JHEP

Published for SISSA by Springer

Received: March 3, 2021
Accepted: April 1, 2021
Published: April 28, 2021

$1/N$ expansion of circular Wilson loop in $\mathcal{N} = 2$ superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ quiver

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Abstract: Localization approach to $\mathcal{N} = 2$ superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ quiver theory leads to a non-Gaussian two-matrix model representation for the expectation value of BPS circular $\mathrm{SU}(N)$ Wilson loop $\langle \mathcal{W} \rangle$. We study the subleading $1/N^2$ term in the large N expansion of $\langle \mathcal{W} \rangle$ at weak and strong coupling. We concentrate on the case of the symmetric quiver with equal gauge couplings which is equivalent to the $\mathbb{Z}_2$ orbifold of the $\mathrm{SU}(2N)$ $\mathcal{N} = 4$ SYM theory. This orbifold gauge theory should be dual to type IIB superstring in $\mathrm{AdS}_5 \times (S^5/\mathbb{Z}_2)$. We present a string theory argument suggesting that the $1/N^2$ term in $\langle \mathcal{W} \rangle$ in the orbifold theory should have the same strong-coupling asymptotics $\lambda^{3/2}$ as in the $\mathcal{N} = 4$ SYM case. We support this prediction on the gauge theory side by a numerical study of the localization matrix model. We also find a relation between the $1/N^2$ term in the Wilson loop expectation value and the derivative of the free energy of the orbifold gauge theory on 4-sphere.

Keywords: $1/N$ Expansion, AdS-CFT Correspondence, Wilson, 't Hooft and Polyakov loops

ArXiv ePrint: 2102.07696

https://doi.org/10.1007/JHEP04(2021)265

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1 Introduction and summary

Supersymmetric Wilson loop operators provide an important class of observables that shed light on the intricate structure of weak-strong coupling interpolation in the context of AdS/CFT duality. In special cases with extended supersymmetry the localization method [1] allows one to represent the expectation value of a supersymmetric loop in terms of a matrix model integral.

Here we will consider a particular $\mathcal{N} = 2$ supersymmetric gauge theory which is the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ quiver with two bi-fundamental hypermultiplets [2–5]. In the "symmetric" case when the two 't Hooft couplings λ_1, λ_2 are equal this theory is equivalent to the $\mathbb{Z}_2$ orbifold of the $\mathrm{SU}(2N)$ $\mathcal{N} = 4$ SYM theory [6]. The orbifold theory has the same planar diagrams as the parent $\mathcal{N} = 4$ SYM theory [7], i.e. the two are closely related at large N. The dual string theory should be the corresponding orbifold of the $\mathrm{AdS}_5 \times S^5$ superstring, i.e. type IIB string on $\mathrm{AdS}_5 \times (S^5/\mathbb{Z}_2)$ [8, 9].[1]

[1] $\mathbb{Z}_2$ acts by flipping 4 of the 6 embedding coordinates of the 5-sphere, reflecting the 2+4 split of the $\mathcal{N} = 4$ SYM scalars between the vector multiplets and the hypermultiplets of the $\mathcal{N} = 2$ theory.

For each of the two SU(N) factors of the quiver theory one may define $\frac{1}{2}$-BPS circular Wilson loops coupled to the corresponding gauge and scalar fields (a $= 1, 2$)

$$\mathcal{W}_{\rm a} = {\rm tr}\, \mathcal{P}\, \exp\left[\oint ds\ (i\, \dot{x}^\mu\, A_{\mu\, {\rm a}} + |\dot{x}|\, \Phi_{\rm a})\right] , \tag{1.1}$$

where we choose not to include the $1/N$ factor in front of the trace. For the orbifold theory their (normalized) expectation values are equal

$$\langle \mathcal{W}_1 \rangle = \langle \mathcal{W}_2 \rangle \equiv \langle \mathcal{W} \rangle^{\rm orb} , \tag{1.2}$$

and at large N coincide [4] with the famous SU(N) $\mathcal{N} = 4$ SYM result [10–12]

$$\begin{aligned} \langle \mathcal{W} \rangle^{\rm orb}_{N\to\infty} &= \langle \mathcal{W} \rangle^{\rm SYM}_{N\to\infty} = \langle \mathcal{W} \rangle_0 , \\ \langle \mathcal{W} \rangle_0 &= \frac{2\, N}{\sqrt{\lambda}}\, I_1\left(\sqrt{\lambda}\right) \overset{\lambda\gg 1}{=} \sqrt{\frac{2}{\pi}}\, N\, \lambda^{-3/4}\, e^{\sqrt{\lambda}} \left[1 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right] . \end{aligned} \tag{1.3}$$

For general N the expression for $\langle \mathcal{W} \rangle^{\rm orb}$ is given by a special non-Gaussian matrix model integral following from the localization approach [12]. In contrast to the $\mathcal{N} = 4$ SYM case where the corresponding matrix model is Gaussian leading to the closed expression [11, 12]

$$\langle \mathcal{W} \rangle^{\rm SYM} = e^{\frac{\lambda}{8N}\left(1-\frac{1}{N}\right)} L^{(1)}_{N-1}\left(-\frac{\lambda}{4N}\right) = N\, e^{\sqrt{\lambda}} \sum_{p=0}^{\infty} \frac{\sqrt{2}}{96^p \sqrt{\pi}\, p!}\, \frac{\lambda^{\frac{6p-3}{4}}}{N^{2p}} \left[1 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right] , \tag{1.4}$$

working out the $1/N$ expansion of $\langle \mathcal{W} \rangle^{\rm orb}$ turns out to be a non-trivial problem. Below we will address the question about the structure of the λ-dependent coefficients in the $1/N$ expansion of $\langle \mathcal{W} \rangle^{\rm orb}$ by considering separately the small and large λ limits.

On the dual string theory side, the $1/N^2$ expansion is the genus expansion, and the λ dependence of the $1/N^{2p}$ coefficient in the analog of (1.4) corresponds to the string tension dependence of the partition function with world surface of topology of a disc with p handles.

As discussed recently in [13], the strong coupling expansions of the $\frac{1}{2}$-BPS circular Wilson loops in $\mathcal{N} = 4$ SYM and ABJM gauge theories with string duals defined on $AdS_5 \times S^5$ and $AdS_4 \times CP^3$ have a remarkably similar structure. The string counterpart of the dominant at large N ($p = 0$) term in (1.3), (1.4) is the open-string partition function on the disk which contains an overall factor of the inverse closed string coupling $g_{\rm s}$

$$\langle \mathcal{W} \rangle_0 = \frac{\sqrt{T}}{2\pi\, g_{\rm s}}\, e^{2\pi T} e^{-\bar{\Gamma}} \left[1 + \mathcal{O}\left(T^{-1}\right)\right] . \tag{1.5}$$

In the SU(N) $\mathcal{N} = 4$ SYM case

$$g_{\rm s} = \frac{g^2_{\rm YM}}{4\pi} = \frac{\lambda}{4\pi N}, \qquad T = \frac{L^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi} , \qquad \bar{\Gamma} = \frac{1}{2}\log(2\pi) , \tag{1.6}$$

so that the leading term in (1.5) is the same as in (1.3).

