we get a nonzero result if $\lambda=\lambda'$ so

$$\begin{aligned} Z_\infty^{S_k}(\boldsymbol{g}) &= \sum_{\lambda\vdash k}\frac{1}{\rho_\lambda^2}\int\prod_{n\geqslant 1}\frac{dt_n^+dt_n^-}{2\pi\, ng_n}e^{-\frac{1-g_n}{ng_n}t_n^+t_n^-}\sum_{p\geqslant 0}n^p p!\binom{r_n}{p}\binom{r_n}{p}(t_n^+t_n^-)^{r_n-p}\\ &= Z_\infty(\boldsymbol{g})\sum_{\lambda\vdash k}\frac{1}{\rho_\lambda^2}n^{r_n}\sum_{p\geqslant 0}p!\binom{r_n}{p}\binom{r_n}{p}(r_n-p)!\left(\frac{g_n}{1-g_n}\right)^{r_n-p}\\ &= Z_\infty(\boldsymbol{g})\sum_{\lambda\vdash k}\frac{1}{\rho_\lambda^2}n^{r_n}r_n!(1-g_n)^{-r_n} = Z_\infty(\boldsymbol{g})\sum_{\lambda\vdash k}\prod_{n\geqslant 1}\frac{1}{n^{r_n}r_n!}\frac{1}{(1-g_n)^{r_n}}\,, \end{aligned}\tag{2.40}$$

where we applied (2.36) with coupling $g = ng_n/(1-g_n)$. This is same as (2.24).

3. Single giant graviton correction: symmetric representation

The leading giant graviton correction to (2.37) follows from the correction to $\widetilde{Z}_N(t^+,t^-)$ computed in [64]. The main result is a determinantal expansion with first term

$$\frac{\widetilde{Z}_N(t^+,t^-)}{\widetilde{Z}_\infty(t^+,t^-)} = 1 - K_N(t^+,t^-) + \cdots, \qquad K_N(t^+,t^-) = \sum_{\substack{N<r \\ r\in\mathbb{Z}+\frac{1}{2}}} \widetilde{K}(r,r;t^+,t^-), \tag{3.1}$$

where

$$\begin{aligned} &\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \widetilde{K}(r,s;t^+,t^-)\, z^r w^{-s} = \frac{J(z;t^+,t^-)}{J(w;t^+,t^-)}\,\frac{\sqrt{zw}}{z-w}, \qquad |w|<|z|, \\ &J(z;t^+,t^-) = \exp\left(\sum_{n=1}^\infty \frac{1}{n}(t_n^+ z^n - t_n^- z^{-n})\right). \end{aligned} \tag{3.2}$$

Thus, the correction to (2.37) reads

$$\begin{aligned} &Z_N^{S_k}(g) = Z_\infty^{S_k}(g) - \sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \delta Z_N^{\lambda,\lambda'}(g) + \cdots, \\ &\delta Z_N^{\lambda,\lambda'}(g) = \int \prod_{n\geqslant 1} \frac{dt_n^+ dt_n^-}{2\pi\, n g_n} e^{-\frac{1}{n g_n} t_n^+ t_n^-} \eta^{r_n + r'_n} \partial_{t_n^+}^{r_n} \partial_{t_n^-}^{r'_n} [\widetilde{Z}_\infty(t^+,t^-) K_N(t^+,t^-)] + \cdots . \end{aligned} \tag{3.3}$$

As we fully discuss in Appendix C, one can prove the following result for the leading correction to the ratio $R_{S_k}^{U(N)}(g)$

$$R_{S_k}^{U(N)}(g) = 1 + \frac{1}{W_k(g)}\, \Phi_k^S(g;\zeta) G(g;\zeta)\Big|_{\zeta^{-N}} + \cdots, \tag{3.4}$$

with

$$G(g;\zeta) = \frac{-\zeta}{(1-\zeta)^2}\, \text{PE}\left[-\frac{g}{1-g}(1-\zeta)(1-\zeta^{-1})\right], \tag{3.5}$$

$$\Phi_k^S(g;\zeta) = \left[\frac{1}{1-\varepsilon_+\varepsilon_-}\,\frac{1-\varepsilon_+}{1-\varepsilon_-}\,\frac{1-\zeta\varepsilon_-}{1-\zeta^{-1}\varepsilon_+}\ \text{PE}\left[\frac{g}{1-g}\left(\varepsilon_+\varepsilon_- + \varepsilon_-(1-\zeta) - \varepsilon_+(1-\zeta^{-1})\right)\right]\right]_{\varepsilon_+^k,\varepsilon_-^k}. \tag{3.6}$$

At fixed k, we get a sum of terms where the dependence on ζ is in simple powers. Each power ζ^p in

$$\frac{1}{W_k(g)}\, \Phi_k^S(g;\zeta), \tag{3.7}$$

simply replaces $G_N(g) \to G_{N-p}(g)$ in (3.4), where $G_N(g) = G(g;\zeta)|_{\zeta^{-N}}$.

3.1. *Explicit ratios $R_{S_k}^{U(N)}$ at fixed k and special limits*

Evaluation of (3.4) at fixed low k is straightforward and reproduces the associated terms in the expansions of $R_{S_k}^{U(N)}$ presented in Section 2.1. To give explicit examples, for the rank-2, 3, 4 representations one finds, *cf.* (1.13),

$$\begin{aligned} G_{S_2}^+(\eta;q) &= \frac{\eta^2}{1-\eta^2} + \frac{2(1+\eta^4)}{\eta(1-\eta^2)}q + \frac{2-3\eta^4-4\eta^6+\eta^8}{\eta^4(1-\eta^2)}q^2 + \frac{2+\eta^2-3\eta^4-3\eta^6-\eta^8-2\eta^{10}+\eta^{12}}{\eta^7(1-\eta^2)}q^3 + \cdots, \\ G_{S_3}^+(\eta;q) &= \frac{\eta^2}{1-\eta^2} + \frac{2(1+\eta^4)}{\eta(1-\eta^2)}q + \frac{2+\eta^2-\eta^4-3\eta^6+2\eta^8}{\eta^4(1-\eta^2)}q^2 + \frac{2+2\eta^2-\eta^4-6\eta^6-2\eta^8-3\eta^{10}+\eta^{12}}{\eta^7(1-\eta^2)}q^3 + \cdots, \\ G_{S_4}^+(\eta;q) &= \frac{\eta^2}{1-\eta^2} + \frac{2(1+\eta^4)}{\eta(1-\eta^2)}q + \frac{2+\eta^2-\eta^4-3\eta^6+2\eta^8}{\eta^4(1-\eta^2)}q^2 + \frac{2(1+\eta^2-2\eta^6-\eta^{10}+\eta^{12})}{\eta^7(1-\eta^2)}q^3 + \cdots, \end{aligned} \tag{3.8}$$

where the above series are from exact expression obtained by (3.4)-(3.6) and we just wrote the first terms. Plugging the above into (1.13) reproduces the expansions (2.9), (2.10). In particular, the first two terms in $G_{S_k}^+$ do not depend on k and give

$$R_{S_k}^{U(N)} = 1 - \left[\frac{\eta(\eta^{-N-1}-\eta^{N+1})}{1-\eta^2} + \frac{2(1+\eta^4)(\eta^{-N}-\eta^{-N})}{\eta(1-\eta^2)}q + \cdots\right] q^N + \cdots . \tag{3.9}$$

