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Projective geometry of homogeneous second-order Hamiltonian operators

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Abstract

We prove the invariance of homogeneous second-order Hamiltonian operators under the action of projective reciprocal transformations. We establish a correspondence between such operators in dimension n and 3-forms in dimension $n+1$. In this way we classify second-order Hamiltonian operators using the known classification of 3-forms in dimensions $\leqslant 9$. As a by-product, we identify such operators as linear line congruences, that are distinguished algebraic varieties in Plücker's space of lines. Systems of first-order conservation laws that are Hamiltonian with respect to such operators are also explicitly found. The geometry and integrability of the systems is discussed in detail.

Keywords: Hamiltonian formalism for PDEs, homogeneous operators, integrable systems, projective transformations, linear line complexes

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1. Introduction

Hamiltonian operators play a fundamental role in the theory of integrability of partial differential equations (PDEs). A distinguished class of Hamiltonian operators was introduced in 1983 by Dubrovin and Novikov [10]. One of the main features of the new class was that it was invariant under diffeomorphisms of the underlying space, thus bringing geometry into the theory of integrable PDEs.

More precisely, let us denote by $u^i = u^i(t,x)$ n unknown functions of two independent variables t and x, $i = 1, \ldots, n$, and denote by u^i_σ the x-derivative of u^i σ times. An element of the above class of operators has the form of a matrix of differential operators where each summand contains the same number of x-derivatives; for example, in the first-order case:

$$P^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x. \tag{1}$$

Then, a PDE of the form $u^i_t = f^i(u^j, u^j_\sigma)$ is Hamiltonian if there exists a density $H = \int h(u^j)\mathrm{d}x$ such that

$$u^i_t = f^i = P^{ij}\frac{\delta H}{\delta u^j}. \tag{2}$$

The typical situation is when $f^i = V^i_j(u^k)u^j_x$, i.e. the system of PDEs is quasilinear and of the first order. In that case, if the system of PDEs is Hamiltonian, it is straightforward to realize that both the system and its Hamiltonian formulation are form-invariant with respect to a transformation involving the dependent variables only: $\bar{u}^i = \bar{u}^i(u^j)$. We recall that the Hamiltonian property of P is the fact that it induces a Poisson bracket on the space of densities:

$$\{F,G\}_P = \int \frac{\delta f}{\delta u^i}P^{ij}\frac{\delta g}{\delta u^j}\mathrm{d}x, \tag{3}$$

where $F = \int f\mathrm{d}x$ and $G = \int g\,\mathrm{d}x$. The Poisson bracket property is also invariant with respect to the above transformations.

The first order case was generalized to the higher order case in [9], the homogeneity degree being equal to the order of the operators. Numerous examples show that homogeneous Hamiltonian operators (HHOs) are ubiquitous, either as stand-alone operators or in linear combinations with operators of different homogeneity degrees. See [22] for many examples of the latter kind.

Recently, it was observed that third-order homogeneous Hamiltonian operators are invariant (when in their canonical form, see [13, equation (2)]) with respect to a non-obvious class of transformations, namely projective reciprocal transformations [13]. They have the form of a projective transformation of dependent variables coupled with a non-local transformation of x:

$$\tilde{u}^i = \frac{A^i_j u^j + A^i_{n+1}}{A^{n+1}_j u^j + A^{n+1}_{n+1}}, \qquad \mathrm{d}\tilde{x} = (A^{n+1}_j u^j + A^{n+1}_{n+1})\mathrm{d}x. \tag{4}$$

Note that the denominator of the projective transformation also defines the non-local part of the transformation.

Third-order homogeneous Hamiltonian operators have been classified in low dimensions ($n \leqslant 4$, [13, 14]) with respect to the action of the above group. It was also proved that there exists a multiparametric family of quasilinear systems of first-order PDEs that are Hamiltonian with respect to any such operator [15].

The goal of the current paper is to prove that second-order homogeneous Hamiltonian operators have the same invariance properties of third-order homogeneous Hamiltonian operators.

The interest in such a result is that projective-geometric invariance is not just an 'isolated' feature of third-order operators: being also a property of second-order operators it is reasonable to think that it is something bound to all higher-order homogeneous operators.

A classification then follows from the above invariance result; the algebraic variety that is identified with a second-order operator has been extensively studied in [26] (linear line congruence defined by a three-form), and has a different nature with respect to the algebraic variety defined by third-order operators (quadratic line complex). We also obtain similar (but not identical!) results concerning associated systems of quasilinear first-order PDEs.

More precisely, second-order homogeneous Hamiltonian operators have the general form

$$P^{ij} = g^{ij}\partial_x^2 + b^{ij}_k u^k_x \partial_x + c^{ij}_k u^k_{xx} + c^{ij}_{kh} u^k_x u^h_x. \tag{5}$$

We will always consider the non-degenerate case $\det(g^{ij}) \neq 0$. Under a coordinate transformation of the type $\bar{u}^i = \bar{u}^i(u^j)$ the symbols $\Gamma^k_{ij} = -g_{ip}c^{pk}_j$ transform as a linear connection. It is proved in [7, 27] (but see also [16, 24]) that the Hamiltonian property of the above operator implies that Γ^k_{ij} is symmetric and flat. With respect to flat coordinates the operator can be rewritten as

$$P^{ij} = \partial_x g^{ij} \partial_x. \tag{6}$$

The Hamiltonian property in flat coordinates is then equivalent to the fact that

$$g_{ij} = T_{ijk}u^k + g^0_{ij}, \tag{7}$$

where T_{ijk} are constant and skew-symmetric with respect to i, j, k and g^0_{ij} is constant and skew-symmetric with respect to i, j. The above equations (6) and (7) have been independently found in [7, 27]. See also [24] for a thorough review on homogeneous Hamiltonian operators, and see [16] for a further differential-geometric analysis of the properties of homogeneous second-order Hamiltonian operators and their pencils.

Here, we will prove the following theorem (theorem 6 on page 7).

Theorem 1. *Second-order homogeneous Hamiltonian operators are invariant under projective reciprocal transformations* (4).

Then, we will prove a result that enables us to classify second-order homogeneous Hamiltonian operators in low dimensions ($n \leqslant 8$) (theorem 8 on page 9).

Theorem 2. *Second-order homogeneous Hamiltonian operators in dimension n can be put in bijection with 3-forms in the $n+1$-dimensional space $\mathbb{C}^{n+1}$.*

A projective reciprocal transformation induces an $SL(n+1)$-transformation on the corresponding 3-form which commutes with the action on the corresponding second-order homogeneous Hamiltonian operator.

It is interesting to observe that, in a generic situation, 3-forms define *linear line congruences* (see [26] for the algebraic geometric description and properties of that correspondence), hence second-order homogeneous Hamiltonian operators are in correspondence with algebraic varieties, as it happened in the third-order case (for different algebraic varieties, i.e. quadratic line complexes).

Let us consider a quasilinear first-order system of PDEs in conservative form

$$u^i_t = (V^i)_x, \tag{8}$$

where $V^i = V^i(u^j)$. We will prove the following Theorem (which is the union of the statements of theorems 11, 19, propositions 15, 18, 20 and corollary 21).

Theorem 3. *Every second-order homogeneous Hamiltonian operator is the Hamiltonian operator of a multiparameter family of systems of conservation laws as in (8). The systems possess a non-local Hamiltonian and have identically vanishing Haantjes tensor. In the generic case, the systems have distinct eigenvalues of multiplicity two and are linearly degenerate, diagonalizable and semi-Hamiltonian.*

Strictly speaking, we do not have a general method to integrate our systems, as the generalized hodograph method [31] can be used in the case of distinct eigenvalues. See the discussion at the end of subsection 3.4. Moreover, we (still) did not prove the existence of Lax pairs or bi-Hamiltonian pairs or other integrability structures for our systems. However, since our systems have so many 'good' geometric properties, and are semi-Hamiltonian in the generic situation we can conjecture that they are 'integrable' in one of the ways that are accepted by the scientific community. We will devote ourselves to this task in a future research work.

We would like to stress that homogeneous Hamiltonian operators are important building blocks in the theory of integrable systems. We can mention several ways in which they are involved:

- Many bi-Hamiltonian systems have a bi-Hamiltonian pair of the form $P = P_1 + R$ and $Q = Q_1$, where P_1, Q_1 are compatible homogeneous first-order Hamiltonian operators and R is a homogeneous second-order or third-order Hamiltonian operator which is compatible with P_1 and Q_1. We call such systems *bi-Hamiltonian systems of KdV type* [22]. Examples include the AKNS (or two-boson) hierarchy, the two-component Camassa–Holm hierarchy [11], a multiparameter family defined in [29] (see [22]) and the Kaup–Broer system [21] when

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_x^2. \tag{9}$$

  Other examples with R a third-order HHO are the KdV equation, the Camassa–Holm equation [22], a dispersive water waves equation [5] and a coupled Harry–Dym hierarchy [4]. See also the recent papers [6, 20], with a differential-geometric focus on the same construction.
- Fewer systems are determined by a pair of Hamiltonian operators of the form $P = P_1$ and R; here we mention the WDVV systems [32], where R is of the third order. No instances of systems that we determined in this paper were previously known to our knowledge. A probable explanation is that the first non-trivial systems (although linearizable) appear in dimension 4, and non-linearizable ones in dimension 6 and greater, and that makes their investigation quite complicated.
- Homogeneous operators play a central role in Dubrovin–Zhang's perturbative approach to the classification of integrable systems under the action of the group of Miura transformations [8]. Since deformations of a first-order Poisson pencil are given as a formal series of homogeneous operators, one might expect that projective transformations and invariance can play a role. See the dedicated section in [23].