In general, the presence of the universal $\sqrt{T}$ prefactor in (1.5) follows from the structure of the 1-loop fluctuation determinants [14] appearing in the string partition function

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expanded near the AdS_2 minimal surface (corresponding to the circular Wilson loop). In the case of genus p surface the UV divergent part of the one-loop effective action $\Gamma = \frac{1}{2}\sum_i \log\det\Delta_i$ reads [13]

$$\Gamma = -\zeta_{\rm tot}(0)\log(L\Lambda) + \bar{\Gamma}\,, \qquad \zeta_{\rm tot}(0) = \chi = 1 - 2p\,, \tag{1.7}$$

where Λ is 2d cutoff, L is the AdS radius ($T = \frac{L^2}{2\pi\alpha'}$) and the $\zeta_{\rm tot}(0)$ coefficient turns out to be equal to the Euler number of the surface. The 2d UV divergence should be canceled by a universal superstring measure contribution $\log(\sqrt{\alpha'}\Lambda)$ involving only the string scale and not the AdS radius. Then the finite part of Γ depends on T through the term $-\chi\log\frac{L}{\sqrt{\alpha'}} = -\chi\log\sqrt{T}$ and thus the string partition function on a genus p surface is proportional to $e^{-\Gamma_{\rm fin}} \sim (\sqrt{T})^{\chi}$, i.e.

$$\langle \mathcal{W}\rangle = \sum_{p=0}^{\infty}\langle \mathcal{W}\rangle_p = e^{2\pi T}\sum_{p=0}^{\infty} c_p\left(\frac{g_{\rm s}}{\sqrt{T}}\right)^{2p-1}\left[1 + \mathcal{O}\left(T^{-1}\right)\right]\,. \tag{1.8}$$

Written in terms of N and λ in (1.6) this matches the structure of the $1/N$ expansion of the exact $\mathcal{N}=4$ SYM result in (1.4). One can also use similar considerations to predict the structure of the string theory expansions for other related observables [15].

It is important to emphasize the universality of the structure of the expansion in (1.8): it relies only on the fact that one expands near the AdS_2 minimal surface embedded into the AdS_3 part of AdS_n space and should thus be valid also for the corresponding partition functions in the $AdS_4\times CP^3$ and $AdS_3\times S^3\times T^4$ superstring theories [13]. It should also apply to the orbifold $AdS_5\times(S^5/\mathbb{Z}_2)$ theory: orbifolding the S^5 should not change the above argument determining the tension dependence from the way how the AdS radius L appears in (1.7).[2]

We thus conjecture that the same form of the large N, strong coupling expansion (1.8) or (1.4) should also apply appear in the $\mathcal{N}=2$ orbifold theory case, i.e.

$$\langle \mathcal{W}\rangle^{\rm orb} = N\, e^{\sqrt{\lambda}}\sum_{p=0}^{\infty}{\rm c}_p\frac{\lambda^{\frac{6p-3}{4}}}{N^{2p}}\left[1 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right]\,, \tag{1.9}$$

where the coefficients ${\rm c}_p = \frac{c_p}{(8\pi)^{p-1/2}}$ will be different from the ones in (1.4).

In order to check the prediction (1.8), (1.9) for the large N, strong-coupling expansion of $\langle\mathcal{W}\rangle^{\rm orb}$ we shall consider the genus one term corresponding to the leading $1/N^2$ correction to the planar part in (1.3). Normalizing to $\langle\mathcal{W}\rangle_0$ in (1.3) we have in both $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ orbifold cases

$$\frac{\langle\mathcal{W}\rangle}{\langle\mathcal{W}\rangle_0} = 1 + \frac{1}{N^2}\,q(\lambda) + \mathcal{O}\left(\frac{1}{N^4}\right), \qquad \langle\mathcal{W}\rangle_0 = \frac{2N}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right)\,, \tag{1.10}$$

[2] In the case of $\mathrm{SU}(N)\times\ldots\times\mathrm{SU}(N)$ $\mathcal{N}=2$ quiver theory which is the $\mathbb{Z}_k$ orbifold of the $\mathrm{SU}(kN)$ $\mathcal{N}=4$ SYM and should be dual to the superstring on $AdS_5\times(S^5/\mathbb{Z}_k)$ one has for the AdS radius $L^4 = 4\pi kN g_{\rm s}\alpha'$ and thus instead of (1.6) we get $g_{\rm s} = \frac{g^2_{\rm YM}}{4\pi} = \frac{\lambda}{4\pi kN}$, $T = \frac{L^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$.

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where the form of the function $q(\lambda)$ will be our main focus in what follows. In the SU(N) SYM case the expression for $q(\lambda)$ follows from the expansion of the exact Laguerre polynomial expression in (1.4) (I_n are modified Bessel functions of the first kind)

$$q^{\rm SYM}(\lambda) = \frac{\lambda}{96}\left[\frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} - 12\right] = \begin{cases} -\frac{1}{8}\lambda + \frac{1}{384}\lambda^2 + \mathcal{O}(\lambda^3), & \lambda \ll 1, \\ \frac{1}{96}\lambda^{3/2} - \frac{9}{64}\lambda + \frac{1}{256}\lambda^{1/2} + \mathcal{O}(1), & \lambda \gg 1\ . \end{cases} \tag{1.11}$$

As discussed above, the leading strong-coupling behaviour of the genus one correction in $\langle \mathcal{W} \rangle^{\rm SYM}$

$$\frac{1}{N^2}\, q^{\rm SYM}(\lambda) \quad \overset{\lambda \gg 1}{\sim} \quad \frac{\lambda^{3/2}}{N^2} \sim \frac{g_{\rm s}^2}{T} \tag{1.12}$$

is consistent with the universal form of the string theory expansion in (1.4). Then according to (3.15) the same should be true also in the orbifold theory case, i.e.

$$q^{\rm orb}(\lambda) \overset{\lambda \gg 1}{=} C\, \lambda^{3/2} + \mathcal{O}(\lambda)\ . \tag{1.13}$$

where the value of the coefficient C may of course be different from $\frac{1}{96}$ in the SYM case in (1.11). Confirming the prediction (1.13) starting from the localization matrix model representation for $\langle \mathcal{W} \rangle^{\rm orb}$ will be one of the aims of the present paper.

Summary of the results. As we shall see below, the matrix model representations for the orbifold $\mathcal{N} = 2$ gauge theory partition function $Z^{\rm orb}(\lambda; N)$ on S^4 and for $\langle \mathcal{W} \rangle^{\rm orb}$ imply a remarkable relation between $\Delta q(\lambda)$

$$\Delta q(\lambda) \equiv q^{\rm orb}(\lambda) - q^{\rm SYM}(\lambda)\,, \tag{1.14}$$

and the $N \to \infty$ limit of the deviation of the orbifold free energy $F^{\rm orb} = -\log Z^{\rm orb}$ from its SYM counterpart

$$\Delta q(\lambda) = -\frac{\lambda^2}{8}\frac{d}{d\lambda}\Delta F(\lambda)\,, \tag{1.15}$$

$$\Delta F(\lambda) \equiv \lim_{N\to\infty}\Big[F^{\rm orb}(\lambda; N) - 2\, F^{\rm SYM}(\lambda; N)\Big] = -\lim_{N\to\infty} \log \frac{Z^{\rm orb}(\lambda; N)}{\big[Z^{\rm SYM}(\lambda; N)\big]^2}\ . \tag{1.16}$$

The leading $\mathcal{O}(N^2)$ term in $F^{\rm orb} - 2\, F^{\rm SYM}$ cancels due to the planar equivalence between the SU(N)$\times$SU(N) orbifold theory and the two decoupled copies of the SU(N) $\mathcal{N} = 4$ SYM theory.[3] Using (1.15) the expected strong-coupling behaviour (1.13) of $q(\lambda)$ translates into the following scaling for the difference of free energies in (1.16) ($c_1 = -16C$)

$$\Delta F(\lambda) \overset{\lambda \gg 1}{=} c_1\, \lambda^{1/2} + \dots\ . \tag{1.17}$$

[3] More precisely, for the Wilson loops (1.2) the planar equivalence means that the normalized expectation value of an operator in one of the two SU(N) factors of the quiver is the same as in the SU(N) $\mathcal{N} = 4$ SYM theory. In general, the planar correlators of operators from $\mathbb{Z}_2$ symmetric (i.e. "untwisted") sector should match between the orbifold and the parent SU($2N$) SYM theory. For "extensive" quantities like the free energy (or conformal anomalies, correlators of total stress tensor, etc.) the $N \to \infty$ results in the SU(N) $\times$ SU(N) orbifold theory should match those of the two copies of the SU(N) $\mathcal{N} = 4$ SYM.

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In view of (1.16) this leads to a prediction about the strong coupling asymptotics of the leading $1/N^2$ correction to the planar $F^{\rm orb}(\lambda;\infty) = 2\,F^{\rm SYM}(\lambda;\infty)$ part of the free energy of the orbifold theory.