It is also possible to consider the generalization of the subtracted ratio in (1.8) corresponding to $k=1$, *i.e.* we can introduce

$$\frac{\mathrm{I}^{U(N)}_{S_k}-W_k\,\mathrm{I}^{U(N)}}{\mathrm{I}^{U(\infty)}}=1+\left(\mathsf{G}^{+}_{S_k}(\eta;q)\,\eta^N+\mathsf{G}^{-}_{S_k}(\eta;q)\,\eta^{-N}\right)q^N+\cdots. \tag{3.10}$$

From the leading correction to the undecorated Schur index,

$$\begin{aligned}
&\frac{\mathrm{I}^{U(N)}(\eta;q)}{\mathrm{I}^{U(\infty)}(\eta;q)}=1+\left[\eta^N G^{+}_{\mathrm{D3}}(\eta;q)+\eta^{-N}G^{-}_{\mathrm{D3}}(\eta;q)\right]q^N+\mathcal{O}(q^{2N}),\\
&G^{+}_{\mathrm{D3}}(\eta;q)=G^{-}_{\mathrm{D3}}(\eta^{-1};q)=-\eta^2 q\,\frac{(\frac{q}{\eta})^3_\infty}{\vartheta(\eta^2,\frac{q}{\eta})}=\mathrm{PE}\left[\frac{\frac{1}{\eta q}-\frac{2}{\eta}q+q^2}{1-\frac{q}{\eta}}\right],
\end{aligned} \tag{3.11}$$

we can write

$$\mathsf{G}^{+}_{S_k}=X^{+}_{S_k}G^{+}_{\mathrm{D3}},\qquad X^{+}_{S_k}=W_k\left(\frac{G^{+}_{S_k}}{G^{+}_{\mathrm{D3}}}-1\right). \tag{3.12}$$

In analogy with the $k=1$ case, the quantity $X^{+}_{S_k}$ represents the contributions from fluctuations of the probe D3 brane that is used to represent the k coinciding fundamental strings building S_k, and attached to the giant graviton. For $k=1$, *cf.* (1.9) and (1.10), it was [48]

$$X^{+}_{S_1}=\frac{1}{\eta q}\frac{(1-q^2)^2}{(1-\eta^{-1}q)^2}=\frac{1}{\eta q}\,\mathrm{PE}[f_{\mathrm{F}}(\eta;q)]. \tag{3.13}$$

For $k>1$ exact expressions are unwieldy and little instructive. For instance, for $k=2$ we get

$$X^{+}_{S_2}=\frac{\eta^3+(1+\eta^2)\,q-\eta(4+\eta^2)\,q^2-(1-2\eta^2)\,q^3+\eta(1-\eta^2)\,q^4+q^5}{\eta^4 q}\times\frac{(1-q^2)^2}{(1-\eta q)(1-\eta^{-1}q)^2(1-\eta^{-2}q^2)^2}. \tag{3.14}$$

The second line may be written as a simple plethystic exponential, but we also have a non trivial sum of monomials in the first line. A similar result was found in [48] for the symmetric 2-power of the fundamental representation.

The result in $\frac{1}{2}$-BPS limit (2.32) is much simpler and one finds the following pattern, valid for all k,

$$X^{+}_{S_k}=\frac{1}{t}\prod_{p=1}^{k-1}\frac{1}{1-t^p}=\frac{1}{t\,(t;t)_{k-1}}. \tag{3.15}$$

3.2. Large k limit and $R^{U(N)}_S$

Taking into account that $g_n\sim q^n$, it will be convenient to organize expressions in powers of the auxiliary counting variable κ with $g_n\to\kappa^n g_n$. The explicit form of $\Phi^S_k(\boldsymbol{g};\zeta)$ turns out to be

$$\begin{aligned}
\Phi^S_k(\boldsymbol{g};\zeta)&=\zeta-\zeta^{1-k}+\zeta^{-k}\\
&+(\zeta-3\zeta^{1-k}+\zeta^{2-k}+2\zeta^{-k})\,g_1\,\kappa\\
&+[g_1^2+\zeta^2 g_1^2+\zeta^{1-k}(-7g_1^2-g_2)+\frac{1}{2}\zeta^{2-k}(7g_1^2-g_2)+\frac{1}{2}\zeta(-g_1^2+g_2)+\frac{1}{2}\zeta^{3-k}(-g_1^2+g_2)\\
&+\zeta^{-k}(4g_1^2+g_2)]\,\kappa^2+\cdots.
\end{aligned} \tag{3.16}$$

Alternatively, after specializing $g_n=f(\eta^n;q^n)$, we can organize the expansion in powers of q. Defining

$$\Phi^S_k(\eta;q;\zeta)=\sum_{p=0}^{\infty}\Phi^S_{k,p}(\eta;\zeta)\,q^p, \tag{3.17}$$

we have

$$\begin{aligned}
\Phi^S_{k,0}(\eta;\zeta)&=\zeta^{-k}(1-\zeta+\zeta^{1+k}),\\
\Phi^S_{k,1}(\eta;\zeta)&=(\eta+\eta^{-1})\zeta^{-k}(2-3\zeta+\zeta^2+\zeta^{k+1}),\\
\Phi^S_{k,2}(\eta;\zeta)&=(\eta^{-2}+2+\eta^2)(1+\zeta^2)-3\zeta-\zeta^{3-k}+\zeta^{2-k}(3\eta^{-2}+5+3\eta^2)\\
&\quad+\zeta^{-k}(5\eta^{-2}+4+5\eta^2)-8\zeta^{1-k}(\eta^{-2}+1+\eta^2),\\
&\cdots.
\end{aligned} \tag{3.18}$$

The above coefficients are valid for large enough k. In more details, we need $k \geqslant p$ in $\Phi^S_{k,p}(\eta;\zeta)$. We can now take the $k\to\infty$ limit at fixed N. We can drop from $\Phi^S_{k,p}$ powers of ζ with exponent dependent on k since these are of the form $q^{k+\cdots}$ and give terms suppressed at large k. Special large k scaling relations like $k\sim N$ or $k\sim N^2$ are equivalent from this point of view.