The results obtained so far show that *the group of projective reciprocal transformations act on hierarchies defined by trios of compatible operators P_1, Q_1, R or by pairs of compatible operators P_1, R*. The action preserves the locality of second-order or third-order HHOs in canonical form, even if it does not preserve the locality of P_1, Q_1. So, a projective geometric study of the above hierarchies makes sense and is potentially interesting.

The chances that the projective invariance properties that are shared by second-order and third-order HHOs might be generalized to HHOs of arbitrary order are high enough to consider that possibility in the framework of the perturbative approach in [8].

More generally, our results might indicate that a projective-geometric approach to integrable systems is starting to emerge in the field. We will pursue that research line in the future.

2. Projective geometry and Hamiltonian operators

Let us consider a projective transformation $T\colon \mathbb{P}^n \to \mathbb{P}^n$. We will treat (u^i) as an affine chart of the homogeneous coordinates $[\mathbf{v}] = [v^1, \dots, v^{n+1}]$, where $u^i = v^i/v^{n+1}$. In homogeneous coordinates we have $T(v) = [a^\lambda_\mu v^\mu]$, where $(a^\lambda_\mu) \in SL(n+1)$. In this Section latin indices i, j, … will run from 1 to n and greek indices λ, μ, … will run from 1 to $n+1$. A projective transformation in the affine chart has the form:

$$\tilde{u}^i = T^i(u^j) = \frac{a^i_j u^j + a^i_{n+1}}{a^{n+1}_j u^j + a^{n+1}_{n+1}}. \tag{10}$$

In this section we will calculate the action of a projective transformation on a second-order homogeneous Hamiltonian operator. We will realize that the transformation (10) alone is not enough to yield invariance, while reciprocal projective transformations guarantee the invariance of the form (6) of our operators.

2.1. Projective invariance of the Hamiltonian operators

We would like to find the change of coordinates formula on the leading coefficient g of a second-order homogeneous Hamiltonian operator; in other words, we are looking for a formula connecting[5]

$$g = (T_{ijk}u^k + g^0_{ij})\mathrm{d}u^i \wedge \mathrm{d}u^j \quad \text{and} \quad \tilde{g} = (\tilde{T}_{ijk}\tilde{u}^k + \tilde{g}^0_{ij})\mathrm{d}\tilde{u}^i \wedge \mathrm{d}\tilde{u}^j. \tag{11}$$

Note that we will work with the covariant version of the leading coefficient; this is possible due to our assumption $\det(g) \neq 0$. As a preliminary remark, note that

$$d\left(\tilde{u}^i\right) = d\left(\frac{a^i_s u^s + a^i_{n+1}}{a^{n+1}_s u^s + a^{n+1}_{n+1}}\right) = \frac{A a^i_s \mathrm{d}u^s - (a^i_s u^s + a^i_{n+1}) a^{n+1}_l \mathrm{d}u^l}{A^2} \tag{12}$$

where $A = a^{n+1}_s u^s + a^{n+1}_{n+1}$.

Theorem 4. *Under the transformation* (10) *we obtain*

$$\begin{aligned} T_{lcs} = \frac{1}{2A^3}\Big(&\tilde{T}_{ijk}(a^i_l a^j_c - a^i_c a^j_l)a^k_s + \tilde{g}^0_{ij}(a^i_l a^j_c - a^i_c a^j_l)a^{n+1}_s \\ &- \tilde{g}^0_{ij}(a^i_l a^{n+1}_c - a^i_c a^{n+1}_l)a^j_s - \tilde{g}^0_{ij}(a^{n+1}_l a^j_c - a^{n+1}_c a^j_l)a^i_s\Big) \end{aligned} \tag{13a}$$

$$\begin{aligned} g^0_{lc} = \frac{1}{2A^3}\Big(&\tilde{T}_{ijk}(a^i_l a^j_c - a^i_c a^j_l)a^k_{n+1} + \tilde{g}^0_{ij}(a^i_l a^j_c - a^i_c a^j_l)a^{n+1}_{n+1} \\ &- \tilde{g}^0_{ij}(a^i_l a^{n+1}_c - a^i_c a^{n+1}_l)a^j_{n+1} - \tilde{g}^0_{ij}(a^j_c a^{n+1}_l - a^j_l a^{n+1}_c)a^i_{n+1}\Big). \end{aligned} \tag{13b}$$

[5] We use Einstein's summation convention throughout our paper.

Proof. Applying the transformation to $\tilde{g}_{ij}\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j=\tilde{T}_{ijk}\tilde{u}^k\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j+\tilde{g}^0_{ij}\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j$ we obtain

$$\begin{aligned}
\tilde{g}^0_{ij}\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j &= \tilde{g}^0_{ij}\left(\frac{Aa^i_s\mathrm{d}u^s-(a^i_su^s+a^i_{n+1})a^{n+1}_l\mathrm{d}u^l}{A^2}\right)\wedge\left(\frac{Aa^j_s\mathrm{d}u^s-(a^j_su^s+a^j_{n+1})a^{n+1}_l\mathrm{d}u^l}{A^2}\right)\\
&=\frac{\tilde{g}^0_{ij}}{A^4}\big(A^2a^i_sa^j_l\mathrm{d}u^s\wedge\mathrm{d}u^l-Aa^i_s(a^j_bu^b+a^j_{n+1})a^{n+1}_c\mathrm{d}u^s\wedge\mathrm{d}u^c\\
&\quad -A(a^i_mu^m+a^i_{n+1})a^{n+1}_la^j_s\mathrm{d}u^l\wedge\mathrm{d}u^s\\
&\quad +(a^i_mu^m+a^i_{n+1})a^{n+1}_l(a^j_bu^b+a^j_{n+1})a^{n+1}_c\mathrm{d}u^l\wedge\mathrm{d}u^c\big)\\
&=\frac{\tilde{g}^0_{ij}}{A^4}\big(A^2a^i_sa^j_l\mathrm{d}u^s\wedge\mathrm{d}u^l-Aa^i_s(a^j_bu^b+a^j_{n+1})a^{n+1}_c\mathrm{d}u^s\wedge\mathrm{d}u^c\\
&\quad -A(a^i_mu^m+a^i_{n+1})a^{n+1}_la^j_s\mathrm{d}u^l\wedge\mathrm{d}u^s\big)\\
&=\frac{\tilde{g}^0_{ij}}{A^3}\big(Aa^i_sa^j_l\mathrm{d}u^s\wedge\mathrm{d}u^l-a^i_s(a^j_bu^b+a^j_{n+1})a^{n+1}_c\mathrm{d}u^s\wedge\mathrm{d}u^c\\
&\quad -(a^i_mu^m+a^i_{n+1})a^{n+1}_la^j_s\mathrm{d}u^l\wedge\mathrm{d}u^s\big).
\end{aligned}$$

Analogously,

$$\begin{aligned}
\tilde{T}_{ijk}\tilde{u}^k\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j &= \tilde{T}_{ijk}\frac{a^i_su^s+a^i_{n+1}}{A^5}A^2a^i_la^j_c\mathrm{d}u^l\wedge\mathrm{d}u^c\\
&=\frac{\tilde{T}_{ijk}}{A^3}(a^k_su^s+a^k_{n+1})a^i_la^j_c\mathrm{d}u^l\wedge\mathrm{d}u^c,
\end{aligned}$$

where three terms cancel due to the skew-symmetry of $\tilde{T}_{ijk}$. We finally obtain

$$\begin{aligned}
\tilde{g}_{ij}\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j=\frac{1}{A^3}\Big[&\tilde{T}_{ijk}(a^k_su^s+a^k_{n+1})a^i_la^j_c+\tilde{g}^0_{ij}a^i_l(a^{n+1}_su^s+a^{n+1}_{n+1})a^j_c\\
&-\tilde{g}^0_{ij}a^i_l(a^j_su^s+a^j_{n+1})a^{n+1}_c-\tilde{g}^0_{ij}a^j_c(a^i_su^s+a^i_{n+1})a^{n+1}_l\Big]\mathrm{d}u^l\wedge\mathrm{d}u^c. \qquad (14)
\end{aligned}$$

Collecting u^s, and comparing the left-hand side with $g_{lc}\mathrm{d}u^l\wedge\mathrm{d}u^c$ with respect to a basis (i.e. $l<c$) we obtain the change of coordinates formula (13) □

Corollary 5. *The indexed families T_{lcs} and g^0_{lc} as obtained from $\tilde{T}_{ijk}$ and $\tilde{g}^0_{ij}$ by means of the above transformation are skew-symmetric with respect to all of their indices. Hence, a projective transformation of the leading coefficient of a second-order HHO preserves its form up to a conformal factor:*

$$\tilde{g}_{ij}\mathrm{d}\tilde{u}^i\wedge\mathrm{d}\tilde{u}^j=\frac{1}{A^3}g_{lc}\mathrm{d}u^l\wedge\mathrm{d}u^c. \qquad (15)$$

Proof. The skew-symmetry of g^0_{lc} is evident, and it is easy to show that $T_{lcs}=-T_{lsc}$ by observing that the skew-symmetry holds separately in the summand $\tilde{T}_{ijk}(a^i_la^j_c-a^i_ca^j_l)a^k_s$ and in the remaining three summands. □

We recall that a reciprocal transformation is a nonlocal change of the independent variables t, x defined as

$$\mathrm{d}\tilde{t}=B(u^j)\mathrm{d}t,\qquad \mathrm{d}\tilde{x}=A(u^j)\mathrm{d}x \qquad (16)$$

where $A(u^j), B(u^j)$ are functions depending on (u^j). Projective reciprocal transformation were introduced in [13] as invariance transformations for the canonical form of third-order HHOs. They are reciprocal transformations of the form

$$\mathrm{d}\tilde{t} = \mathrm{d}t, \qquad \mathrm{d}\tilde{x} = A\,\mathrm{d}x = (a^{n+1}_k u^k + a^{n+1}_{n+1})\mathrm{d}x \tag{17}$$

coupled with a projective transformation T as in (10). We are going to prove that projective reciprocal transformations preserve the canonical form (6) of second-order HHOs. The proof follows the lines of the proof of the analogous result for third-order HHOs [13].