Let us first recall that the free energy of SU(N) $\mathcal{N}=4$ SYM on S^4 should not be renormalized, i.e. should be given exactly by the familiar one-loop expression $F^{\rm SYM}(\lambda;N;\Lambda) = 4\,\mathrm{a}\,\big[\log(\mathrm{r}\Lambda) + f_0\big]$. Here a is the conformal anomaly coefficient, $\mathrm{a} = \frac{1}{4}(N^2-1)$, r is the radius of S^4, Λ is a 4d UV cutoff and f_0 is a constant. Since the free energy is UV divergent, its finite part is not universal depending on a particular regularization scheme. The localization procedure [12] representing the free energy in terms of a finite matrix model integral with a simple λ-independent measure implicitly assumed a special regularization in which the renormalized SU(N) SYM free energy is given by

$$F^{\rm SYM}(\lambda;N) = -2\mathrm{a}\log\lambda = -\frac{1}{2}\Big(N^2-1\Big)\log\lambda \tag{1.18}$$

as this expression follows simply from the Gaussian matrix model integral [16] (we drop an additive numerical constant).

From the dual string theory point of view, the gauge theory free energy is expected to be given by the $\mathrm{AdS}_5\times S^5$ string partition or (more precisely, at the tree level) by the IIB string effective action $S = S_0 + S_1 + \ldots = \frac{1}{(2\pi)^7 g_{\rm s}^2 \alpha'^4}\int d^{10}x\sqrt{G}\big[(R+\ldots)+\alpha'^3 R^4+\ldots\big] + \mathcal{O}(g_{\rm s}^0)+\ldots$. Evaluated on $\mathrm{AdS}_5\times S^5$ (using (1.6) and $R+\ldots = -8L^{-2}$) the leading supergravity term here is $S_0 = \frac{1}{\pi^2}N^2 V_{\mathrm{AdS}_5}$ where V_{AdS_5} is the (logarithmically) IR divergent volume of unit-radius AdS_5. Subtracting the IR divergence in V_{AdS_5} using a particular AdS/CFT motivated prescription gives $V_{\mathrm{AdS}_5} \to \bar{V}_{\mathrm{AdS}_5} = -\pi^2\log\sqrt{\lambda}$ and thus one reproduces [16] the $N^2\log\lambda$ term in (1.18).[4]

The -1 shift of N^2 in (1.18) should come from the 1-loop superstring correction (again proportional to V_{AdS_5}): this should follow the same pattern as found for the $\mathcal{N}=4$ SYM conformal anomaly and S^3 Casimir energy in [17] (with only loops of short supergravity supermultiplets contributing). Other string α'^n tree level (e.g. $\alpha'^3 R^4$, cf. [18]) and string loop corrections should vanish on maximally supersymmetric $\mathrm{AdS}_5\times S^5$ background.

Let us now turn to the orbifold theory that should be dual to the superstring on $\mathrm{AdS}_5\times(S^5/\mathbb{Z}_2)$. Combining (1.16), (1.17) and (1.18) we get the following prediction

$$F^{\rm orb}(\lambda;N) \overset{\lambda\gg 1}{=} -N^2\log\lambda + \Big[c_1\,\lambda^{1/2} + \mathcal{O}(\log\lambda)\Big] + \mathcal{O}\left(\frac{1}{N^2}\right)\,. \tag{1.19}$$

The leading N^2 term here is implied by the planar equivalence to the SU($2N$) SYM and should follow again from the leading type IIB supergravity term evaluated on

[4] One way to understand the origin of the $\log\sqrt{\lambda}$ term is as follows. On AdS side the IR cutoff is measured in units of AdS radius L. On the gauge theory side viewed as originating from the flat-space open string theory the natural UV cutoff is inverse of the string length $\sqrt{\alpha'}$. Thus the two cutoffs are related by the ratio $\frac{L}{\sqrt{\alpha'}} = \lambda^{1/4}$.

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$\mathrm{AdS}_5\times(S^5/\mathbb{Z}_2)$.[5] The planar equivalence also means that like in (1.18) this leading N^2 term should not get string tree level α'-corrections, i.e. they should still vanish when evaluated on $\mathrm{AdS}_5\times(S^5/\mathbb{Z}_2)$.

One may attempt to give an independent string theory explanation of the subleading $\lambda^{1/2}$ term in (1.19) without using the connection (1.15) to the Wilson loop. The string one-loop (genus one or order $g_{\rm s}^0\sim N^0$) type IIB effective action is known to start with $S_1\sim\frac{1}{\alpha'}\int d^{10}x\sqrt{G}\,R^4+\dots$ [19–21]. If we conjecture that when evaluated on the orbifold $\mathrm{AdS}_5\times(S^5/\mathbb{Z}_2)$ it is no longer zero then on dimensional grounds it should scale as $S_1\sim\frac{L^2}{\alpha'}\sim\sqrt{\lambda}$, reproducing the subleading term in (1.19). If a non-zero contribution comes just due to the curvature singularity then it may not be proportional to the AdS_5 volume so there will be no extra $\log\lambda$ factor. The remaining puzzle is why the one-loop R^4 term may contribute to $F^{\rm orb}$ while the tree-level one should not, even though the two invariants have the same structure in type IIB string theory [22].

Starting with the matrix model representation for $\langle\mathcal{W}\rangle^{\rm orb}$ we shall first study the structure of the function $q^{\rm orb}(\lambda)$ in (1.10) or $\Delta q(\lambda)$ in (1.14) at weak coupling. While the small λ expansion of $q^{\rm SYM}(\lambda)$ in (1.11) has only rational coefficients, the coefficients in the expansion of $\Delta q(\lambda)$ in powers of λ are transcendental — proportional to the values $\zeta_n\equiv\zeta(n)$ of the Riemann ζ-function

$$\frac{1}{8\pi^2}\,\Delta q(\lambda)=-\frac{3}{4}\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^3+\frac{45}{8}\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^4+\left[\frac{9}{2}\zeta_3^2-\frac{315}{8}\zeta_7\right]\left(\frac{\lambda}{8\pi^2}\right)^5+\mathcal{O}\left(\lambda^6\right),\quad(1.20)$$

Thus ζ_n may be formally used to parametrise the deviation of the orbifold theory result from the $\mathcal{N}=4$ SYM one.

To find the strong-coupling expansion of $q^{\rm orb}(\lambda)$ requires a resummation of the weak coupling expansion. We shall study resummations of particular subclasses of terms proportional to monomials built out of ζ_n. While this will not be enough to determine the correct strong coupling asymptotics of $q^{\rm orb}(\lambda)$, this may still help to shed light on the general structure of this function.[6]

Exploiting the relation (1.15) we shall compute $\Delta q(\lambda)$ up to order $\mathcal{O}(\lambda^{20})$ and also determine the resummation of all terms with the following types of coefficients involving particular ζ_n and their powers

$$\text{I: }\zeta_{2n+1},\qquad\text{II: }\zeta_3\,\zeta_{2n+1},\qquad\text{III: }\zeta_3^p,\qquad\text{IV: }\zeta_3^p\zeta_5^q,\qquad\text{V: }\zeta_3^p\zeta_5^q\zeta_7^r\,,\qquad(1.21)$$

[5] To recall, the orbifold projection of $\mathrm{SU}(2N)$ SYM giving the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ theory with two bi-fundamental hypermultiplets reduces the number of degrees of freedom and thus also the leading term in the conformal anomaly coefficient from $\mathrm{a}=\frac{1}{4}[(2N)^2-1]$ to $2\times\frac{1}{4}(N^2-1)$ which is twice the anomaly of a single copy of $\mathrm{SU}(N)$ SYM theory. On the supergravity side, replacing the N^2 coefficient in the above discussion by $(2N)^2$ and noting that the volume of $(S^5/\mathbb{Z}_2)$ is half of the volume of S^5 we end up with $2N^2$ as an overall coefficient.

[6] A similar approach was applied [23] to $\mathrm{SU}(N)$ superconformal $\mathcal{N}=2$ theories admitting a large N limit. Also, the use of sufficiently many terms in the perturbative series as a guide towards some non-perturbative features like singularities was emphasised in [24, 25].