To isolate powers of ζ with exponent independent on k, we start from

$$\Phi^S_k(\mathbf{g};\zeta) = \oint \frac{d\varepsilon_+}{2\pi i\varepsilon_+^{k+1}} \oint \frac{d\varepsilon_-}{2\pi i\varepsilon_-^{k+1}} \frac{1}{1-\varepsilon_+\varepsilon_-}\frac{1-\varepsilon_+}{1-\varepsilon_-}\frac{1-\zeta\varepsilon_-}{1-\zeta^{-1}\varepsilon_+}\ \mathrm{PE}\left[\frac{\mathbf{g}}{1-\mathbf{g}}\left(\varepsilon_+\varepsilon_- + \varepsilon_-(1-\zeta)-\varepsilon_+(1-\zeta^{-1})\right)\right]. \tag{3.19}$$

We integrate over ε_- by taking minus the residues at $\varepsilon_- = 1, \varepsilon_+^{-1}$. The first pole leaves a k dependence. So, taking the residue at the second pole, we define

$$\widetilde{\Phi}^S(\mathbf{g};\zeta) = \zeta \oint \frac{d\varepsilon}{2\pi i\varepsilon}\ \mathrm{PE}\left[\frac{\mathbf{g}}{1-\mathbf{g}}\left(1+\varepsilon^{-1}(1-\zeta)-\varepsilon(1-\zeta^{-1})\right)\right]. \tag{3.20}$$

One checks that indeed it is correct, *i.e.* generates all terms in (3.16) that are k-independent

$$\widetilde{\Phi}^S(\mathbf{g};\zeta) = \zeta + \zeta\, g_1 + [g_1^2 + \zeta^2 g_1^2 + \frac{1}{2}\zeta(-g_1^2+g_2)] + \cdots . \tag{3.21}$$

Each power ζ^p simply shifts $G_N(\eta;q)\to G_{N-p}(\eta;q)$ where, *cf.* (3.11),

$$G_N(\eta;q) = -\left[\eta^{N+2}\frac{(\frac{q}{\eta})^3_\infty}{\vartheta(\eta^2,\frac{q}{\eta})} + \eta^{-N-2}\frac{(\eta q)^3_\infty}{\vartheta(\eta^{-2},\eta q)}\right] q^{N+1}. \tag{3.22}$$

This gives

$$R_S^{U(N)}(\eta;q) = 1 + [\eta^N G^+_S(\eta;q) + \eta^{-N} G^-_S(\eta;q)]\, q^N + \cdots, \qquad G^-_S(\eta;q) = G^+_S(\eta^{-1};q), \tag{3.23}$$

with

$$\begin{aligned}\frac{G^+_S(\eta;q)}{G^+_{\mathrm{D3}}(\eta;q)} &= \mathrm{PE}\left[-\frac{f}{1-f}\right]\frac{1}{\eta q}\oint \frac{d\varepsilon}{2\pi i\varepsilon}\ \mathrm{PE}\left[\frac{f}{1-f}\left(1+\varepsilon^{-1}(1-\eta^{-1}q^{-1})-\varepsilon(1-\eta q)\right)\right] \\ &= \frac{1}{\eta q}\oint\frac{d\varepsilon}{2\pi i\varepsilon}\ \mathrm{PE}\left[\frac{f}{1-f}\left(\frac{1}{\varepsilon}\left(1-\frac{1}{\eta q}\right)-\varepsilon(1-\eta q)\right)\right].\end{aligned} \tag{3.24}$$

The plethystic exponential may be expressed in terms of q-Pochhammer symbols

$$\begin{aligned}\mathrm{PE}\left[\frac{f}{1-f}\left(\frac{1}{\varepsilon}\left(1-\frac{1}{\eta q}\right)-\varepsilon(1-\eta q)\right)\right] &= \mathrm{PE}\left[-\frac{(1+\eta q\varepsilon^2)(1-2\eta q+\eta^2)}{\varepsilon\eta^2(1-\eta^{-1}q)}\right] \\ &= \mathrm{PE}[\sum_{n=0}^\infty(-\frac{1}{\varepsilon}+2q^2\varepsilon-\frac{1}{\varepsilon\eta^2}+\frac{2q}{\varepsilon\eta}-\frac{q\varepsilon}{\eta}-q\eta\varepsilon)(\eta^{-1}q)^n] \\ &= \frac{(\varepsilon^{-1};\eta^{-1}q)_\infty\,(\varepsilon^{-1}\eta^{-2};\eta^{-1}q)_\infty\,(\varepsilon\eta^{-1}q;\eta^{-1}q)_\infty\,(\varepsilon\eta q;\eta^{-1}q)_\infty}{(\varepsilon q^2;\eta^{-1}q)^2_\infty\,(\varepsilon^{-1}\eta^{-1}q;\eta^{-1}q)^2_\infty}\end{aligned} \tag{3.25}$$

The general ratio is thus

$$\frac{G^+_S(\eta;q)}{G^+_{\mathrm{D3}}(\eta;q)} = \frac{1}{\eta q}\oint\frac{d\varepsilon}{2\pi i\varepsilon}\frac{(\varepsilon^{-1};\eta^{-1}q)_\infty\,(\varepsilon^{-1}\eta^{-2};\eta^{-1}q)_\infty\,(\varepsilon\eta^{-1}q;\eta^{-1}q)_\infty\,(\varepsilon\eta q;\eta^{-1}q)_\infty}{(\varepsilon q^2;\eta^{-1}q)^2_\infty\,(\varepsilon^{-1}\eta^{-1}q;\eta^{-1}q)^2_\infty}. \tag{3.26}$$

This may be written in a more compact form by using q-theta functions, *cf.* (A.3),

$$\frac{G^+_S(\eta;q)}{G^+_{\mathrm{D3}}(\eta;q)} = \frac{1}{\eta q^2}\oint\frac{d\varepsilon}{2\pi i\varepsilon}\frac{(\varepsilon;\frac{q}{\eta})^2_\infty}{(\varepsilon q^2;\frac{q}{\eta})^2_\infty}\frac{\vartheta(\varepsilon;\frac{q}{\eta})\,\vartheta(\varepsilon\eta^2;\frac{q}{\eta})}{\vartheta(\varepsilon\frac{\eta}{q};\frac{q}{\eta})^2}. \tag{3.27}$$

From (A.4), we have

$$\vartheta\left(\varepsilon\frac{\eta}{q};\frac{q}{\eta}\right) = -(q/\eta)^{-1/2}\varepsilon\,\vartheta\left(\varepsilon;\frac{q}{\eta}\right), \tag{3.28}$$

and thus we get[10]