Theorem 6. *Projective reciprocal transformations preserve the canonical form* (6) *of second-order HHOs.*

Proof. It is enough to prove the result for a transformation of the type $\tilde{u}^i = u^i/A$, where $A = a^{n+1}_k u^k + a^{n+1}_{n+1}$. It is easy to see that $\int u^i \mathrm{d}x$ transform as $\int \frac{u^i}{A}\mathrm{d}\tilde{x} = \int \tilde{u}^i \mathrm{d}\tilde{x}$, and, more generally, two densities $F = \int f(u)\mathrm{d}x$ and $H = \int h(u)\mathrm{d}x$ transform as $f = A\tilde{f}$ and $h = A\tilde{h}$. Moreover, we have:

$$f_j = \frac{\partial f}{\partial u^j} = \frac{\partial A}{\partial u^j}\tilde{f} + A\frac{\partial \tilde{f}}{\partial u^j} = a^{n+1}_j \tilde{f} + A\tilde{f}_j \tag{18}$$

and analogously $h_j = a^{n+1}_j \tilde{h} + A\tilde{h}_j$, then:

$$\{F,H\} = \int f_i P^{ij} h_j \mathrm{d}x = \int (a^{n+1}_i \tilde{f} + A\tilde{f}_i) A\partial_{\tilde{x}} \left(g^{ij} A \partial_{\tilde{x}} (a^{n+1}_j \tilde{h} + A\tilde{h}_j) \right) \frac{1}{A}\mathrm{d}\tilde{x}, \tag{19}$$

where we used $\partial_x = A\partial_{\tilde{x}}$. We can cancel A once and obtain a new second-order HHO with leading term $A^3 g^{ij}$. Let us first observe that $a^{n+1}_{n+1} + a^{n+1}_k u^k = A = a^{n+1}_{n+1}\frac{1}{1 - a^{n+1}_l \tilde{u}^l}$. Then, we have

$$\frac{\partial \tilde{u}^i}{\partial u^j} = \frac{\delta^i_j A - a^{n+1}_j u^i}{A^2} = \frac{\delta^i_j - a^{n+1}_i \tilde{u}^j}{A} \tag{20}$$

$$A\frac{\partial \tilde{f}}{\partial u^j} = A\frac{\partial \tilde{u}^k}{\partial u^j}\frac{\partial \tilde{f}}{\partial \tilde{u}^k} = (\delta^k_j - a^{n+1}_j \tilde{u}^k)\frac{\partial \tilde{f}}{\partial \tilde{u}^k}. \tag{21}$$

Now, let us consider again the bracket in (19) and carry out the coordinate change:

$$\begin{aligned}\{F,H\} &= \int \left(A\frac{\partial \tilde{f}}{\partial u^i} + a^{n+1}_i \tilde{f} \right) \partial_{\tilde{x}} \left(g^{ij} A \partial_{\tilde{x}} \left(A\frac{\partial \tilde{h}}{\partial u^j} + a^{n+1}_j \tilde{h} \right) \right) \mathrm{d}\tilde{x} \\ &= \int \left((\delta^k_i - a^{n+1}_i \tilde{u}^k)\frac{\partial \tilde{f}}{\partial \tilde{u}^k} + a^{n+1}_i \tilde{f} \right) \partial_{\tilde{x}} \left(g^{ij} A \partial_{\tilde{x}} \left((\delta^l_j - a^{n+1}_j \tilde{u}^l)\frac{\partial \tilde{h}}{\partial \tilde{u}^l} + a^{n+1}_j \tilde{h} \right) \right) \mathrm{d}\tilde{x}.\end{aligned}$$

Using the identity:

$$\partial_{\tilde{x}} \left((\delta^l_j - a^{n+1}_j \tilde{u}^l)\frac{\partial \tilde{h}}{\partial \tilde{u}^l} + a^{n+1}_j \tilde{h} \right) = (\delta^l_j - a^{n+1}_j \tilde{u}^l)\partial_{\tilde{x}}\frac{\partial \tilde{h}}{\partial \tilde{u}^l}$$

we obtain

$$\begin{aligned}\{F,H\} &= \int \left((\delta_i^k - a_i^{n+1}\tilde{u}^k)\frac{\partial \tilde{f}}{\partial \tilde{u}^k} + a_i^{n+1}\tilde{f} \right) \partial_{\tilde{x}} \left(g^{ij}A(\delta_j^l - a_j^{n+1}\tilde{u}^l)\partial_{\tilde{x}} \frac{\partial \tilde{h}}{\partial \tilde{u}^l} \right) \mathrm{d}\tilde{x} \\ &= \int \left((\delta_i^k - a_i^{n+1}\tilde{u}^k)\frac{\partial \tilde{f}}{\partial \tilde{u}^k} \right) \partial_{\tilde{x}} \left(g^{ij}A(\delta_j^l - a_j^{n+1}\tilde{u}^l)\partial_{\tilde{x}} \frac{\partial \tilde{h}}{\partial \tilde{u}^l} \right) \mathrm{d}\tilde{x} \\ &\quad - \int a_i^{n+1}\partial_{\tilde{x}}\tilde{f} \cdot \left(g^{ij}A(\delta_j^l - a_j^{n+1}\tilde{u}^l)\partial_{\tilde{x}} \frac{\partial \tilde{h}}{\partial \tilde{u}^l} \right) \mathrm{d}\tilde{x}.\end{aligned}$$

Finally, observing that $\partial_{\tilde{x}}\tilde{f} = \tilde{f}_{,m}\tilde{u}^m_{\tilde{x}}$ and by using the identity

$$(\delta_i^k - a_i^{n+1}\tilde{u}^k)\frac{\partial \tilde{f}}{\partial \tilde{u}^k}\partial_{\tilde{x}} - a_i^{n+1}\frac{\partial \tilde{f}}{\partial \tilde{u}^k}\tilde{u}^k_{\tilde{x}} = \frac{\partial \tilde{f}}{\partial \tilde{u}^k}\partial_{\tilde{x}}(\delta_i^k - a_i^{n+1}\tilde{u}^k)$$

we have

$$\{F,H\} = \int \frac{\partial \tilde{f}}{\partial \tilde{u}^k}\tilde{P}^{kl}\frac{\partial \tilde{h}}{\partial \tilde{u}^l}\mathrm{d}\tilde{x} \tag{22}$$

with

$$\tilde{P}^{kl} = \partial_{\tilde{x}}(\delta_i^k - a_i^{n+1}\tilde{u}^k)g^{ij}A(\delta_j^l - a_j^{n+1}\tilde{u}^l)\partial_{\tilde{x}} = \partial_{\tilde{x}}\tilde{g}^{ij}\partial_{\tilde{x}}. \tag{23}$$

where $\tilde{P}$ is again a local homogeneous operator of second order in view of corollary 5.

□

Remark 1. Recent results show that the above arguments can be generalized to prove that all operators of the form $\partial_x \circ Q \circ \partial_x$ are invariant under projective reciprocal transformations, see [23].

2.2. Projective interpretation of the Hamiltonian operators

The action of the projective group on second-order HHOs allows us to classify such operators. Indeed, we exhibit a bijective correspondence of the leading term of the operator (in dimension n) with a projective 3-form (in dimension $n+1$). Such geometric objects are well-known in algebraic geometry [26] and there exist a classification in dimensions up to $n+1=9$. Of course, we are interested in the even cases $n = 2, 4, 6, 8$ due to the assumption $\det(g) \neq 0$.

Let us set

$$T_{n+1\,jk} = -T_{j\,n+1\,k} = T_{jk\,n+1} = g^0_{jk}. \tag{24}$$

Then, we have a skew-symmetric indexed family $T_{\lambda\mu\nu}$ with (greek) indices running from 1 to $n+1$, extending T_{ijk} (recall that latin indices run from 1 to n). We have the following statement.

Lemma 7. *A projective reciprocal transformation induces the transformation*

$$T_{\lambda\mu\nu} = \frac{1}{A^3}\tilde{T}_{\alpha\beta\gamma}a^\alpha_\lambda a^\beta_\mu a^\gamma_\nu. \tag{25}$$

Thus, $T_{\lambda\mu\nu}$ transforms as a tensor in $\mathbb{C}^{n+1}$ up to a conformal factor.