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i.e. $q_{\rm I}^{\rm orb} = \sum c_{n,m}\zeta_{2n+1}\lambda^m$, etc. We find that they have the following behaviour at strong coupling

$$q_{\rm I}^{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} -\frac{3}{128\,\pi^2}\,\lambda^2 + \mathcal{O}\left(\lambda^{3/2}\right), \qquad q_{\rm II}^{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} -\frac{9\,\zeta_3^2}{4096\,\pi^8}\,\lambda^5 + \mathcal{O}\left(\lambda^4\right), \quad (1.22)$$

$$q_{\rm III,IV,V}^{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} -\frac{(k+1)(2k-1)}{16}\,\lambda + \mathcal{O}(1), \qquad k=1,2,3\,. \quad (1.23)$$

The difference in these asymptotics implies that to find the correct strong coupling behaviour of the full $q^{\rm orb}$ one needs first to sum together different subsets of terms and only then expand the total at large λ.

Lacking an analytic method to compute $q^{\rm orb}(\lambda)$ at strong coupling we performed extensive numerical simulations of the SU(N) × SU(N) orbifold matrix model to measure it. This required an extrapolation to large N for finite λ, followed by an analysis of large λ region. We confirmed that the deviation from the $\mathcal{N}=4$ SYM case starts only at the non-planar level. The numerical data agrees with the Padé-Borel resummation of the weak-coupling expansion up to moderate $\lambda \sim 50$. At larger values of the coupling λ we found that the data is compatible with the following asymptotics

$$q^{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} C\,\lambda^{\eta}\left[1 + a_1\lambda^{-1/2} + \dots\right], \quad (1.24)$$

$$\eta = 1.49(2)\,, \qquad C \simeq -0.0049(5)\,, \qquad a_1 \simeq 15.5(5)\,. \quad (1.25)$$

The power of the leading asymptotics $\eta \approx 1.5$ is thus consistent with the string theory prediction (1.13).[7] It is interesting to notice that the values of the coefficients C and Ca_1 are very close to the values of the corresponding coefficients in (1.11) in the SYM case up to a factor of $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively. This suggests a conjecture that the exact form of the strong coupling expansion of $q^{\rm orb}(\lambda)$ is given by

$$q^{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} -\frac{1}{192}\lambda^{3/2} - \frac{9}{128}\lambda + \mathcal{O}\left(\lambda^{1/2}\right), \quad (1.26)$$

It remains to be seen if one can prove this analytically.

One may wonder if the coefficient of the leading $\frac{\lambda^{3/2}}{N^2}$ correction in $\langle \mathcal{W}\rangle^{\rm orb}$ may be found, as in the $\mathcal{N}=4$ SYM case [26], also by considering the circular Wilson loop in k-symmetric representation which should be described (for $k \gg 1$ and $\frac{k\sqrt{\lambda}}{N}$=fixed) by a classical D3-brane solution. This does not seem possible as the D3-brane solution of [26] is restricted to AdS_5 and thus the $\frac{k^3\lambda^{3/2}}{N^2}$ term in its action should have the same coefficient as in the $\mathcal{N}=4$ SYM case, in contradiction with (1.25), (1.26). In fact, the D3-brane solution of [26] should be related not to the SU(N) Wilson loop (1.2) of the SU(N)×SU(N) orbifold theory but to the orbifold projection of the original circular Wilson loop in the SU($2N$) SYM theory. The projection of the latter is represented by the correlator $\langle \mathcal{W}_1\mathcal{W}_2\rangle$ where $\mathcal{W}_{1,2}$ in (1.1) correspond to the two SU(N) factors of the orbifold theory. Starting with the SU($2N$) Wilson loop in k-symmetric representation one is to split it into the sum of products of the two SU(N) representations. Then the D3-brane description may apply only to a special combination of the $\langle \mathcal{W}_1\mathcal{W}_2\rangle$ correlators where $\mathcal{W}_1$ and $\mathcal{W}_2$ are taken in the particular representations of the SU(N) appearing in the product.[8] Assuming that

[7] If η is set to be exactly 3/2, then the best fit value of the coefficient C slightly changes to $-4.7\cdot 10^{-3}$.

[8] We thank N. Drukker for this suggestion.

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k-symmetric representation may be replaced by the k-fundamental one (corresponding to multiply wrapped circle, cf. [27, 28]) one would expect to get the sum of the correlators

$$\sum_{m=0}^{k} \binom{k}{m} \langle \mathcal{W}_1^{(m)} \mathcal{W}_2^{(k-m)} \rangle \tag{1.27}$$

where $\mathcal{W}_{\rm a}^{(m)}$ is the SU(N) Wilson loop in m-fundamental representation. Rescaling the fields in (1.2) (or the corresponding matrices in the matrix model representation as in [11]) one would then end up with the sum of the correlators $\langle \mathcal{W}_1 \mathcal{W}_2 \rangle$ of the two fundamental SU(N) Wilson loops in the SU(N) $\times$ SU(N) quiver theory with the two 't Hooft couplings $\lambda_1 = m^2\lambda$, $\lambda_2 = (m-k)^2\lambda$. The resulting expression should simplify in the large k limit and one expects it to be dominated by the "diagonal" term ($m = k/2$) with $\mathcal{W}_{\rm a}$ in the same representation. In section 6 we shall present numerical data indicating that the $1/N^2$ term in this correlator (with both Wilson loops taken in the fundamental representation of SU(N)) has a similar strong-coupling behaviour to $\frac{1}{96}\lambda^{3/2}$ found in the SYM case in (1.11).

Below we also computed numerically the individual SU(N) Wilson loop (1.1) expectation values $\langle \mathcal{W}_1 \rangle$, $\langle \mathcal{W}_2 \rangle$ in the SU(N) $\times$ SU(N) quiver with unequal couplings λ_1, λ_2 starting with the localization matrix model representation. Guided by the discussion in [4] here we considered the following analog of the ratio in (1.10)[9]

$$\frac{\langle \mathcal{W}_1 \rangle}{w(\theta)\, \langle \mathcal{W} \rangle_0} = p\,(\lambda;\theta) + \frac{1}{N^2}\, q(\lambda;\theta) + \mathcal{O}\left(\frac{1}{N^4}\right) , \tag{1.28}$$

where

$$\lambda = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} , \qquad \theta = 2\pi\, \frac{\lambda_1}{\lambda_1+\lambda_2} , \qquad w(\theta) = \frac{1 - \frac{\theta}{2}\cot\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} , \tag{1.29}$$

The strong coupling result of [4] then implies that $p(\lambda;\theta)\big|_{\lambda\to\infty} \to 1$ for any $\theta \neq 0, 2\pi$. We confirmed this prediction numerically by considering a particular value of the ratio of the two couplings $\lambda_2/\lambda_1 = 3$ (i.e. $\theta = \frac{\pi}{2}$) and measuring both Wilson loops $\langle \mathcal{W}_1 \rangle$ and $\langle \mathcal{W}_2 \rangle$, thus effectively probing also the value $\theta = \frac{3\pi}{2}$. We found that the strong-coupling expansion of the function p has the form $p(\lambda;\theta) = 1 + h(\theta)/\sqrt{\lambda} + \dots$ where h has a non-trivial dependence on θ. The numerical data for the function $q(\lambda;\theta)$ turns out to be compatible with the strong-coupling asymptotics in (1.24) with the exponent η being again close to 3/2 independently of θ, i.e. for both Wilson loops.

The rest of this paper is organized as follows. In section 2 we shall present the matrix model integral representation for the Wilson loop expectation value in the quiver theory which will be our starting point. In section 3 we shall consider the weak coupling expansion of the leading non-planar term the case of the orbifold theory organising it in terms of monomials of transcendental ζ_n factors. We shall also derive the relation between the function $q(\lambda)$ and the free energy. In section 4 we shall perform a resummation of some subsets of terms and then expand them at strong coupling finding non-universal behaviour. In section 5 we shall consider the weak-coupling expansion in the case of non-symmetric quiver theory. Finally, in section 6 we shall present the results of the numerical computation

[9] $\langle \mathcal{W}_2 \rangle$ is found by interchanging λ_1 and λ_2 or $\theta \to 2\pi - \theta$.