[10] It would be interesting to see if methods of [19] may deal with the contour integral (3.27) to give an explicit η dependent q-series, although this is not essential for our applications.

$$\frac{G^+_S(\eta;q)}{G^+_{\rm D3}(\eta;q)} = \frac{1}{\eta^2 q}\oint \frac{d\varepsilon}{2\pi i\varepsilon}\frac{1}{\varepsilon^2}\frac{(\varepsilon;\frac{q}{\eta})^2_\infty}{(\varepsilon q^2;\frac{q}{\eta})^2_\infty}\frac{\vartheta(\varepsilon\eta^2;\frac{q}{\eta})}{\vartheta(\varepsilon;\frac{q}{\eta})}. \tag{3.29}$$

The explicit expansion in powers of q is straightforward and reads

$$\begin{aligned} G^+_S(\eta;q) &= \frac{\eta^2}{1-\eta^2}+\frac{2(1+\eta^4)}{\eta(1-\eta^2)}\,q+\frac{2+\eta^2-\eta^4-3\eta^6+2\eta^8}{\eta^4(1-\eta^2)}\,q^2 \\ &+(2\eta^{-7}+4\eta^{-5}+4\eta^{-3}-2\eta^3)\,q^3 \\ &+(2\eta^{-10}+4\eta^{-8}+7\eta^{-6}-3\eta^{-4}-3\eta^{-2}+1-2\eta^4)\,q^4 \\ &+(2\eta^{-13}+4\eta^{-11}+8\eta^{-9}-6\eta^{-5}+2\eta^{-3}-2\eta^5)\,q^5+\cdots. \end{aligned} \tag{3.30}$$

3.2.1. *Unflavored limit*

In unflavored limit $\eta\to 1$, the plethystic exponential (3.25) simplifies to

$$\text{PE}\left[-\frac{(1+q\varepsilon^2)(1-2q+1)}{\varepsilon(1-q)}\right] = \text{PE}\left[-\frac{2}{\varepsilon}-2q\varepsilon\right] = \left(1-\frac{1}{\varepsilon}\right)^2(1-q\varepsilon)^2, \tag{3.31}$$

and thus

$$\lim_{\eta\to1}\frac{G^+_S(\eta;q)}{G^+_{\rm D3}(\eta;q)} = \frac{1}{q}\oint\frac{d\varepsilon}{2\pi i\varepsilon}\left(1-\frac{1}{\varepsilon}\right)^2(1-q\varepsilon)^2 = q^{-1}+4+q. \tag{3.32}$$

The first correction in $\eta-1$ can be computed as follows

$$\begin{aligned} &\frac{1}{\eta q}\ \text{PE}\left[\frac{f}{1-f}\left(\frac{1}{\varepsilon}\left(1-\frac{1}{\eta q}\right)-\varepsilon(1-\eta q)\right)\right] \\ &=\frac{1}{q}\Big(1-(\eta-1)+\cdots\Big)\,\text{PE}\left[-\frac{2}{\varepsilon}-2q\varepsilon+\frac{2(1+\varepsilon^2q^2)}{\varepsilon(1-q)}(\eta-1)+\cdots\right] \\ &=\frac{1}{q}\left(1-\frac{1}{\varepsilon}\right)^2(1-q\varepsilon)^2\left[1+\left(-1+\sum_{n=0}^\infty\left(\frac{2q^n}{\varepsilon-q^n}+\frac{2\varepsilon q^{n+2}}{1-\varepsilon q^{n+2}}\right)\right)(\eta-1)+\cdots\right] \\ &=\frac{1}{q}\left(1-\frac{1}{\varepsilon}\right)^2(1-q\varepsilon)^2\left[1+\left(-1+2\sum_{n,p=0}^\infty\left(\frac{q^n}{\varepsilon}\frac{q^{np}}{\varepsilon^p}+\varepsilon q^{n+2}(\varepsilon q^{n+2})^p\right)\right)(\eta-1)+\cdots\right] \\ &=\frac{1}{q}\left(1-\frac{1}{\varepsilon}\right)^2(1-q\varepsilon)^2\left[1+\left(-1+\sum_{p=0}^\infty\frac{2(\varepsilon^{-1-p}+\varepsilon^{1+p}q^{2+2p})}{1-q^{p+1}}\right)(\eta-1)+\cdots\right]. \end{aligned} \tag{3.33}$$

The coefficient of ε^0 receives contributions from a finite number of terms in the sum over p, and this gives

$$\frac{G^+_S(\eta;q)}{G^+_{\rm D3}(\eta;q)} = q^{-1}+4+q-\frac{1+8q+10q^2+8q^3+q^4}{q(1-q^2)}(\eta-1)+\mathcal{O}((\eta-1)^2). \tag{3.34}$$

Using the near unflavored expansion [36]

$$G^+_{\rm D3}(\eta;q) = -\frac{1}{2}q\,\frac{1}{\eta-1}-\frac{5}{4}q+\mathcal{O}((\eta-1)^2), \tag{3.35}$$

gives the total correction

$$\eta^N G^+_S(\eta;q)+\eta^{-N}G^-_S(\eta;q) \overset{\eta\to1}{=} -N(1+4q+q^2)-\frac{1-10q^2-16q^3-3q^4}{1-q^2}. \tag{3.36}$$

Extra powers of N in the unrefined limit are commonly referred to as a wall-crossing effect [6,7,65]. In similar contexts, it was shown that they arise on gravity side from zero mode of fluctuations [24,25,66]. In this case, it would be important to clarify whether they arise by the same mechanisms and determine the details of the physics, in particular what symmetries are responsible for zero modes. This would require a detailed analysis of the boundary conditions for the brane probe attached to the giant.