Proof. It follows from

$$
\begin{aligned}
T_{lcs} &= \frac{1}{2A^3}\Big(\tilde{T}_{ijk}(a^i_l a^j_c - a^i_c a^j_l)a^k_s + \tilde{T}_{ij\,n+1}(a^i_l a^j_c - a^i_c a^j_l)a^{n+1}_s \\
&\quad - \tilde{T}_{ij\,n+1}(a^i_l a^{n+1}_c - a^i_c a^{n+1}_l)a^j_s - \tilde{T}_{ij\,n+1}(a^{n+1}_l a^j_c - a^{n+1}_c a^j_l)a^i_s\Big) \\
&= \frac{1}{2A^3}\Big(\tilde{T}_{ij\nu}(a^i_l a^j_c - a^i_c a^j_l)a^\nu_s + \tilde{T}_{i\,n+1\,k}(a^i_l a^{n+1}_c - a^i_c a^{n+1}_l)a^k_s \\
&\quad + \tilde{T}_{n+1\,jk}(a^{n+1}_l a^j_c - a^{n+1}_c a^j_l)a^k_s\Big) \\
&= \frac{1}{2A^3}\tilde{T}_{\lambda\mu\nu}(a^\lambda_l a^\mu_c - a^\lambda_c a^\mu_l)a^\nu_s \\
&= \frac{1}{A^3}\tilde{T}_{\lambda\mu\nu}a^\lambda_l a^\mu_c a^\nu_s.
\end{aligned}
$$

A similar proof holds for $T_{lc\,n+1} = g^0_{lc}$. □

In what follows we will identify three-forms $\omega \in \wedge^3(\mathbb{C}^{n+1})^*$ on a vector space $\mathbb{C}^{n+1}$ with maps of the form (see also [26] for more details)

$$
i(\omega)\colon \mathbb{C}^{n+1} \to \wedge^2(\mathbb{C}^{n+1})^*, v \mapsto \frac{1}{3}i_v(\omega). \tag{26}
$$

Clearly, the map $\omega \mapsto i(\omega)$ is an isomorphism onto its image. If (v^i) are coordinates on $\mathbb{C}^{n+1}$, then $(\mathrm{d}v^i)$ is a basis of $(\mathbb{C}^{n+1})^*$ and the above isomorphism reads as

$$
\omega_{\lambda\mu\nu}\mathrm{d}v^\lambda \wedge \mathrm{d}v^\mu \wedge \mathrm{d}v^\nu \mapsto \omega_{\lambda\mu\nu}v^\lambda \mathrm{d}v^\mu \wedge \mathrm{d}v^\nu. \tag{27}
$$

Theorem 8. *There is a bijective correspondence between leading coefficients of second-order HHOs $g = (T_{ijk}u^k + g^0_{ij})\mathrm{d}u^i \wedge \mathrm{d}u^j$ as in (7), and three-forms $\omega = \omega_{\lambda\mu\nu}\mathrm{d}v^\lambda \wedge \mathrm{d}v^\mu \wedge \mathrm{d}v^\nu$. The bijective correspondence is preserved by projective reciprocal transformations up to a conformal factor.*

Proof. Let us consider a three-form $\omega = \omega_{\lambda\mu\nu}\mathrm{d}v^\lambda \wedge \mathrm{d}v^\mu \wedge \mathrm{d}v^\nu$. Using the isomorphism (27) we can rewrite the form as

$$
\begin{aligned}
i(\omega) &= \omega_{\lambda\mu\nu}v^\nu \mathrm{d}v^\lambda \wedge \mathrm{d}v^\mu \\
&= \omega_{i\mu\nu}v^\nu \mathrm{d}v^i \wedge \mathrm{d}v^\mu + \omega_{n+1\,\mu\nu}v^\nu \mathrm{d}v^{n+1} \wedge \mathrm{d}v^\mu \\
&\quad + \omega_{\lambda i\nu}v^\nu \mathrm{d}v^\lambda \wedge \mathrm{d}v^i + \omega_{\lambda\,n+1\,\nu}v^\nu \mathrm{d}v^\lambda \wedge \mathrm{d}v^{n+1} \\
&\quad + \omega_{\lambda\mu i}v^i \mathrm{d}v^\lambda \wedge \mathrm{d}v^\mu + \omega_{\lambda\mu\,n+1}v^{n+1}\mathrm{d}v^\lambda \wedge \mathrm{d}v^\mu \\
&= \omega_{ij\nu}v^\nu \mathrm{d}v^i \wedge \mathrm{d}v^j + \omega_{i\,n+1\,j}v^j \mathrm{d}v^i \wedge \mathrm{d}v^{n+1} + \omega_{n+1\,ij}v^j \mathrm{d}v^{n+1} \wedge \mathrm{d}v^i \\
&\quad + \omega_{ij\nu}v^\nu \mathrm{d}v^i \wedge \mathrm{d}v^j + \omega_{n+1\,ij}v^j \mathrm{d}v^{n+1} \wedge \mathrm{d}v^i + \omega_{i\,n+1\,j}v^j \mathrm{d}v^i \wedge \mathrm{d}v^{n+1} \\
&\quad + \omega_{j\mu i}v^i \mathrm{d}v^j \wedge \mathrm{d}v^\mu + \omega_{n+1\,ij}v^j \mathrm{d}v^{n+1} \wedge \mathrm{d}v^i + \omega_{ij\,n+1}v^{n+1}\mathrm{d}v^i \wedge \mathrm{d}v^j \\
&= 3\omega_{ijk}v^k \mathrm{d}v^i \wedge \mathrm{d}v^j + 3\omega_{ij\,n+1}v^{n+1}\mathrm{d}v^i \wedge \mathrm{d}v^j + 6\omega_{ij\,n+1}v^i \mathrm{d}v^j \wedge \mathrm{d}v^{n+1}.
\end{aligned}
$$

Using the affine chart restriction $v^{n+1} = 1, \mathrm{d}v^{n+1} = 0$ we obtain a second-order HHO by setting

$$
T_{ijk} = 3\omega_{ijk} \quad \text{and} \quad g^0_{ij} = T_{ij\,n+1} = 3\omega_{ij\,n+1}. \tag{28}
$$

On the other hand, from a second-order HHO g as in the statement one can define the form in homogeneous coordinates

$$
G = (T_{ijk}v^k + g^0_{ij}v^{n+1})\mathrm{d}v^i \wedge \mathrm{d}v^j. \tag{29}
$$

Reversing the steps of the first part of the proof we get the desired three-form ω.

The fact that the correspondence is preserved by projective reciprocal transformation up to the conformal factor $1/A^3$ follows from lemma 7.

□

There is an immediate and important consequence of the above Theorem.

Corollary 9. *There is a bijective correspondence between homogeneous second-order Hamiltonian operators in dimension n and three-forms in dimension $n+1$. The bijective correspondence is preserved by projective reciprocal transformations.*

At this point we observe two important facts:

- from a *geometric* viewpoint, second-order HHOs yield algebraic varieties using the corresponding three-forms and the mechanism explained in [26].
- from an *algebraic* viewpoint, second-order HHOs can be classified under the action of the projective reciprocal transformations by means of the classification of three-forms under the action of $SL(n+1,\mathbb{C})$.

Let us first summarize the main features of the geometric properties of second-order HHOs. Our main source is [26]. Let ω be a three-form as above. A line L in $\mathbb{C}^{n+1}$ can be identified with the skew-symmetric tensor $L = p^{\lambda\mu}\frac{\partial}{\partial v^\lambda}\wedge\frac{\partial}{\partial v^\mu}$, where $(p^{\lambda\mu})$ are the Plücker coordinates. The system

$$i_L\omega = 0 \Leftrightarrow \omega_{\lambda\mu\nu}p^{\mu\nu} = 0 \tag{30}$$

is a system of $n+1$ linear equations whose solutions constitute a linear subspace $\Lambda_\omega \subset \mathbb{P}(\wedge^2\mathbb{C}^{n+1})$. If ω is a generic 3-form, then the intersection of Λ_ω with the Grassmannian $\mathbb{G}$, $X_\omega = \mathbb{G}\cap\Lambda_\omega$ is an $n-1$-dimensional variety, i.e. it is a *linear line congruence*.

It is interesting to remark that the algebraic variety $\det(g_{ij}) = 0$, where g is the leading coefficient of a second-order HHO, is the singular locus of the linear line congruence, and has been studied in detail in [26, proposition 4.4].

As a direct consequence of theorem 8, the problem of classifying non-degenerate n-components second-order HHOs under the action of projective reciprocal transformations is solved by means of the classification of 3-forms in $\mathbb{C}^{n+1}$ under the action of the group $SL(n+1,\mathbb{C})$. This is what will be exposed in next section.

2.3. *Projective classification of Hamiltonian operators*

The following results are a direct consequence of the classification of 3-forms in $\mathbb{C}^{n+1}$ under the action of the group $SL(n+1,\mathbb{C})$. Such a classification can be found in the book [17] for $n\leqslant 7$, while the case $n=8$ is covered in [35]. It should be remarked that the latter paper presents the classification of trivectors in dimension 9, i.e. elements of $\wedge^3\mathbb{C}^9$, under the natural action of $SL(n+1,\mathbb{C})$. It is easy to realize that the classification of 3-forms (i.e. the set of orbits) is put in bijective correspondence with the classification of trivectors by any isomorphism, for example, the correspondence defined by the passage from a basis to its dual $e_i\mapsto e^i$.

2.3.1. The case $n=2$. There is only one (nontrivial) 3-form, namely $\omega = \mathrm{d}v^1\wedge\mathrm{d}v^2\wedge\mathrm{d}v^3$. We can rewrite it as

$$i(\omega) = \frac{1}{3}(v^1\mathrm{d}v^2\wedge\mathrm{d}v^3 - v^2\mathrm{d}v^3\wedge\mathrm{d}v^1 + v^3\mathrm{d}v^1\wedge\mathrm{d}v^2). \tag{31}$$

The affine projection $v^3 = 1$, $\mathrm{d}v^3 = 0$, yields, up to a factor, the leading coefficient $\mathrm{d}u^1 \wedge \mathrm{d}u^2$ of the second-order HHO

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2. \tag{32}$$

2.3.2. The case n = 4. There are two (nontrivial) orbits. The open orbit is generated by

$$\omega = \mathrm{d}v^5 \wedge (\mathrm{d}v^1 \wedge \mathrm{d}v^2 + \mathrm{d}v^3 \wedge \mathrm{d}v^4), \tag{33}$$

that corresponds to the leading coefficient $\mathrm{d}u^1 \wedge \mathrm{d}u^2 + \mathrm{d}u^3 \wedge \mathrm{d}u^4$ of the second-order HHO

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \partial_x^2 \tag{34}$$

the closed orbit is totally decomposable and generated by

$$\omega = \mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge \mathrm{d}v^3; \tag{35}$$

the corresponding leading coefficient is degenerate: $\det(g_{ij}) = 0$.