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of the matrix model integrals. Appendices A and B will contain some technical details of the computations in sections 3 and 4. In appendix C we shall briefly discuss similar weak-coupling analysis of the Wilson loop in SU(N) "orientifold" $\mathcal{N}=2$ superconformal theory.

2 Matrix model representation

Our starting point will be the localization matrix model representation for the S^4 partition function and the expectation values of the circular Wilson loops (1.1) in the SU(N)×SU(N) $\mathcal{N}=2$ superconformal quiver theory [12] (see also [2–4]). The partition function may be written as the integral over two sets of eigenvalues ($\mathrm{a}=1,2;\ i=1,\dots,N$)

$$Z=\int\prod_{\mathrm{a}=1}^{2}\Big[\prod_{i=1}^{N}da_{\mathrm{a}i}\ \delta\Big(\sum_i a_{\mathrm{a}i}\Big)\prod_{i<j}^{N}(a_{\mathrm{a}i}-a_{\mathrm{a}j})^2\ e^{-\frac{8\pi^2 N}{\lambda_{\mathrm{a}}}\sum_i a_{\mathrm{a}i}^2}\Big]\ f[a_1,a_2]\ , \tag{2.1}$$

where the δ-functions reflect the fact that we are considering the SU(N) case (they may be ignored in strict planar limit) and[10]

$$f[a_1,a_2]=\frac{\prod_{\mathrm{a}}\prod_{i<j}H^2(a_{\mathrm{a}i}-a_{\mathrm{a}j})}{\prod_{i,j}H^2(a_{1i}-a_{2j})}\,,\qquad H(x)\equiv\prod_{n=1}^{\infty}\Big(1+\frac{x^2}{n^2}\Big)^n\ e^{-\frac{x^2}{n}}\,. \tag{2.2}$$

Below we shall use $\langle\cdots\rangle_0$ to denote the (normalized) expectation value in the matrix model with the Gaussian measure so that (2.1) may be written as

$$Z=Z_0\,\langle f\rangle_0\ ,\qquad Z_0=Z^{\rm SYM}(\lambda_1;N)Z^{\rm SYM}(\lambda_2;N)\,, \tag{2.3}$$

$$Z_0=\int\prod_{\mathrm{a}=1}^{2}\Big[\prod_{i=1}^{N}da_{\mathrm{a}i}\ \delta\Big(\sum_i a_{\mathrm{a}i}\Big)\prod_{i<j}^{N}(a_{ai}-a_{aj})^2\ e^{-\frac{8\pi^2 N}{\lambda_{\mathrm{a}}}\sum_i a_{\mathrm{a}i}^2}\Big]\ . \tag{2.4}$$

Here $Z^{\rm SYM}(\lambda;N)$ is the SU(N) SYM partition function on S^4.

[10] $H(x)$ has the following representation in terms of the Barnes function $G(x)$

$$\log H(x)=\log\big[G(1+ix)G(1-ix)\big]-(1+\gamma_{\rm E})\,x^2.$$

The partition function is invariant under $H(x)\to H(x)\,e^{Cx^2}$ [3]. Note also that we ignored the instanton factor [12] in the integrand as we will be interested in perturbative $1/N$ expansion (see [3]).

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The expectation values of the two Wilson loops (1.1) are given by

$$\langle \mathcal{W}_{\rm a} \rangle = \langle \sum_{i=1}^{N} e^{2\pi a_{{\rm a}i}} \rangle = \frac{\langle\, f \,{\rm tr}\, e^{2\pi a_{\rm a}} \rangle_0}{\langle f \rangle_0} \ , \tag{2.5}$$

where $\langle \ldots \rangle$ is given by the same integral as in (2.1) and is normalized so that $\langle 1 \rangle = 1$. We use the notation $a_{\rm a}$ for the diagonal matrix $a_{\rm a} = {\rm diag}(a_{{\rm a}1}, \ldots, a_{{\rm a}N})$. The two expectation values (2.5) are equal (cf. (1.2)) at the orbifold point $\lambda_1 = \lambda_2 = \lambda$.

For large N one may study the saddle points of the "effective action" in (2.1)

$$S[a_1, a_2] = N \sum_{\rm a=1}^{2} \frac{8\pi^2}{\lambda_{\rm a}} \sum_i a_{{\rm a}i}^2 - \log f[a_1, a_2] \ . \tag{2.6}$$

Differentiating over $a_{{\rm a}i}$ and introducing the densities

$$\rho_{\rm a}(x) = \langle\, \frac{1}{N} \sum_{i=1}^{N} \delta(x - a_{{\rm a}i}) \,\rangle \,, \tag{2.7}$$

one finds the following saddle point equations [2, 3]

$$\int_{-\mu_1}^{\mu_1} dy\, \rho_1(y) \left(\frac{1}{x-y} - K(x-y) \right) + \int_{-\mu_2}^{\mu_2} dy\, \rho_2(y)\, K(x-y) = \frac{8\pi^2}{\lambda_1}\, x, \tag{2.8}$$

$$\int_{-\mu_2}^{\mu_2} dy\, \rho_2(y) \left(\frac{1}{x-y} - K(x-y) \right) + \int_{-\mu_1}^{\mu_1} dy\, \rho_1(y)\, K(x-y) = \frac{8\pi^2}{\lambda_2}\, x, \tag{2.9}$$

$$K(x) \equiv -\frac{H'(x)}{H(x)} = x\, [\psi(1+ix) + \psi(1-ix) + 2\gamma_{\rm E}] = -2 \sum_{n=1}^{\infty} (-1)^n\, \zeta_{2n+1}\, x^{2n+1} \ . \tag{2.10}$$

The large N equivalence of the orbifold theory with the $\mathcal{N} = 4$ SYM follows [2, 4] from the fact that for $\lambda_1 = \lambda_2 = \lambda$ the equations (2.8), (2.9) admit the symmetric Ansatz $\rho_1 = \rho_2 = \rho$, $\mu_1 = \mu_2 = \mu$ and reduce to the saddle point equation for the Gaussian matrix model corresponding to the $\mathcal{N} = 4$ SYM case for which

$$\rho(x) = \frac{2}{\pi\mu^2} \sqrt{\mu^2 - x^2} \,, \qquad \mu = \frac{\sqrt{\lambda}}{2\pi} \,. \tag{2.11}$$

The solution of the two integral equations (2.8), (2.9) in the large λ limit was studied in [2], showing that $\langle \mathcal{W}_a \rangle \sim e^{\sqrt{\lambda}}$, $\lambda = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$, and more recently in [4] where it was found that (see (1.29))

$$\langle \mathcal{W}_1 \rangle \overset{\lambda \gg 1}{=} w(\theta)\, W_0 \,, \qquad \langle \mathcal{W}_2 \rangle \overset{\lambda \gg 1}{=} w(2\pi - \theta)\, W_0 \,, \qquad W_0 = N \sqrt{\frac{2}{\pi}}\, \lambda^{-3/4}\, e^{\sqrt{\lambda}} \,. \tag{2.12}$$

W_0 is the leading large N, strong coupling term in the SYM result in (1.3).