3.2.2. $\frac{1}{2}$*-BPS limit*

In $\frac{1}{2}$-BPS limit (2.32), the plethystic exponential (3.25) is

$$\text{PE}\left[-\frac{(1+\eta q\varepsilon^2)(1-2\eta q+\eta^2)}{\varepsilon\eta^2(1-\eta^{-1}q)}\right] = \text{PE}\left[-\frac{1}{\varepsilon}-t\varepsilon\right] = \left(1-\frac{1}{\varepsilon}\right)(1-t\varepsilon), \tag{3.37}$$

and thus

$$\frac{G_S^+(t)}{G_{D3}^+(t)} = \frac{1}{t}\oint \frac{d\varepsilon}{2\pi i\varepsilon}\left(1-\frac{1}{\varepsilon}\right)(1-t\varepsilon) = \frac{1+t}{t}. \tag{3.38}$$

Using

$$G_{D3}^+(t) = -\frac{t}{1-t}, \tag{3.39}$$

gives

$$G_S^+(t) = -\frac{1+t}{1-t}. \tag{3.40}$$

This is consistent with (3.15). Indeed, for large k

$$X_S^+ = W_\infty\left(\frac{G_S^+}{G_{D3}^+}-1\right) = \left(\frac{1+t}{t}-1\right)\prod_{n\geqslant 1}\frac{1}{1-t^n} = \frac{1}{t}\prod_{n\geqslant 1}\frac{1}{1-t^n} = \lim_{k\to\infty} X_{S_k}^+. \tag{3.41}$$

4. Single giant graviton correction: antisymmetric representation

For the antisymmetric representation, *cf.* (2.4), we can repeat the derivation in Appendix C.1 by taking into account relation (2.7). This readily gives, *cf.* (3.4),

$$R_{A_k}^{U(N)}(\boldsymbol{g}) = 1 + \frac{1}{W_k(\boldsymbol{g})}\,\Phi_k^A(\boldsymbol{g};\zeta)G(\boldsymbol{g};\zeta)\Big|_{\zeta^{-N}} + \cdots, \tag{4.1}$$

with

$$\Phi_k^A(\boldsymbol{g};\zeta) = \left[\frac{1}{1-\varepsilon_+\varepsilon_-}\,\frac{1-\varepsilon_-}{1-\varepsilon_+}\,\frac{1-\zeta^{-1}\varepsilon_+}{1-\zeta\varepsilon_-}\ \text{PE}\left[-\frac{\boldsymbol{g}}{1-\boldsymbol{g}}\left(-\varepsilon_+\varepsilon_- + \varepsilon_-(1-\zeta) - \varepsilon_+(1-\zeta^{-1})\right)\right]\right]_{\varepsilon_+^k,\varepsilon_-^k}. \tag{4.2}$$

In fact, this expression implies the very simple relation between the Φ functions for symmetric and antisymmetric representations

$$\Phi_k^A(\boldsymbol{g};\zeta) = \Phi_k^S(\boldsymbol{g};\zeta^{-1}). \tag{4.3}$$

Expanding in q after specialization to the Schur index gives then, *cf.* (3.18),[11]

$$\begin{aligned}
\Phi_{k,0}^A(\eta;\zeta) &= \frac{1}{\zeta} - \zeta^{k-1} + \zeta^k,\\
\Phi_{k,1}^A(\eta;\zeta) &= (\eta+\eta^{-1})\left[\frac{1}{\zeta} + \zeta^{k-2} - 3\zeta^{k-1} + 2\zeta^k\right],\\
&\dots
\end{aligned} \tag{4.4}$$

and so on. Now, due to (4.3), exponents of ζ include cases $k-n$ with a non-negative integer n. These replace $G_N \to G_{N-k+n}$. It makes sense to consider $k \gg 1$ only with $k < N$. We may select here all powers of ζ with an exponent of that form. In the contour integral

$$\begin{aligned}
\Phi_k^A(\boldsymbol{g};\zeta) = \oint \frac{d\varepsilon_+}{2\pi i\varepsilon_+^{k+1}}\oint \frac{d\varepsilon_-}{2\pi i\varepsilon_-^{k+1}}\\
\frac{1}{1-\varepsilon_+\varepsilon_-}\,\frac{1-\varepsilon_-}{1-\varepsilon_+}\,\frac{1-\zeta^{-1}\varepsilon_+}{1-\zeta\varepsilon_-}\ \text{PE}\left[-\frac{\boldsymbol{g}}{1-\boldsymbol{g}}\left(-\varepsilon_+\varepsilon_- + \varepsilon_-(1-\zeta) - \varepsilon_+(1-\zeta^{-1})\right)\right],
\end{aligned} \tag{4.5}$$

we pick now minus the pole at $\varepsilon_+ = 1$ and then the pole at $\varepsilon_- = 1/\zeta$. This defines

$$\widetilde{\Phi}_k^A(\boldsymbol{g};\zeta) = \zeta^{k-1}(\zeta-1)\ \text{PE}\left[\frac{\boldsymbol{g}}{1-\boldsymbol{g}}\left(2-\frac{1}{\zeta}\right)\right]. \tag{4.6}$$

In conclusion, we can write (up to subleading corrections $\sim q^{N+\delta}$ with δ being a fixed k-independent integer)

$$R_{A_k}^{U(N)}(\eta;q) = 1 + [\eta^{N-k}G_A^+(\eta;q) + \eta^{k-N}G_A^-(\eta;q)]\,q^{N-k} + \cdots, \tag{4.7}$$

with

$$\begin{aligned}
\frac{G_A^+(\eta;q)}{G_{D3}^+(\eta;q)} &= \text{PE}\left[-\frac{f}{1-f}\right](1-\eta q)\ \text{PE}\left[\frac{f}{1-f}(2-\eta q)\right]\\
&= (1-\eta q)\ \text{PE}\left[\frac{f}{1-f}(1-\eta q)\right] = \frac{1}{1-\eta q}\,\frac{(\eta q;\frac{q}{\eta})_\infty}{(\frac{q}{\eta})_\infty}.
\end{aligned} \tag{4.8}$$

[11] See comments after (3.18) for the validity of these relations.

Notice also the expression

$$G_A^+(\eta;q) = -\frac{\eta q}{1-\eta q}\frac{(\frac{q}{\eta})_\infty}{(\frac{1}{\eta^2};\frac{q}{\eta})_\infty}. \tag{4.9}$$

4.1. Special limits

Unrefined limit We start with

$$\frac{(aq;q)_\infty}{(a^b q;q)_\infty} = \exp\sum_{n\geqslant 0}[\log(1-aq^{n+1}) - \log(1-a^b q^{n+1})] = 1 + (b-1)(a-1)\sum_{n\geqslant 1}\frac{q^n}{1-q^n} + \mathcal{O}((a-1)^2). \tag{4.10}$$

Using (4.9), we get

$$G_A^+(\eta;q) = -\frac{q}{2(1-q)}\frac{1}{\eta-1} + \frac{q(3q-5)}{4(1-q)^2} + \frac{q}{1-q}\sum_{n\geqslant 1}\frac{q^n}{1-q^n} + \mathcal{O}(\eta-1), \tag{4.11}$$

and this gives the total correction

$$\eta^{N-k}G_A^+(\eta;q) + \eta^{-N+k}G_A^-(\eta;q) \overset{\eta\to 1}{=} -\frac{q}{1-q}(N-k) + \frac{2q}{1-q}\sum_{n\geqslant 1}\frac{q^n}{1-q^n} - \frac{q(2-q)}{(1-q)^2}. \tag{4.12}$$

To give an example, for $N=5$ and $k=3$ we get

$$1 + (4.12)\times q^{N-k} = 1 - 4q^3 - 3q^4 + 0\times q^5 + 3q^6 + \cdots, \tag{4.13}$$

where we wrote only terms that are not affected by higher giant graviton contributions. This is in agreement with the $\eta = 1$ limit of (2.13).