2.3.3. The case n = 6. The classification in this case is due to Schouten (see [17]). There are nine nontrivial orbits. We list below the generators of the orbits which lead to a non-degenerate 2-form $i(\omega)$.

(a) The open orbit is generated by

$$\begin{aligned} \omega = {} & \mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge \mathrm{d}v^3 + \mathrm{d}v^4 \wedge \mathrm{d}v^5 \wedge \mathrm{d}v^6 \\ & + \mathrm{d}v^7 \wedge (\mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5 + \mathrm{d}v^3 \wedge \mathrm{d}v^6) \end{aligned} \tag{36}$$

(case X in [17]). By using the map $i(\omega)$:

$$\begin{aligned} i(\omega) = \frac{1}{3}(& v^1 \mathrm{d}v^2 \wedge \mathrm{d}v^3 - v^2 \mathrm{d}v^1 \wedge \mathrm{d}v^3 + v^3 \mathrm{d}v^1 \wedge \mathrm{d}v^2 \\ & + v^4 \mathrm{d}v^5 \wedge \mathrm{d}v^6 - v^5 \mathrm{d}v^4 \wedge \mathrm{d}v^6 + v^6 \mathrm{d}v^4 \wedge \mathrm{d}v^5 \\ & + v^7 \mathrm{d}v^1 \wedge \mathrm{d}v^4 - v^1 \mathrm{d}v^7 \wedge \mathrm{d}v^4 + v^4 \mathrm{d}v^7 \wedge \mathrm{d}v^1 \\ & + v^7 \mathrm{d}v^2 \wedge \mathrm{d}v^5 - v^2 \mathrm{d}v^7 \wedge \mathrm{d}v^5 + v^5 \mathrm{d}v^7 \wedge \mathrm{d}v^2 \\ & + v^7 \mathrm{d}v^3 \wedge \mathrm{d}v^6 - v^3 \mathrm{d}v^7 \wedge \mathrm{d}v^6 + v^6 \mathrm{d}v^7 \wedge \mathrm{d}v^3). \end{aligned} \tag{37}$$

Then with the affine projection $v^7 = 1, \mathrm{d}v^7 = 0$:

$$
\begin{aligned}
i(\omega) = \frac{1}{3}(&v^3 \mathrm{d}v^1 \wedge \mathrm{d}v^2 - v^2 \mathrm{d}v^1 \wedge \mathrm{d}v^3 + v^3 \mathrm{d}v^1 \wedge \mathrm{d}v^2 \\
&+ v^4 \mathrm{d}v^5 \wedge \mathrm{d}v^6 - v^5 \mathrm{d}v^4 \wedge \mathrm{d}v^6 + v^6 \mathrm{d}v^4 \wedge \mathrm{d}v^5 \\
&+ \mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5 + \mathrm{d}v^3 \wedge \mathrm{d}v^6).
\end{aligned} \tag{38}
$$

Then, the associated 2-form is (up to a factor)

$$
g^1_{ij} = \begin{pmatrix}
0 & v^3 & -v^2 & 1 & 0 & 0 \\
-v^3 & 0 & v^1 & 0 & 1 & 0 \\
v^2 & -v^1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & v^6 & -v^5 \\
0 & -1 & 0 & -v^6 & 0 & v^4 \\
0 & 0 & -1 & v^5 & -v^4 & 0
\end{pmatrix} \tag{39}
$$

and $\det(g^1_{ij}) = (v^1 v^4 + v^2 v^5 + v^3 v^6 - 1)^2$.

(b) We have the 3-form

$$
\begin{aligned}
\omega = &\mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge \mathrm{d}v^3 + \mathrm{d}v^4 \wedge \mathrm{d}v^5 \wedge \mathrm{d}v^6 \\
&+ (\mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5) \wedge \mathrm{d}v^7
\end{aligned} \tag{40}
$$

(case IX in [17]). In the affine chart (removing the factor $1/3$),

$$
\begin{aligned}
3i(\omega) = &v^1 \mathrm{d}v^2 \wedge \mathrm{d}v^3 - v^2 \mathrm{d}v^1 \wedge \mathrm{d}v^3 + v^3 \mathrm{d}v^1 \wedge \mathrm{d}v^2 \\
&+ v^4 \mathrm{d}v^5 \wedge \mathrm{d}v^6 - v^5 \mathrm{d}v^4 \wedge \mathrm{d}v^6 + v^6 \mathrm{d}v^4 \wedge \mathrm{d}v^5 \\
&+ \mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5.
\end{aligned} \tag{41}
$$

The leading coefficient of the associated operator is

$$
g^2_{ij} = \begin{pmatrix}
0 & v^3 & -v^2 & 1 & 0 & 0 \\
-v^3 & 0 & v^1 & 0 & 1 & 0 \\
v^2 & -v^1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & v^6 & -v^5 \\
0 & -1 & 0 & -v^6 & 0 & v^4 \\
0 & 0 & 0 & v^5 & -v^4 & 0
\end{pmatrix} \tag{42}
$$

we have $\det(g^2_{ij}) = (v^1 v^4 + v^2 v^5)^2$.

(c) We have the 3-form

$$
\omega = \mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge \mathrm{d}v^3 + \mathrm{d}v^4 \wedge \mathrm{d}v^5 \wedge \mathrm{d}v^6 + \mathrm{d}v^1 \wedge \mathrm{d}v^4 \wedge \mathrm{d}v^7 \tag{43}
$$

(case VIII in [17]). In the affine chart,

$$3i(\omega) = v^1 \mathrm{d}v^2 \wedge \mathrm{d}v^3 - v^2 \mathrm{d}v^1 \wedge \mathrm{d}v^3 + v^3 \mathrm{d}v^1 \wedge \mathrm{d}v^2 + v^4 \mathrm{d}v^5 \wedge \mathrm{d}v^6 - v^5 \mathrm{d}v^4 \wedge \mathrm{d}v^6 + v^6 \mathrm{d}v^4 \wedge \mathrm{d}v^5 + \mathrm{d}v^1 \wedge \mathrm{d}v^4. \tag{44}$$

The leading coefficient of the associated operator is

$$g^3_{ij} = \begin{pmatrix} 0 & v^3 & -v^2 & 1 & 0 & 0 \\ -v^3 & 0 & v^1 & 0 & 0 & 0 \\ v^2 & -v^1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & v^6 & -v^5 \\ 0 & 0 & 0 & -v^6 & 0 & v^4 \\ 0 & 0 & 0 & v^5 & -v^4 & 0 \end{pmatrix} \tag{45}$$

we have $\det(g^3_{ij}) = (v^1 v^4)^2$.

(d) We have the 3-form

$$\omega = \mathrm{d}v^4 \wedge \mathrm{d}v^5 \wedge \mathrm{d}v^6 + \mathrm{d}v^7 (\mathrm{d}u^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5 + \mathrm{d}v^3 \wedge \mathrm{d}v^6) \tag{46}$$

(case VII in [17]). In the affine chart,

$$3i(\omega) = v^4 \mathrm{d}v^5 \wedge \mathrm{d}v^6 - v^5 \mathrm{d}v^4 \wedge \mathrm{d}v^6 + v^6 \mathrm{d}v^4 \wedge \mathrm{d}v^5 + \mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5 + \mathrm{d}v^3 \wedge \mathrm{d}v^6. \tag{47}$$

The leading coefficient of the associated operator is

$$g^4_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & v^6 & -v^5 \\ 0 & -1 & 0 & -v^6 & 0 & v^4 \\ 0 & 0 & -1 & v^5 & -v^4 & 0 \end{pmatrix} \tag{48}$$

we have $\det(g^4_{ij}) = 1$.

(e) We have the 3-form

$$\omega = \mathrm{d}v^7 \wedge (\mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5 + \mathrm{d}v^3 \wedge \mathrm{d}v^6) \tag{49}$$

(case VI in [17]). In the affine chart we have,

$$3i(\omega) = \mathrm{d}v^1 \wedge \mathrm{d}v^4 + \mathrm{d}v^2 \wedge \mathrm{d}v^5 + \mathrm{d}v^3 \wedge \mathrm{d}v^6. \tag{50}$$

The leading coefficient of the associated operator is

$$g^6_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{51}$$

we have $\det(g^6_{ij}) = 1$.

2.3.4. The case $n=8$. We will follow the classification of trivectors in dimension 9 [35]. We will use the isomorphism between $\mathbb{C}^{n+1}$ and $(\mathbb{C}^{n+1})^*$ defined by a basis and its dual in order to put trivectors and 3-forms into correspondence. We recall that a trivector is said to be *semisimple* if its equivalence class is closed in the space of all trivectors, whereas it is said to be *nilpotent* if the closure of this class contains the zero form. Every trivector u can be uniquely written as the sum $u=p+e$, where p is a semisimple trivector and e is a nilpotent trivector such that $p\wedge e=0$.