3 Weak coupling expansion in the orbifold theory

Considering the orbifold theory case $\lambda_1 = \lambda_2$ one can work out the weak-coupling expansion of $\langle \mathcal{W} \rangle^{\rm orb}$ by starting with the integral representation (2.1), (2.5) for finite N. It may be

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formally written as a sum of functions $W_{\zeta_3}(\lambda,N)$, $W_{\zeta_5}(\lambda,N),\dots$, multiplying particular products of $\zeta_n=\zeta(n)$ values[11]

$$\langle\mathcal{W}\rangle^{\rm orb}=\langle\mathcal{W}\rangle^{\rm SYM}+W(\lambda,N)\,,\qquad W=\zeta_3W_{\zeta_3}+\zeta_5W_{\zeta_5}+\zeta_7W_{\zeta_7}\ +\zeta_3^2W_{\zeta_3^2}+\cdots\,. \tag{3.1}$$

Here $\langle\mathcal{W}\rangle^{\rm SYM}$ is given by (1.4) so that W_{ζ_3}, etc., scale as $1/N$ at large N. For small λ one has

$$W_{\zeta_3}=\mathcal{O}\left(\lambda^3\right),\quad W_{\zeta_5}=\mathcal{O}\left(\lambda^4\right),\quad W_{\zeta_7}=\mathcal{O}\left(\lambda^5\right),\quad W_{\zeta_3^2}=\mathcal{O}\left(\lambda^5\right),\quad \text{etc.} \tag{3.2}$$

To compute these functions starting with (2.1) let us note that using (2.2), (2.10) we get

$$\begin{aligned}\log f&=\sum_{i,j}\left[\frac{1}{2}\sum_{\rm a}\log H^2(a_{{\rm a}i}-a_{{\rm a}j})-\log H^2(a_{1i}-a_{2j})\right] \\ &=2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2N}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{2n+2}C_{n+2}^k\left[\frac{1}{2}\sum_{\rm a}{\rm tr}A_{\rm a}^k\,{\rm tr}A_{\rm a}^{2n+2-k}-{\rm tr}A_1^k\,{\rm tr}A_2^{2n+2-k}\right],\end{aligned} \tag{3.3}$$

where we defined $C_{n+2}^k\equiv(-1)^k\binom{2n+2}{k}$, $a_{\rm a}={\rm diag}(a_{{\rm a}1},\dots,a_{{\rm a}N})$ (with ${\rm tr}\,a_{\rm a}=0$) and we also introduced the rescaled matrices $A_{\rm a}$ (appearing in the exponent in (2.1))

$$A_{\rm a}\equiv\sqrt{\frac{8\pi^2N}{\lambda}}\,a_{\rm a}\ . \tag{3.4}$$

3.1 Direct perturbative expansion

Separating different ζ_n terms we may write f in (3.3) as an expansion in $\frac{\lambda}{8\pi^2N}=\frac{g_{\rm YM}^2}{8\pi^2}$

$$\begin{aligned}f=1&-3\zeta_3\left(\frac{\lambda}{8\pi^2N}\right)^2(T^{(1)}_{2,2}+T^{(2)}_{2,2}-2T^{(1)}_2T^{(2)}_2)\\&+\frac{10}{3}\zeta_5\left(\frac{\lambda}{8\pi^2N}\right)^3\left(3T^{(1)}_{2,4}-2T^{(1)}_{3,3}-3T^{(1)}_4T^{(2)}_2-3T^{(1)}_2T^{(2)}_4+3T^{(2)}_{2,4}-2T^{(2)}_{3,3}\right)+\cdots,\end{aligned} \tag{3.5}$$

$$T^{({\rm a})}_{n_1,n_2,\dots,n_r}\equiv{\rm tr}A_{\rm a}^{n_1}{\rm tr}A_{\rm a}^{n_2}\cdots{\rm tr}A_{\rm a}^{n_r}\ . \tag{3.6}$$

Using (3.3) and computing (2.5) by first integrating out the A_2 dependence with the help of

$$\left\langle{\rm tr}A_2^2\right\rangle_0=\frac{N^2-1}{2},\qquad\left\langle({\rm tr}A_2^2)^2\right\rangle_0=\frac{N^4-1}{4},\qquad\cdots, \tag{3.7}$$

we obtain for the coefficient functions in (3.1)

$$W_{\zeta_3}=-3\Big(\frac{\lambda}{8\pi^2N}\Big)^2\Big\langle{\rm tr}e^{\sqrt{\frac{\lambda}{2N}}A_1}[\,:({\rm tr}A_1^2)^2:+2:{\rm tr}A_1^2:]\Big\rangle_0\,, \tag{3.8}$$

$$\begin{aligned}W_{\zeta_5}=\frac{5}{3N}\Big(\frac{\lambda}{8\pi^2N}\Big)^3\Big\langle{\rm tr}e^{\sqrt{\frac{\lambda}{2N}}A_1}\Big[&9(-2+3N^2):{\rm tr}A_1^2:+\frac{6N^3(6+N^2)}{18-6N^2+N^4}:{\rm tr}A_1^4:\\&+6(-3+2N^2):({\rm tr}A_1^2)^2:+6N:{\rm tr}A_1^2{\rm tr}A_1^4:-4N:({\rm tr}A_1^3)^2:\Big]\Big\rangle_0\,,\ \dots\,,\end{aligned} \tag{3.9}$$

[11] Similar expansions are found in other similar $\mathcal{N}=2$ models, cf. [29, 30].

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where

$$:\mathrm{tr}A_1^2: = \mathrm{tr}A_1^2 - \frac{N^2-1}{2}\,, \qquad \left\langle :\mathrm{tr}A_1^2: \right\rangle_0 = 0\,, \qquad \text{etc.} \tag{3.10}$$

This reduces the problem to evaluating correlators in one-matrix ($A = A_1$) Gaussian model.

Computing the Wilson loop correlator with normal ordered operators using that for the SU(N) SYM case [11]

$$\langle \mathcal{W} \rangle^{\mathrm{SYM}} = \langle \mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}} A_1} \rangle_0 = e^{\frac{\lambda}{8N}\left(1-\frac{1}{N}\right)} L_{N-1}^1\left(-\frac{\lambda}{4N}\right)\,, \tag{3.11}$$

and applying the method described in appendix A (see also [15]), we find ($g \equiv g_{\mathrm{YM}} = \sqrt{\frac{\lambda}{N}}$)

$$\left\langle \mathrm{tr} e^{\sqrt{\frac{\lambda}{2N}} A_1} : \mathrm{tr}A_1^2 : \right\rangle_0 = \frac{g}{2}\partial_g \langle \mathcal{W} \rangle^{\mathrm{SYM}} = \lambda \partial_\lambda \langle \mathcal{W} \rangle^{\mathrm{SYM}}\,, \tag{3.12}$$

$$\left\langle \mathrm{tr} e^{\sqrt{\frac{\lambda}{2N}} A_1} : (\mathrm{tr}A_1^2)^2 : \right\rangle = \left(\frac{g^2}{4}\partial_g^2 - \frac{g}{4}\partial_g\right) \langle \mathcal{W} \rangle^{\mathrm{SYM}} = \lambda^2 \partial_\lambda^2 \langle \mathcal{W} \rangle^{\mathrm{SYM}}\,,$$

$$\begin{aligned}\left\langle \mathrm{tr} e^{\sqrt{\frac{\lambda}{2N}} A_1} : \mathrm{tr}A_1^4 : \right\rangle_0 = \Bigg[&\frac{\lambda(-1+N)(1+N)}{16N^3} + \frac{\lambda(\lambda+8N^2-8N^4)}{8N^3}\partial_\lambda \\ &- \frac{\lambda(\lambda-12N^2)}{N}\partial_\lambda^2 + 8\lambda^2 N \partial_\lambda^3 \Bigg] \langle \mathcal{W} \rangle^{\mathrm{SYM}}\,,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\left\langle \mathrm{tr} e^{\sqrt{\frac{\lambda}{2N}} A_1} : \mathrm{tr}A_1^2 \mathrm{tr}A_1^4 : \right\rangle_0 = \Bigg[&-\frac{\lambda(-1+N)(1+N)}{16N^3} + \frac{\lambda(-1+N)(1+N)(\lambda+16N^2)}{16N^3}\partial_\lambda \\ &+ \frac{\lambda(\lambda^2+8\lambda N^2-96N^4-8\lambda N^4)}{8N^3}\partial_\lambda^2 \\ &- \frac{\lambda^2(\lambda-12N^2)}{N}\partial_\lambda^3 + 8\lambda^3 N \partial_\lambda^4 \Bigg] \langle \mathcal{W} \rangle^{\mathrm{SYM}}\,,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\left\langle \mathrm{tr} e^{\sqrt{\frac{\lambda}{2N}} A_1} : (\mathrm{tr}A_1^3)^2 : \right\rangle_0 = \Bigg[&\frac{\lambda(-1+N)(1+N)(-108+72N^2-9N^4+N^6)}{32N^3(18-6N^2+N^4)} \\ &+ \frac{\lambda(-45\lambda-432N^2+42\lambda N^2+612N^4-7\lambda N^4-216N^6+\lambda N^6+10N^8)}{8N^3(18-6N^2+N^4)}\partial_\lambda \\ &+ \frac{\lambda(18\lambda^2-432\lambda N^2-6\lambda^2 N^2-5184N^4-72\lambda N^4+\lambda^2 N^4+2592N^6+12\lambda N^6-144N^8-8\lambda N^8)}{8N^3(18-6N^2+N^4)}\partial_\lambda^2 \\ &- \frac{2\lambda^2(18\lambda+36N^2-6\lambda N^2-48N^4+\lambda N^4-4N^6)}{N(18-6N^2+N^4)}\partial_\lambda^3 + 8\lambda^3 N \partial_\lambda^4 \Bigg] \langle \mathcal{W} \rangle^{\mathrm{SYM}}\,.\end{aligned} \tag{3.15}$$