$\frac{1}{2}$-BPS limit In this limit, we have simply

$$(1-\eta q)\ \text{PE}\left[\frac{f}{1-f}(1-\eta q)\right] = (1-t)\,\text{PE}[t] = 1\,, \tag{4.14}$$

that implies $G_A^+(t) = G_{\text{D3}}^+(t) = -t/(1-t)$, *cf.* (3.39), see also Section 2.4 of [49].

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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Appendix A. Conventions for q-functions

We collect in this appendix the definition of special q-functions appearing in the text.

q-Pochhammer symbol

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-a\,q^k)\,, \qquad (a^\pm;q)_\infty = (a;q)_\infty(a^{-1};q)_\infty\,, \tag{A.1}$$

$$(q)_\infty \equiv (q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k)\,. \tag{A.2}$$

q-theta function The q-theta function is defined as

$$\vartheta(x,q) = -x^{-\frac{1}{2}}(q)_\infty(x;q)_\infty(qx^{-1};q)_\infty\,, \tag{A.3}$$

with

$$\vartheta(x;q) = -\vartheta(x^{-1};q), \qquad \vartheta(q^m x;q) = (-1)^m q^{-\frac{m^2}{2}} x^{-m}\vartheta(x;q)\,. \tag{A.4}$$

Appendix B. A useful differentiation identity

In our analysis it is useful to give an explicit formula for the double multiple derivative

$$\partial_a^n\partial_b^m e^{Xab}\,. \tag{B.1}$$

It may be obtained by introducing two auxiliary Gaussian variables to have linear dependence on a, b in the exponent:

$$\begin{aligned}
\partial_a^n\partial_b^m e^{Xab} &= \partial_a^n\partial_b^m \frac{1}{\pi}\int dA\,dB\,e^{-A^2-B^2+a\sqrt{X}(A+iB)+b\sqrt{X}(A-iB)}\\
&= X^{\frac{n+m}{2}}\frac{1}{\pi}\int dA\,dB(A+iB)^n(A-iB)^m e^{-A^2-B^2+a\sqrt{X}(A+iB)+b\sqrt{X}(A-iB)}\\
&= e^{Xab}X^{\frac{n+m}{2}}\frac{1}{\pi}\int dA\,dB(A+iB+b\sqrt{X})^n(A-iB+a\sqrt{X})^m e^{-A^2-B^2}\\
&= e^{Xab}X^{\frac{n+m}{2}}\langle(A+iB+b\sqrt{X})^n(A-iB+a\sqrt{X})^m\rangle,
\end{aligned} \tag{B.2}$$

with

$$\langle A^{2p}\rangle = \frac{1}{\sqrt{\pi}}\Gamma(p+\tfrac{1}{2})\,. \tag{B.3}$$

Introducing complex combinations

$$z = A+iB, \qquad \bar{z} = A-iB, \tag{B.4}$$

with the non-zero average

$$\langle(z\bar{z})^p\rangle = \frac{1}{\pi}\sum_{q=0}^{p}\Gamma(q+\tfrac{1}{2})\Gamma(p-q+\tfrac{1}{2})\binom{p}{q} = p!\,, \tag{B.5}$$

gives

$$\langle(z+x)^n(\bar{z}+y)^m\rangle = \sum_{a,b}\binom{n}{a}\binom{m}{b}\langle z^a\bar{z}^b\rangle x^{n-a}y^{m-b} = x^n y^m\sum_{p\geqslant 0}p!\binom{n}{p}\binom{m}{p}(xy)^{-p}\,, \tag{B.6}$$

and thus

$$\partial_a^n\partial_b^m e^{Xab} = e^{Xab}a^m b^n X^{n+m}\sum_{p\geqslant 0}p!\binom{n}{p}\binom{m}{p}(Xab)^{-p}. \tag{B.7}$$

If additional sources linear in a, b are present, the same procedure leads to

$$\partial_a^n\partial_b^m e^{Xab+\sqrt{X}(\alpha a+\beta b)} = e^{Xab+\sqrt{X}(\alpha a+\beta b)}X^{\frac{n+m}{2}}(\alpha+b\sqrt{X})^n(\beta+a\sqrt{X})^m\sum_{p\geqslant 0}p!\binom{n}{p}\binom{m}{p}[(\alpha+b\sqrt{X})(\beta+a\sqrt{X})]^{-p}\,, \tag{B.8}$$

reducing to (B.7) when $\alpha=\beta=0$.

Appendix C. Full details of the main calculation

In this Appendix we present the detailed derivation of (3.3), (3.5), (3.6). Following [64], we begin by introducing

$$\begin{aligned}
U &= \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} z^r w^{-s}\langle\prod_{n\geqslant 1} n^{r_n+r'_n}\partial_{t_n^+}^{r_n}\partial_{t_n^-}^{r'_n}[\widetilde{Z}_\infty(t^+,t^-)\,\widetilde{K}(r,s;t^+,t^-)]\rangle_g\\
&= \frac{\sqrt{zw}}{z-w}\int\prod_{n=1}\frac{dt_n^+dt_n^-}{2\pi n g_n}\,e^{-\frac{1}{ng_n}t_n^+t_n^-}\,n^{r_n+r'_n}\partial_{t_n^+}^{r_n}\partial_{t_n^-}^{r'_n}\exp\frac{1}{n}\left(t_n^+t_n^- + t_n^+(z^n-w^n) - t_n^-(z^{-n}-w^{-n})\right).
\end{aligned} \tag{C.1}$$

We use (B.8) with

$$X=\frac{1}{n},\qquad \alpha=\frac{1}{\sqrt{n}}(z^n-w^n)=\frac{1}{\sqrt{n}}T_n^-,\qquad \beta=-\frac{1}{\sqrt{n}}(z^{-n}-w^{-n})=-\frac{1}{\sqrt{n}}T_n^+, \tag{C.2}$$

and obtain

$$U=\frac{\sqrt{zw}}{z-w}\int\prod_{n=1}\frac{dt_n^+dt_n^-}{2\pi n g_n}\sum_{p\geqslant 0}n^p p!\binom{r_n}{p}\binom{r'_n}{p}(t_n^+-T_n^+)^{r'_n-p}(t_n^-+T_n^-)^{r_n-p}$$
$$\exp\frac{1}{n}\left(-\frac{1-g_n}{g_n}t_n^+t_n^-+t_n^+T_n^--t_n^-T_n^+\right). \tag{C.3}$$