Semisimple trivectors p are divided into seven different families for each of which all possible nilpotent parts are provided. Let us introduce the following 3-forms:

$$p_1=\mathrm{d}v^1\wedge\mathrm{d}v^2\wedge\mathrm{d}v^3+\mathrm{d}v^4\wedge\mathrm{d}v^5\wedge\mathrm{d}v^6+\mathrm{d}v^7\wedge\mathrm{d}v^8\wedge\mathrm{d}v^9 \tag{52}$$

$$p_2=\mathrm{d}v^1\wedge\mathrm{d}v^4\wedge\mathrm{d}v^7+\mathrm{d}v^2\wedge\mathrm{d}v^5\wedge\mathrm{d}v^8+\mathrm{d}v^3\wedge\mathrm{d}v^6\wedge\mathrm{d}v^9 \tag{53}$$

$$p_3=\mathrm{d}v^1\wedge\mathrm{d}v^5\wedge\mathrm{d}v^9+\mathrm{d}v^2\wedge\mathrm{d}v^6\wedge\mathrm{d}v^7+\mathrm{d}v^3\wedge\mathrm{d}v^4\wedge\mathrm{d}v^8 \tag{54}$$

$$p_4=\mathrm{d}v^1\wedge\mathrm{d}v^6\wedge\mathrm{d}v^8+\mathrm{d}v^2\wedge\mathrm{d}v^4\wedge\mathrm{d}v^9+\mathrm{d}v^3\wedge\mathrm{d}v^5\wedge\mathrm{d}v^7. \tag{55}$$

Every semisimple trivector is equivalent to a trivector whose corresponding 3-form is of the type

$$p=\lambda_1p_1+\lambda_2p_2+\lambda_3p_3+\lambda_4p_4, \tag{56}$$

where the coefficients are determined up to a linear transformation from a group generated by complex reflections of order 3 [35].

The first family of 3-forms is generated by p only as in (56); more precisely, it consists only of semisimple trivectors ($e=0$). The coefficients λ_i must satisfy a complicated system of algebraic inequalities [35]. The stabilizer subgroup S of this class is a cyclic Abelian group of order 81. The corresponding non-degenerate 2-form in this class is

$$g^{(1)}_{ij}=\begin{pmatrix} 0 & \lambda_1v^3 & -\lambda_1v^2 & \lambda_2v^7 & \lambda_3 & \lambda_4v^8 & -\lambda_2v^4 & -\lambda_4v^6\\ -\lambda_1v^3 & 0 & \lambda_1v^1 & \lambda_4 & \lambda_2v^8 & \lambda_3v^7 & -\lambda_3v^6 & -\lambda_2v^5\\ \lambda_1v^2 & -\lambda_1v^1 & 0 & \lambda_3v^8 & \lambda_4v^7 & \lambda_2 & -\lambda_4v^5 & -\lambda_3v^4\\ -\lambda_2v^7 & -\lambda_4 & -\lambda_3v^8 & 0 & \lambda_1v^6 & -\lambda_1v^5 & \lambda_2v^1 & \lambda_3v^3\\ -\lambda_3 & -\lambda_2v^8 & -\lambda_4v^7 & -\lambda_1v^6 & 0 & \lambda_1v^4 & \lambda_4v^3 & \lambda_2v^2\\ -\lambda_4v^8 & -\lambda_3v^7 & -\lambda_2 & \lambda_1v^5 & -\lambda_1v^4 & 0 & \lambda_3v^2 & \lambda_4v^1\\ \lambda_2v^4 & \lambda_3v^6 & \lambda_4v^5 & -\lambda_2v^1 & -\lambda_4v^3 & -\lambda_3v^2 & 0 & \lambda_1\\ \lambda_4v^6 & \lambda_2v^5 & \lambda_3v^4 & -\lambda_3v^3 & -\lambda_2v^2 & -\lambda_4v^1 & -\lambda_1 & 0 \end{pmatrix}. \tag{57}$$

The second family is generated by the semisimple trivector

$$p=\lambda_1p_1+\lambda_2p_2-\lambda_3p_3, \tag{58}$$

again with λ_i fulfilling an algebraic constraint. The coefficients are determined up to a linear transformation generated by complex reflections. The possible nontrivial nilpotent parts are two:

$$e_1=\mathrm{d}v^1\wedge\mathrm{d}v^6\wedge\mathrm{d}v^8+\mathrm{d}v^2\wedge\mathrm{d}v^4\wedge\mathrm{d}v^9 \tag{59}$$

$$e_2=\mathrm{d}v^1\wedge\mathrm{d}v^6\wedge\mathrm{d}v^8. \tag{60}$$

Here, the dimension of the stabilizer S is 0 for e_1 and 1 for e_2. By summing $p+e_i$ and applying the correspondence, we finally obtain the following two 2-forms:

$$g^{(2)}_{ij} = \begin{pmatrix} 0 & \lambda_1 u^3 & -\lambda_1 u^2 & \lambda_2 u^7 & -\lambda_3 & u^8 & -\lambda_2 u^4 & -u^6 \\ -\lambda_1 u^3 & 0 & \lambda_1 u^1 & 1 & \lambda_2 u^8 & -\lambda_3 v^7 & \lambda_3 u^6 & -\lambda_2 u^5 \\ \lambda_1 u^2 & -\lambda_1 u^1 & 0 & -\lambda_3 u^8 & 0 & \lambda_2 & 0 & \lambda_3 u^4 \\ -\lambda_2 v^7 & -1 & \lambda_3 u^8 & 0 & \lambda_1 u^6 & -\lambda_1 u^5 & \lambda_2 u^1 & -\lambda_3 u^3 \\ \lambda_3 & -\lambda_2 u^8 & 0 & -\lambda_1 u^6 & 0 & \lambda_1 u^4 & 0 & \lambda_2 u^2 \\ -u^8 & \lambda_3 v^7 & -\lambda_2 & \lambda_1 u^5 & -\lambda_1 u^4 & 0 & -\lambda_3 u^2 & u^1 \\ \lambda_2 u^4 & -\lambda_3 u^6 & 0 & -\lambda_2 u^1 & 0 & \lambda_3 u^2 & 0 & \lambda_1 \\ u^6 & \lambda_2 u^5 & -\lambda_3 u^4 & \lambda_3 u^3 & -\lambda_2 u^2 & -u^1 & -\lambda_1 & 0 \end{pmatrix} \tag{61}$$

$$g^{(3)}_{ij} = \begin{pmatrix} 0 & \lambda_1 u^3 & -\lambda_1 u^2 & \lambda_2 v^7 & -\lambda_3 & u^8 & -\lambda_2 u^4 & -u^6 \\ -\lambda_1 u^3 & 0 & \lambda_1 u^1 & 0 & \lambda_2 u^8 & -\lambda_3 v^7 & \lambda_3 u^6 & -\lambda_2 u^5 \\ \lambda_1 u^2 & -\lambda_1 u^1 & 0 & -\lambda_3 u^8 & 0 & \lambda_2 & 0 & \lambda_3 u^4 \\ -\lambda_2 v^7 & 0 & \lambda_3 u^8 & 0 & \lambda_1 u^6 & -\lambda_1 u^5 & \lambda_2 u^1 & -\lambda_3 u^3 \\ \lambda_3 & -\lambda_2 u^8 & 0 & -\lambda_1 u^6 & 0 & \lambda_1 u^4 & 0 & \lambda_2 u^2 \\ -u^8 & \lambda_3 v^7 & -\lambda_2 & \lambda_1 u^5 & -\lambda_1 u^4 & 0 & -\lambda_3 u^2 & u^1 \\ \lambda_2 u^4 & -\lambda_3 u^6 & 0 & -\lambda_2 u^1 & 0 & \lambda_3 u^2 & 0 & \lambda_1 \\ u^6 & \lambda_2 u^5 & -\lambda_3 u^4 & \lambda_3 u^3 & -\lambda_2 u^2 & -u^1 & -\lambda_1 & 0 \end{pmatrix}. \tag{62}$$

In both cases the determinants are non-zero.

The total number of non-degenerate two-forms in the classification is 132. For reasons of space, we will not list elements in the families 3–7; however, we are ready to privately provide the list of non-degenerate two-forms to the interested reader.

Remark 2. The dimension of the space of 3-forms $\wedge^3(\mathbb{C}^{n+1})^*$ grows with the dimension n in a much faster way than the dimension of $SL(\mathbb{C}^{n+1})$. However, for small values of n the dimension of the group is prevailing: this is the reason for the lack of non-trivial classes when $n \leqslant 4$. The same argument shows that a classification for higher values of n does not make sense, in view of the huge number of free parameters that the generic element would depend on.

3. Systems of PDEs with second-order Hamiltonian structure

In [34, theorem 10] it was proved that the necessary conditions for a second-order HHO P (6) to be a Hamiltonian operator for a quasilinear system of first-order conservation laws (8) are

$$g_{qj}V^j_{,p} + g_{pj}V^j_{,q} = 0, \tag{63a}$$

$$g_{qk}V^k_{,pl} + g_{pq,k}V^k_{,l} + g_{qk,l}V^k_{,p} = 0, \tag{63b}$$

where $V^i_{,j} = \frac{\partial V^i}{\partial u^j}$. The above result is analogue to the results in [30] concerning first-order HHOs and quasilinear systems of first-order PDEs, and is obtained by a method introduced in [19] and later adapted to HHOs [15] for third-order homogeneous Hamiltonian operators.

We observe that the above conditions have no direct differential-geometric interpretation as they are derived in flat coordinates of the connection Γ^i_{jk} (see the Introduction). However, we will be able to parameterise the space of solutions of the above equations, thus exhibiting large families of systems of PDEs that are Hamiltonian with respect to second-order Hamiltonian operators. Interesting properties of such systems will be thoroughly investigated.