Since according to (3.11)

$$\langle \mathcal{W} \rangle^{\mathrm{SYM}} = \frac{2N}{\sqrt{\lambda}} I_1 + \frac{1}{N}\frac{\lambda}{48}\left(I_2 - \frac{12}{\sqrt{\lambda}} I_1\right) + \mathcal{O}\left(\frac{1}{N^3}\right)\,, \qquad I_n \equiv I_n\left(\sqrt{\lambda}\right)\,, \tag{3.16}$$

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we find for (3.8), (3.9)

$$W_{\zeta_3} = -3\left(\frac{\lambda}{8\pi^2 N}\right)^2\left[N\frac{\sqrt{\lambda}}{2}I_1+\frac{1}{N}\left(\frac{\lambda(\lambda-48)}{192}I_0-\frac{\lambda^{3/2}}{24}I_1\right)+\mathcal{O}\left(\frac{1}{N^3}\right)\right], \tag{3.17}$$

$$W_{\zeta_5} = \left(\frac{\lambda}{8\pi^2 N}\right)^3\left[\frac{45}{4}N^2\sqrt{\lambda}I_1+\frac{5}{128}\lambda(-160+3\lambda)I_0-\frac{5}{64}\sqrt{\lambda}(80+9\lambda)I_1+\mathcal{O}\left(\frac{1}{N^2}\right)\right]. \tag{3.18}$$

The expressions (3.17) and (3.18) generate all terms proportional to ζ_3 and ζ_5 at leading and subleading order in $1/N$ in (3.1) within the *weak coupling* expansion. As expected, these functions scale as $1/N$ for large N, i.e.

$$W_{\zeta_3} = -\frac{1}{N}\frac{3}{2}\left(\frac{\lambda}{8\pi^2}\right)^2\sqrt{\lambda}I_1\left(\sqrt{\lambda}\right)+\mathcal{O}\left(\frac{1}{N^3}\right), \qquad W_{\zeta_5} = \frac{1}{N}\frac{45}{4}\left(\frac{\lambda}{8\pi^2}\right)^3\sqrt{\lambda}I_1(\sqrt{\lambda})+\mathcal{O}\left(\frac{1}{N^3}\right). \tag{3.19}$$

Similarly, the leading large N terms in other coefficient functions in (3.1) are given by

$$\begin{aligned}
W_{\zeta_7} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^4\left[-\frac{315}{4}\sqrt{\lambda}I_1+\frac{1}{N^2}\left(-\frac{105}{128}(-56+\lambda)\lambda I_0+\frac{105}{32}\sqrt{\lambda}(28+\lambda)I_1\right)+\cdots\right],\\
W_{\zeta_3^2} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^4\left[9\sqrt{\lambda}I_1+\frac{1}{N^2}\left(\frac{3}{32}\lambda(24+\lambda)I_0-\frac{3}{16}(-24+\lambda)\sqrt{\lambda}I_1\right)+\cdots\right],\\
W_{\zeta_3\zeta_5} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^5\left[-150\sqrt{\lambda}I_1+\frac{1}{N^2}\left(-\frac{5}{16}\lambda(96+5\lambda)I_0+\frac{5}{16}(-288+\lambda)\sqrt{\lambda}I_1\right)+\cdots\right],\\
W_{\zeta_9} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^5\left[\frac{2205}{4}\sqrt{\lambda}I_1+\frac{1}{N^2}\left(\frac{105}{128}\lambda(-384+7\lambda)I_0-\frac{105}{64}\sqrt{\lambda}(576+7\lambda)I_1\right)+\cdots\right],\\
W_{\zeta_3^3} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^6\left[-54\sqrt{\lambda}I_1+\frac{1}{N^2}\left(-\frac{9}{16}\lambda(96+\lambda)I_0-\frac{9}{4}\sqrt{\lambda}(96+\lambda)I_1\right)+\cdots\right],\\
W_{\zeta_5^2} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^6\left[675\sqrt{\lambda}I_1+\frac{1}{N^2}\left(\frac{25}{32}\lambda(80+9\lambda)I_0+\frac{25}{64}\sqrt{\lambda}(640+27\lambda)I_1\right)+\cdots\right],\\
W_{\zeta_3\zeta_7} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^6\left[\frac{2205}{2}\sqrt{\lambda}I_1+\frac{1}{N^2}\left(\frac{105}{64}\lambda(160+7\lambda)I_0+\frac{105}{16}\sqrt{\lambda}(160+3\lambda)I_1\right)+\cdots\right],\\
W_{\zeta_{11}} &= \frac{1}{N}\left(\frac{\lambda}{8\pi^2}\right)^6\left[-\frac{31185}{8}\sqrt{\lambda}I_1+\frac{1}{N^2}\left(-\frac{1155}{256}\lambda(-448+9\lambda)I_0+8085\sqrt{\lambda}I_1\right)+\cdots\right].
\end{aligned} \tag{3.20}$$

3.2 Leading non-planar correction and relation to free energy

Remarkably, the dependence on λ of the leading term $\mathcal{O}(1/N)$ in the W functions in (3.1) follows the same pattern, i.e. is proportional to the Bessel function $I_1(\sqrt{\lambda})$ that appears in the leading order term in the SYM expression (3.16)

$$W_{\prod_{n=1}^r \zeta_{2n+1}^{k_n}} = \frac{c_{k_1\ldots k_r}}{N}\left(\frac{\lambda}{8\pi^2}\right)^{\sum_{n=1}^r k_n(n+1)}\sqrt{\lambda}\,I_1\left(\sqrt{\lambda}\right)+\mathcal{O}\left(\frac{1}{N^3}\right). \tag{3.21}$$

The power of $\frac{\lambda}{8\pi^2}$ is coming from the ζ_{2n+1} factors in (3.3) while the extra factor of λ (from the Bessel function factor $\sqrt{\lambda}\,I_1(\sqrt{\lambda}) = \frac{\lambda}{2}+\cdots$) has its origin in the Wilson loop

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operator insertion into the Gaussian matrix model integral at the leading order in large N (cf. (2.5)). In particular, it comes from the A_1^2 term in ($\mathrm{tr} A_{\mathrm{a}} = 0$)

$$\mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}}\, A_1} = N + \frac{\lambda}{4N} \mathrm{tr}\, A_1^2 + \cdots . \tag{3.22}$$

Separating this SYM Bessel function factor, the expression for the leading large N term in $W = \langle \mathcal{W} \rangle^{\mathrm{orb}} - \langle \mathcal{W} \rangle^{\mathrm{SYM}}$ in (3.1) can be written in terms of the function $q(\lambda)$ defined in (1.10), (1.14)

$$q^{\mathrm{orb}}(\lambda) = q^{\mathrm{SYM}}(\lambda) + \Delta q(\lambda), \qquad \Delta q(\lambda) = \lambda \sum_{n=2}^{\infty} d_n \left(\frac{\lambda}{8\pi^2}\right)^n . \tag{3.23}$$