To integrate over $t^\pm$, it is convenient to first translate t^+, t^- according to

$$t_n^+\to t_n^+-\frac{g_n}{1-g_n}T_n^+,\qquad t_n^-\to t_n^-+\frac{g_n}{1-g_n}T_n^-. \tag{C.4}$$

This gives

$$U=\frac{\sqrt{zw}}{z-w}\int\prod_{n=1}\frac{dt_n^+dt_n^-}{2\pi n g_n}\sum_{p\geqslant 0}n^p p!\binom{r_n}{p}\binom{r'_n}{p}\left(t_n^+-\frac{1}{1-g_n}T_n^+\right)^{r'_n-p}\left(t_n^-+\frac{1}{1-g_n}T_n^-\right)^{r_n-p}\exp\frac{1}{n}\left(-\frac{1-g_n}{g_n}t_n^+t_n^--\frac{g_n}{1-g_n}T_n^+T_n^-\right). \tag{C.5}$$

We need the diagonal terms in

$$\sum_{p\geqslant 0}n^p p!\binom{r_n}{p}\binom{r'_n}{p}\left(t_n^+-\frac{1}{1-g_n}T_n^+\right)^{r'_n-p}\left(t_n^-+\frac{1}{1-g_n}T_n^-\right)^{r_n-p}, \tag{C.6}$$

where we will replace

$$(t_n^+t_n^-)^m\to m!\left(\frac{ng_n}{1-g_n}\right)^m. \tag{C.7}$$

Let us compute the quantity

$$\begin{aligned}
D_{r_n,r'_n}&=\sum_{p\geqslant 0}n^p p!\binom{r_n}{p}\binom{r'_n}{p}\left(t_n^+-\frac{1}{1-g_n}T_n^+\right)^{r'_n-p}\left(t_n^-+\frac{1}{1-g_n}T_n^-\right)^{r_n-p}\\
&=\sum_{p,a,b\geqslant 0}n^p p!\binom{r_n}{p}\binom{r'_n}{p}\binom{r'_n-p}{a}\binom{r_n-p}{b}(t_n^+)^a(t_n^-)^b\left(-\frac{1}{1-g_n}T_n^+\right)^{r'_n-p-a}\left(\frac{1}{1-g_n}T_n^-\right)^{r_n-p-b}\\
&=\left(-\frac{1}{1-g_n}T_n^+\right)^{r'_n}\left(\frac{1}{1-g_n}T_n^-\right)^{r_n}\sum_{p,a\geqslant 0}n^p p!\binom{r_n}{p}\binom{r'_n}{p}\binom{r_n-p}{a}\binom{r'_n-p}{a}a!\left(\frac{ng_n}{1-g_n}\right)^a\left(-\frac{1}{(1-g_n)^2}T_n^+T_n^-\right)^{-p-a}\\
&=\left(-\frac{1}{1-g_n}T_n^+\right)^{r'_n}\left(\frac{1}{1-g_n}T_n^-\right)^{r_n}\sum_{p,a\geqslant 0}a!p!\binom{r_n}{p}\binom{r'_n}{p}\binom{r_n-p}{a}\binom{r'_n-p}{a}\left(\frac{g_n}{1-g_n}\right)^a\left(-\frac{1}{n(1-g_n)^2}T_n^+T_n^-\right)^{-p-a}.
\end{aligned} \tag{C.8}$$

The product over n has a factor

$$\prod_{n\geqslant 1}(T_n^+)^{r'_n}(T_n^-)^{r_n}. \tag{C.9}$$

The sums of $\pm$ indices are $\sum_n r'_n n=|\lambda'|$ and $\sum_n r_n n=|\lambda|$, respectively. Consider now

$$\begin{aligned}
f_{r,r'}&=\sum_{p,a\geqslant 0}a!p!\binom{r}{p}\binom{r'}{p}\binom{r-p}{a}\binom{r'-p}{a}X^aY^p\\
&=\sum_{p,a\geqslant 0}a!p!\frac{r!}{p!(r-p)!}\frac{r'!}{p!(r'-p)!}\frac{(r-p)!}{a!(r-p-a)!}\frac{(r'-p)!}{a!(r'-p-a)!}X^aY^p\\
&=\sum_{p,a\geqslant 0}\frac{1}{a!p!}\frac{r!r'!}{(r-p-a)!(r'-p-a)!}X^aY^p.
\end{aligned} \tag{C.10}$$

Changing summation variable $p+a=q$, we get

$$\begin{aligned}
f_{r,r'}&=\sum_{a,q\geqslant 0}\frac{1}{a!(q-a)!}\frac{r!r'!}{(r-q)!(r'-q)!}X^aY^{q-a}\\
&=\sum_{q\geqslant 0}q!\binom{r}{q}\binom{r'}{q}(X+Y)^q={}_2F_0(-r,-r',X+Y).
\end{aligned} \tag{C.11}$$

Setting $w = \zeta\, z$, the object we need to evaluate is thus

$$\begin{aligned}\Phi_k^S(\boldsymbol{g};\zeta) &= \sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \prod_{n\geqslant 1} D_{r_n, r'_n} \\ &= \sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \prod_{n\geqslant 1} \Big(-\frac{1}{1-g_n} T_n^+\Big)^{r'_n} \Big(\frac{1}{1-g_n} T_n^-\Big)^{r_n} \sum_{p\geqslant 0} p! \binom{r_n}{p}\binom{r'_n}{p}\Big(-\frac{n(1-g_n)}{T_n^+ T_n^-}\Big)^p,\end{aligned} \tag{C.12}$$

which is a function of ζ because, after taking the product over n, only terms of the form

$$T^+_{n_1^+} T^+_{n_2^+} \cdots T^-_{n_1^-} T^-_{n_2^-} \cdots \tag{C.13}$$

with

$$n_1^+ + n_2^+ + \cdots = n_1^- + n_2^- + \cdots, \tag{C.14}$$

survive, and thus all dependence on z, w is through the ratio $\zeta = w/z$. The leading giant graviton correction in (3.3) is then[12]

$$\sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \delta Z_N^{\lambda,\lambda'}(\boldsymbol{g}) = Z_\infty(\boldsymbol{g})\, \Phi_k^S(\boldsymbol{g};\zeta) \frac{\zeta}{(1-\zeta)^2} \,\mathrm{PE}\Big[-\frac{\boldsymbol{g}}{1-\boldsymbol{g}}(1-\zeta)(1-\zeta^{-1})\Big]\Big|_{\zeta^{-N}}, \tag{C.15}$$

and this proves (3.4). It remains to put in a more explicit form the quantity $\Phi_k^S(\boldsymbol{g};\zeta)$ which was defined in (C.12).