3.1. Solution of the compatibility conditions

We will now solve completely the system of compatibility conditions between a quasilinear system of first-order PDEs (63). We will first prove that the system is in involution, then we will parameterise its solutions.

Proposition 10. *The system (63) is in involution.*

Proof. Let us derive (11):

$$g_{qj,l}V^j_{,p} + g_{qj}V^j_{,pl} + g_{pj,l}V^j_{,q} + g_{pj}V^j_{,ql} = 0 \tag{64}$$

then, by using condition (63*b*) we can substitute

$$g_{qj}V^j_{,pl} + g_{qj,l}V^j_{,p} = -g_{pq,j}V^j_{,l} \tag{65}$$

$$g_{pj}V^j_{,ql} + g_{pj,l}V^j_{,q} = -g_{qp,j}V^k_{,l} \tag{66}$$

in (64), which yields

$$-g_{pq,j}V^j_{,l} - g_{qp,j}V^j_{,l} = 0, \tag{67}$$

which is an identity.

The condition (64) can be rewritten as

$$(g_{qk}V^k_{,p})_{,l} + g_{pq,k}V^k_{,l} = 0. \tag{68}$$

From the consistency condition $V^k_{,plm} = V^k_{,pml}$ we obtain

$$(g_{qk}V^k_{,p})_{,lm} + (g_{pq,k}V^k_{,l})_{,m} = (g_{qk}V^k_{,p})_{,ml} + (g_{pq,k}V^k_{,m})_{,l} \tag{69}$$

which yields the identity $g_{pq,k}V^k_{,lm} = g_{pq,k}V^k_{,ml}$ in view of $g_{pq,kl} = 0$.

□

The above Proposition shows that, since (64) expresses all second-order derivatives, the general solution of the system depends on no more than $n + n^2$ parameters. The equations (11) impose further $n(n-1)/2$ additional restrictions, so that the total number of arbitrary constants in the general solution is

$$n + n^2 - \frac{n(n-1)}{2} = n(n+3)/2. \tag{70}$$

Now, we will solve the system (63).

Theorem 11. *Let C be a second-order HHO in canonical form (6). Then, the (explicit) solution of the system (63) is the vector V^i given by*

$$V^i = g^{ij}W_j, \tag{71}$$

where W_j is the covector

$$W_j = A_{jl}u^l + B_j \tag{72}$$

where $A_{ij} = -A_{ji}$, B_i are arbitrary constants.

Proof.

$$(W_j)_{,ab} = g_{jk}V^k_{,ab} + g_{jk,b}V^k_{,a} + g_{jk,ab}V^k + g_{jk,a}V^k_{,b}. \tag{73}$$

Since $g_{jk,ab} = 0$ and $g_{bj,k} = g_{jk,b}$ the right-hand side of the above equation becomes (63*b*), hence it vanishes. Then $W_j = A_{jl}u^l + B_j$. Moreover, we have the following identity:

$$W_{j,p} + W_{p,j} = g_{jl}V^l_{,p} + g_{pl}V^l_{,j} \tag{74}$$

which is the right-hand side of (11). But we have

$$W_{j,p} + W_{p,j} = A_{jp} + A_{pj}, \tag{75}$$

which completes the proof. □

Remark 3. The above solution of the system (63) is the most general: indeed, W_i depends on $n(n+1)/2$ arbitrary constants, and V^i is defined up to n arbitrary constants (as it enters the right-hand side of (8)). The total figure is equal to the dimension count following the proof of proposition 10.

Corollary 12. *The fluxes V^i are rational functions of the form:*

$$V^i = \frac{Q}{\mathrm{Pf}(g)} \tag{76}$$

where Q is a polynomial of degree $n/2$ and the denominator is $\mathrm{Pf}(g)$, the Pfaffian of g.

Proof. We have $V^i = g^{is}W_s$, where g^{ij} is the inverse matrix of g_{ij}. By means of properties of the determinant of skew-symmetric matrices [25] the inverse matrix has rational functions entries where the numerator has degree $(n-2)/2$ and the denominator is the Pfaffian of g_{ij}, whose degree is at most $n/2$. Since W_s are linear functions, the statement follows. □

Corollary 13. *The eigenvalues of the matrix $V^i_{,j}$ have even algebraic multiplicity.*

Proof. The eigenvalues are computed by the characteristic polynomial: $\det(V^i_{,j} - \lambda\delta^i_j)$. Lowering one index we obtain a skew-symmetric matrix:

$$\begin{aligned} g_{hi}(V^i_{,j} - \lambda\delta^i_j) &= g_{hi}(g^{ik}W_k)_{,j} - \lambda g_{hj} \\ &= -g_{hi,j}g^{ik}W_k + g_{hi}g^{ik}A_{kj} - \lambda g_{hj} \\ &= T_{hji}g^{ik}W_k + A_{hj} - \lambda g_{hj} \end{aligned}$$

whose determinant is the square of its Pfaffian. Since $\det(g_{ij})$ is also a perfect square, we obtain the result. □

Due to the above considerations, it turns out that the case when all eigenvalues have algebraic multiplicity 2 is generic. However, we cannot exclude cases with higher multiplicity: for example, it is easy to realize that, when g_{ij} is a constant non-degenerate skew-symmetric matrix and $A_{ij} = g_{ij}$, $B_i = 0$, we have $V^i = u^i$ and $V^i_{,j} = I$, the identity matrix.

Remark 4. The above result has important consequences on the integrability of the system (8). Indeed, in [31] the generalized hodograph method for the solution of semi-Hamiltonian quasilinear first-order systems is developed. However, one of the hypothesis in the above paper is that all eigenvalues of V^i_j are distinct. We will see in section 3.4 that, at least in the generic case, this does not prevent the semi-Hamiltonianity of the systems determined by second-order HHOs.

3.2. Projective geometry of the systems of PDEs

Let us recall that for every conservative quasilinear system of first-order PDEs of the form

$$u^i_t = (V(u^j))_x = V^i_{,j}(u^j)u^j_x \tag{77}$$

it is possible to associate a congruence

$$y^i = u^i y^{n+1} + V^i y^{n+2} \tag{78}$$

in auxiliary projective space $\mathbb{P}^{n+1}$ with homogeneous coordinates $(y^1 : \cdots : y^{n+2})$.

Proposition 14. *Let $u^i_t = V^i_{,j}u^j_x$ be a system compatible with a second-order HHO in canonical form (6). Then the associated congruence is linear.*

Proof. By theorem 11 we obtain that

$$A_{jl}u^l + B_j = (T_{jkl}u^l + g^0_{jk})V^k \tag{79}$$

then

$$\frac{1}{2}T_{jkl}\left(u^lV^k - u^kV^l\right) + g^0_{jk}V^k = A_{jl}u^l + B_j. \tag{80}$$

This yields a system of n linear relations between Plücker's coordinates describing the line congruence, hence the statement is proved. □

More interesting facts about the geometry of quasilinear systems of the first order can be found in [1–3, 31]. In particular, in the case of distinct eigenvalues, it was proved in [3] that the linearity of the associated congruence implies that the systems are linearly degenerate. We recall that a strictly hyperbolic quasilinear first-order system of PDEs of the form

$$u^i_t = V^i_j(u^k)u^j_x \tag{81}$$

is said to be *linearly degenerate* (or *weakly non-linear*) if its eigenvalues $\lambda^i = \lambda^i(u^k)$ are constant along the corresponding eigenvector X^i: $L_{X^i}\lambda^i = 0$ (no sum in i), where L stands for the Lie derivative. In our case the eigenvalues have multiplicity higher than one; following [33], we define a (not necessarily strictly hyperbolic) quasilinear first-order system of PDEs to be linearly degenerate if

$$\nabla P(\lambda)|_{\lambda=A} = 0, \tag{82}$$

where ∇ stands for the gradient.[6]

One might repeat the arguments of [3] to prove that our systems are linearly degenerate, but a more simple argument yields the proof in the generic case.

Proposition 15. *Let $u^i_t = V^i_{,j}u^j_x$ be a system compatible with a second-order HHO in canonical form (6). Suppose that $V^i_{,j}$ is diagonalizabile. Then, the system is linearly degenerate.*

Proof. In our case, the eigenvalues of $V^i_{,j}$ have multiplicity $m \geqslant 2$. Then, repeating the proof of proposition 2 in [33] for the case of diagonal systems we obtain that the criterion for linear degeneracy is fulfilled. □

As we already observed, the case when all eigenvalues of $V^i_{,j}$ have multiplicity exactly equal to 2 is generic. We will see in section 3.4 that in this case our systems are diagonalizable, and

[6] We stress that this definition does not coincide with the standard one if the system is not strictly hyperbolic.

we will discuss the impact of diagonalizability on integrability. Note that the linearity of the congruence does not prevent diagonalizability (see the remark in [2, p. 1771]).

We can have a look at the case $n=2$ (already considered in [34]) using the theory of congruences.

Proposition. *Let $n=2$. Then, a system of quasilinear first-order conservation laws that is compatible with a second-order HHO in canonical form (6) is linearisable.*

Proof. Let $n=2$. Then in $\mathbb{P}^3$ linear congruences can be brought (modulo projective transformations) to the form

$$y^1 = u^2y^3 + u^2y^3 \qquad y^2 = u^2y^3 + u^1y^4. \tag{83}$$

By using the affine chart $y^4 = 1$ we have only two cases $(y^1 = y^2, y^3 = 1)$ or $(y^1 = -y^2, y^3 = -1)$. But by condition (63*a*) we obtain that the system is skew-symmetric, then

$$\begin{cases} u^1_t = u^2_x \\ u^2_t = -u^1_x \end{cases}. \tag{84}$$

In particular, every system $u^i_t = (V^i)_x$ compatible with a second-order operator is linearisable. □

We stress that an obvious alternative proof immediately follows from the classification of second-order HHOs in section 2.3 or from theorem 11. The same argument yields the following Proposition.