C.1. Plethystic representation of $\Phi_k^S(\boldsymbol{g};\zeta)$

We start with the following straightforward manipulations

$$\begin{aligned}\Phi_k^S(\boldsymbol{g};\zeta) &= \sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \prod_{n\geqslant 1} \Big(-\frac{T_n^+}{1-g_n}\Big)^{r'_n} \Big(\frac{T_n^-}{1-g_n}\Big)^{r_n} \sum_{p\geqslant 0} p! \binom{r_n}{p}\binom{r'_n}{p}\Big(-\frac{n(1-g_n)}{T_n^+ T_n^-}\Big)^p \\ &= \sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \prod_{n\geqslant 1} \sum_{p=0}^\infty \frac{1}{p!}(-n(1-g_n))^p \partial^p_{T_n^+}\partial^p_{T_n^-}\Big(-\frac{T_n^+}{1-g_n}\Big)^{r'_n} \Big(\frac{T_n^-}{1-g_n}\Big)^{r_n} \\ &= \prod_{n\geqslant 1} \sum_{p=0}^\infty \frac{1}{p!}(-n(1-g_n))^p \partial^p_{T_n^+}\partial^p_{T_n^-} \sum_{\lambda,\lambda'\vdash k} \frac{1}{\rho_\lambda \rho_{\lambda'}} \prod_{n\geqslant 1}\Big(-\frac{T_n^+}{1-g_n}\Big)^{r'_n} \Big(\frac{T_n^-}{1-g_n}\Big)^{r_n} \\ &= \prod_{n\geqslant 1} \sum_{p=0}^\infty \frac{1}{p!}(-n(1-g_n))^p \partial^p_{T_n^+}\partial^p_{T_n^-} \exp\sum_{n\geqslant 1}\frac{1}{n}\Big(\varepsilon_-^n \frac{T_n^-}{1-g_n}\Big)\Big|_{\varepsilon_-^k} \exp\sum_{n\geqslant 1}\frac{1}{n}\Big(-\varepsilon_+^n \frac{T_n^+}{1-g_n}\Big)\Big|_{\varepsilon_+^k}.\end{aligned} \tag{C.16}$$

This may be written

$$\begin{aligned}\Phi_k^S(\boldsymbol{g};\zeta) &= \Big[\prod_{n\geqslant 1} \sum_{p=0}^\infty \frac{1}{p!}(-n(1-g_n))^p \partial^p_{T_n^+}\partial^p_{T_n^-} \exp\frac{1}{n(1-g_n)}\Big(\varepsilon_-^n T_n^- - \varepsilon_+^n T_n^+\Big)\Big]\Big|_{\varepsilon_+^k,\varepsilon_-^k} \\ &= \Big[\prod_{n\geqslant 1} \sum_{p=0}^\infty \frac{1}{p!}(-n(1-g_n))^p \Big(-\frac{\varepsilon_+^n}{n(1-g_n)}\frac{\varepsilon_-^n}{n(1-g_n)}\Big)^p \exp\frac{1}{n(1-g_n)}\Big(\varepsilon_-^n T_n^- - \varepsilon_+^n T_n^+\Big)\Big]\Big|_{\varepsilon_+^k,\varepsilon_-^k} \\ &= \Big[\prod_{n\geqslant 1} \exp\frac{1}{n(1-g_n)}\Big(\varepsilon_+^n\varepsilon_-^n + \varepsilon_-^n T_n^- - \varepsilon_+^n T_n^+\Big)\Big]\Big|_{\varepsilon_+^k,\varepsilon_-^k}.\end{aligned} \tag{C.17}$$

Adding and subtracting $\frac{1}{1-g_n} = 1 + \frac{g_n}{1-g_n}$, evaluating the plethystic of the part independent on g_n, and finally rescaling $\varepsilon_\pm \to z^{\pm 1}\varepsilon_\pm$, we get

$$\begin{aligned}\Phi_k^S(\boldsymbol{g};\zeta) &= \Big[\frac{1}{1-\varepsilon_+\varepsilon_-}\frac{1-\varepsilon_+}{1-\varepsilon_-}\frac{1-\zeta\varepsilon_-}{1-\zeta^{-1}\varepsilon_+} \prod_{n\geqslant 1} \exp\frac{g_n}{n(1-g_n)}\Big(\varepsilon_+^n\varepsilon_-^n + \varepsilon_-^n z^{-n} T_n^- - \varepsilon_+^n z^n\, T_n^+\Big)\Big]_{\varepsilon_+^k,\varepsilon_-^k} \\ &= \Big[\frac{1}{1-\varepsilon_+\varepsilon_-}\frac{1-\varepsilon_+}{1-\varepsilon_-}\frac{1-\zeta\varepsilon_-}{1-\zeta^{-1}\varepsilon_+} \prod_{n\geqslant 1} \exp\frac{g_n}{n(1-g_n)}\Big(\varepsilon_+^n\varepsilon_-^n + \varepsilon_-^n(1-\zeta^n) - \varepsilon_+^n(1-\zeta^{-n})\Big)\Big]_{\varepsilon_+^k,\varepsilon_-^k},\end{aligned} \tag{C.18}$$

which is formula (3.6) used in the text. Notice that in free limit $\boldsymbol{g} = 0$ one has

[12] From $\sum_{s\in\mathbb{Z}+\frac{1}{2}} \zeta^{-s} H_s = f(\zeta)$, we get $\sum_{N<s\, s\in\mathbb{Z}+\frac{1}{2}} H_s = \sum_{N<s\, s\in\mathbb{Z}+\frac{1}{2}} \oint d\zeta\, \zeta^{s-1} f(\zeta) = \oint d\zeta\, f(\zeta) \sum_{n=0}^\infty \zeta^{N+\frac{1}{2}+n-1} = \int d\zeta \zeta^{N-1} \frac{\sqrt{\zeta}}{1-\zeta} f(\zeta) = \frac{\sqrt{\zeta}}{1-\zeta} f(\zeta)\Big|_{\zeta^{-N}}$, and an extra $\frac{\sqrt{\zeta}}{1-\zeta}$ comes from $\sqrt{wz}/(z-w)$ in (C.1).

$$\Phi_k^S(0;\zeta) = \left[\frac{1}{1-\varepsilon_+\varepsilon_-}\frac{1-\varepsilon_+}{1-\varepsilon_-}\frac{1-\zeta\varepsilon_-}{1-\zeta^{-1}\varepsilon_+}\right]_{\varepsilon_+^k,\varepsilon_-^k} = \frac{1}{\zeta^k} - \frac{1}{\zeta^{k-1}} + \zeta\,, \tag{C.19}$$

where we did contour integration, *cf.* the first line in (3.18).

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