Proposition 17. *Systems of quasilinear first-order conservation laws that admit a second-order HHO in canonical form (6) with constant coefficient matrix (g^{ij}) are linearisable by projective reciprocal transformations.*

In particular, the above result holds for all systems in the case $n=4$, while if $n \geqslant 6$ there are systems that are *not* linearizable by projective reciprocal transformations.

3.3. Hamiltonian systems

We would like to find an Hamiltonian for the systems that we obtained in section 3.1. To this aim we observe that, in potential coordinates $u^i = b^i_x$, the second-order HHO P (6) undergoes the coordinate change

$$P^{(b)} = \ell_{(b^i)} \circ P^{(u)} \circ \ell^*_{(b^i)}, \tag{85}$$

where $\ell_{(b^i)} = \partial_x^{-1}$ is the linearization of the coordinate change $b^i = \partial_x^{-1}u^i$, and $\ell^*_{(b^i)} = -\partial_x^{-1}$ is its formal adjoint. Hence, in potential coordinates P becomes the ultralocal operator ${P^{(b)}}^{ij} = -g^{ij}(b^k_x)$, and the system of first-order conservation laws (8) becomes $b^i_t = V^i(b^k_x)$ (see [15]). It is then easy to make an ansatz for the form of H, and prove the following result.

Proposition 18. *We have*

$$H = -\int \left(\frac{1}{2}A_{sl}b^l_x + B_s\right) b^s \mathrm{d}x. \tag{86}$$

Proof. Let us consider the variational derivative of H:

$$\frac{\delta H}{\delta b^k} = -A_{ks}b^s_x - B_k \tag{87}$$

then

$$-g^{ik}\frac{\delta H}{\delta b^k} = -g^{ik}(-A_{ks}b^s_x - B_k) = g^{ik}W_k = V^i(b_x). \tag{88}$$

□

Remark 5. At difference with the third-order case [15] we observe that there are no non-trivial nonlocal Casimirs, as the equation

$$-g^{ik}\frac{\delta F^j}{\delta b^k} = 0 \tag{89}$$

has no non-trivial solutions.

3.4. The Haantjes tensor and integrability

According to many Authors (e.g. [28, 31]) integrability of quasilinear first-order systems (81) is strictly related to the dimension of the space of conservation laws. In particular, semi-Hamiltonian systems [31] (or *systèmes riches* [28, p. 20]) have a maximal space of conservation laws and coincide with diagonalizable conservative systems, according with proposition 5 in [28, p. 19, 20].

Then, semi-Hamiltonian systems whose velocity matrix has *distinct* eigenvalues are integrable via the generalized hodograph method, that was first introduced in [31].

We consider conservative systems (8) that admit a second-order HHO in the canonical form (6). According with a well-known result [18] quasilinear first-order systems (81) are diagonalizable if and only if: 1 – the Haantjes tensor

$$H^i_{jk} = N^i_{pr}V^p_jV^r_k - N^p_{jr}V^i_pV^r_k - N^p_{rk}V^i_pV^r_j + N^p_{jk}V^i_rV^r_p, \tag{90}$$

identically vanishes, where N^i_{jk} is the Nijenhuis tensor

$$N^i_{jk} = V^p_jV^i_{kp} - V^p_kV^i_{jp} - V^i_p(V^p_{kj} - V^p_{jk}), \tag{91}$$

and 2 – the dimension of the eigenspaces is equal to the algebraic multiplicity of the eigenvalues.

We stress that the vanishing of the Haantjes tensor is necessary in order to have an *holonomic* basis of eigenvectors, which is a non-trivial fact and implies the existence of Riemann invariants, i.e. the coordinates that diagonalize the velocity matrix.

It is possible to compute the Haantjes tensor for all our conservative Hamiltonian systems (8); it is remarkable that it identically vanishes.

Theorem 19. *The Haantjes tensor of a conservative quasilinear system* (8) *that admits a second-order HHO in canonical form* (6) *is identically vanishing.*

Proof. It is easy to prove the following identity

$$g^{ia}_{,k}g_{aj} = -g^{ia}T_{ajk}. \tag{92}$$

Then, from (63*b*) we have $V^a_{pl} = -g^{aq}(T_{pqk}V^k_{,l} + T_{qkl}V^k_{,p})$, hence the Nijenhuis tensor can be written as

$$N^i_{jk} = g^{ia}\left(T_{jal}V^l_pV^p_k - T_{kal}V^l_pV^p_j - 2T_{alp}V^l_kV^p_j\right). \tag{93}$$

Using (93) we obtain

$$\begin{aligned} H^i_{kj} = {} & g^{ia}\left(T_{pal}V^l_sV^s_r - T_{ral}V^l_sV^s_p - 2T_{als}V^l_rV^s_p\right)V^p_kV^r_j \\ & - g^{pa}\left(T_{kal}V^l_sV^s_r - T_{ral}V^l_sV^s_k - 2T_{als}V^l_rV^s_k\right)V^i_pV^r_j \\ & - g^{pa}\left(T_{ral}V^l_sV^s_j - T_{jal}V^l_sV^s_r - 2T_{als}V^l_jV^s_r\right)V^i_pV^r_k \\ & + g^{pa}\left(T_{kal}V^l_sV^s_j - T_{jal}V^l_sV^s_k - 2T_{als}V^l_jV^s_k\right)V^i_rV^r_p. \end{aligned} \tag{94}$$

Let us consider, for example, the summand

$$S = -2g^{ia}T_{als}V^l_rV^s_pV^p_kV^r_j + 2g^{pa}T_{als}V^l_rV^s_kV^i_pV^r_j. \tag{95}$$

It is clear that $S = 0$ if $-2g^{ia}T_{als}V^s_pV^p_k + 2g^{pa}T_{als}V^s_kV^i_p = 0$. Now, we use the identities (92), (11) and the upper indices version $g^{il}V^j_{,l} + g^{jl}V^i_{,l} = 0$ to prove that $g^{ia}T_{als}V^s_pV^p_k = g^{pa}T_{als}V^s_kV^i_p$, so that $S = 0$.

With similar algebraic manipulations it is easy to prove that the following pairs of summands annihilate:

$$+2g^{pa}T_{als}V^l_jV^s_rV^i_pV^r_k - 2g^{pa}T_{als}V^l_jV^s_kV^i_rV^r_p = 0, \tag{96}$$

$$+g^{ia}T_{pal}V^l_sV^s_rV^p_kV^r_j - g^{pa}T_{kal}V^l_sV^s_rV^i_pV^r_j = 0, \tag{97}$$

$$-g^{ia}T_{ral}V^l_sV^s_pV^p_kV^r_j + g^{pa}T_{jal}V^l_sV^s_rV^r_kV^i_p = 0. \tag{98}$$

$$+g^{pa}T_{kal}V^l_sV^s_jV^i_rV^r_p - g^{pa}T_{ral}V^l_sV^s_jV^i_pV^r_k = 0, \tag{99}$$

$$-g^{pa}T_{jal}V^l_sV^s_kV^i_rV^r_p + g^{pa}T_{ral}V^l_sV^s_kV^i_pV^r_j = 0. \tag{100}$$

□

In the generic case (where the multiplicity of all eigenvalues is exactly equal to two), our systems are diagonalizable.

Proposition 20. *Let $u^i_t = V^i_{,j}u^j_x$ be a Hamiltonian system with respect to a second-order HHO in canonical form (6). If $V^i_{,j}$ admits $n/2$ distinct eigenvalues (i.e. each eigenvalue has multiplicity two), then $V^i_{,j}$ is diagonalizable.*

Proof. First of all, we observe that $V^i_{,j}$ is the product $V^i_{,j} = g^{ik}M_{kj}$ of two skew-symmetric matrices, where $M_{ij} = A_{ij} + T_{ijk}g^{ks}(A_{sl}u^l + B_s)$. Then, one can easily prove that $V^i_{,j}$ is self-adjoint with respect to the bilinear form given by M_{ij}. At each point (u^j), the Jordan form of $V^i_{,j}$ is made by 2×2 blocks whose corresponding subspaces are mutually orthogonal with respect to M; it turns out that both M and L have 2×2 block-diagonal structure. This implies that $V^i_{,j}$ is diagonal (at each point), and its eigenspaces are 2-dimensional. This fact, together with the vanishing of the Haantjes tensor (by the above Theorem), implies that the diagonalising transformation can be integrated to a transformation of the dependent variables which diagonalises $V^i_{,j}$. □

Corollary 21. *In the above hypotheses, the systems are linearly degenerate and semi-Hamiltonian.*

Proof. Indeed, the systems are diagonalizable, and that implies that they are linearly degenerate, according with proposition 15, and admit a conservative form. Hence, proposition 5 in [28, p. 19] yields the result. □

Even if the systems are generically semi-Hamiltonian, we cannot use the results of [31] and conclude that the systems can be solved by the generalized hodograph method. Such a method was developed for semi-Hamiltonian systems whose velocity matrix has *distinct* eigenvalues (also see other methods in [12]).

Despite the lack of general integration results on systems of our type (even if we assume that they are diagonalizable), in a recent paper [36] a family of Jordan-block-type systems was integrated using an argument that is similar of that of Tsarev's generalized hodograph method. So, it might be that our systems are solvable in some way. We plan to devote ourselves to the task of integrating our systems in a future paper.

Data availability statement

No new data were created or analysed in this study.

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