Here $q^{\mathrm{SYM}}(\lambda)$ was given in (1.11) and the coefficients d_n are found to be (cf. (3.20))

$$\begin{aligned}
&d_2 = -\frac{3}{4}\zeta_3, \qquad d_3 = \frac{45}{8}\zeta_5 \qquad d_4 = -\frac{315}{8}\zeta_7 + \frac{9}{2}\zeta_3^2, \qquad d_5 = -75\zeta_3\zeta_5 + \frac{2205}{8}\zeta_9, \\
&d_6 = -27\zeta_3^3 + \frac{675}{2}\zeta_5^2 + \frac{2205}{4}\zeta_3\zeta_7 - \frac{31185}{16}\zeta_{11}, \\
&d_7 = 630\zeta_3^2\zeta_5 - \frac{41895}{8}\zeta_5\zeta_7 - 3969\zeta_3\zeta_9 + \frac{891891}{64}\zeta_{13}, \\
&d_8 = 162\zeta_3^4 - 4950\zeta_3\zeta_5^2 - 4410\zeta_3^2\zeta_7 + \frac{337365}{16}\zeta_7^2 + \frac{78435}{2}\zeta_5\zeta_9 + \frac{114345}{4}\zeta_3\zeta_{11} - \frac{6441435}{64}\zeta_{15},
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
d_9 = &-4860\zeta_3^3\zeta_5 + \frac{25875}{2}\zeta_5^3 + 69930\zeta_3\zeta_5\zeta_7 + 30618\,\zeta_3^2\zeta_9 - \frac{5190885}{16}\,\zeta_7\zeta_9 \\
&- \frac{4655475}{16}\,\zeta_5\zeta_{11} - \frac{1656369}{8}\,\zeta_3\zeta_{13} + \frac{46930455}{64}\,\zeta_{17}.
\end{aligned}$$

In general, starting from the definition (2.5) of the Wilson loop expectation value (2.5) (e.g. for a = 1), plugging in the expansion of f in (3.5) and taking N large we get, using that the integration over the "decoupled" variable A_2 gives an extra SU(N) SYM factor (cf. (1.10), (1.14) and (2.3), (3.4), (3.11))

$$N \to \infty : \qquad \langle \mathcal{W} \rangle^{\mathrm{orb}} = \langle \mathcal{W} \rangle^{\mathrm{SYM}} \left[1 + \frac{\frac{\sqrt{\lambda}}{2N} I_1 \, \langle : \mathrm{tr} A_1^2 : \, f \rangle_0}{\langle f \rangle_0} \frac{1}{\frac{2N}{\sqrt{\lambda}} I_1} + \mathcal{O}\left(\frac{1}{N^4}\right) \right] , \tag{3.25}$$

$$\Delta q(\lambda) = \lim_{N\to\infty} \frac{\lambda}{4} \frac{\langle : \mathrm{tr} A_1^2 : \, f \rangle_0}{\langle f \rangle_0} = \lim_{N\to\infty} \frac{\lambda^2}{8} \frac{d}{d\lambda} \log \langle f \rangle_0 \ . \tag{3.26}$$

We used (2.2), (3.3) to represent $\Delta q(\lambda)$ in (1.10), (1.14), (3.23) in terms of the Gaussian matrix model expectation value, traded the insertion of $: \mathrm{tr} A_1^2 + \mathrm{tr} A_2^2 :$ for the application of $\lambda \partial_\lambda$ and used the $A_1 \leftrightarrow A_2$ symmetry of the integration measure. The expression (3.26) is equivalent to (1.15), (1.16) representing $\Delta q(\lambda)$ in terms of the free energy $F = -\log Z$ of the orbifold theory (see (2.3)).

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As a check, using (3.3) one can compute the leading terms in the expansion of $\langle f\rangle_0$

$$\begin{aligned}\langle f\rangle_0 &= \frac{Z^{\rm orb}}{Z_0} = 1 - \frac{3(N^2-1)}{N^2}\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{5(N^2-1)(3N^2-2)}{N^4}\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^3 \\ &\quad + \left[\frac{27(N^2-1)(N^2+1)}{2N^4}\zeta_3^2 - \frac{105(N^2-1)(3N^4-4N^2+2)}{4N^6}\zeta_7\right]\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots \\ &\overset{N\to\infty}{=} 1 - 3\,\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + 15\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^3 + \left[\frac{27}{2}\zeta_3^2 - \frac{315}{4}\zeta_7\right]\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots . \end{aligned} \tag{3.27}$$

Hence, for (3.26) we get

$$\Delta q = \frac{\lambda^2}{8}\frac{d}{d\lambda}\lim_{N\to\infty}\log\langle f\rangle_0 = -\frac{3}{4}\zeta_3\lambda\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{45}{8}\zeta_5\lambda\left(\frac{\lambda}{8\pi^2}\right)^3 + \left[\frac{9}{2}\zeta_3^2 - \frac{315}{8}\zeta_7\right]\lambda\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots , \tag{3.28}$$

which is in agreement with (3.23), (3.24).

Thus the problem of computing Δq in (3.23) is reduced to the calculation of the large N limit of the free energy of the $\mathcal{N}=2$ orbifold theory (which is finite for $N\to\infty$ after the subtraction of the planar $\mathcal{N}=4$ SYM term). This method is rather efficient as the direct computation of Δq to higher orders without exploiting the resummation of the λ dependence in the factor $\sqrt{\lambda}I_1(\sqrt{\lambda})$ would be prohibitively difficult. Using it we were able to push the calculation of the ζ_n expansion of $q(\lambda)$ up to $\mathcal{O}(\lambda^{20})$. In particular, the next five terms beyond (3.23) read

$$\begin{aligned} d_{10} &= -972\zeta_3^5 + 55125\zeta_3^2\zeta_5^2 + 33075\zeta_3^3\zeta_7 - \frac{2197125}{8}\zeta_5^2\zeta_7 - \frac{3980025}{16}\zeta_3\zeta_7^2 - \frac{978075}{2}\zeta_3\zeta_5\zeta_9 \\ &\quad + \frac{40730445}{32}\zeta_9^2 - \frac{1715175}{8}\zeta_3^2\zeta_{11} + \frac{157224375}{64}\zeta_7\zeta_{11} + \frac{69015375}{32}\zeta_5\zeta_{13} \\ &\quad + \frac{96621525}{64}\zeta_3\zeta_{15} - \frac{1378048815}{256}\zeta_{19}, \\ d_{11} &= 35640\zeta_3^4\zeta_5 - 280500\zeta_3\zeta_5^3 - 755370\zeta_3^2\zeta_5\zeta_7 + \frac{62375775}{32}\zeta_5\zeta_7^2 - 224532\zeta_3^3\zeta_9 + \frac{15401925}{8}\zeta_5^2\zeta_9 \\ &\quad + \frac{6995835}{2}\zeta_3\zeta_7\zeta_9 + \frac{6887925}{2}\zeta_3\zeta_5\zeta_{11} - \frac{1252497015}{64}\zeta_9\zeta_{11} + \frac{6073353}{4}\zeta_3^2\zeta_{13} \\ &\quad - \frac{2368300935}{128}\zeta_7\zeta_{13} - \frac{2050523475}{128}\zeta_5\zeta_{15} - \frac{88646415}{8}\zeta_3\zeta_{17} + \frac{20364499155}{512}\zeta_{21}, \\ d_{12} &= 5832\zeta_3^6 - 526500\zeta_3^3\zeta_5^2 + \frac{1085625}{2}\zeta_5^4 - 238140\zeta_3^4\zeta_7 + \frac{11618775}{2}\zeta_3\zeta_5^2\zeta_7 + \frac{10405395}{4}\zeta_3^2\zeta_7^2 \\ &\quad - \frac{147697515}{32}\zeta_7^3 + 5154030\zeta_3^2\zeta_5\zeta_9 - \frac{109609605}{4}\zeta_5\zeta_7\zeta_9 - \frac{98765163}{8}\zeta_3\zeta_9^2 + \frac{3087315}{2}zeta_3^3\zeta_{11} \\ &\quad - \frac{108709425}{8}\zeta_5^2\zeta_{11} - \frac{395748045}{16}\zeta_3\zeta_7\zeta_{11} + \frac{9743451795}{128}\zeta_{11}^2 - \frac{196003665}{8}\zeta_3\zeta_5\zeta_{13} \\ &\quad + \frac{4773806037}{32}\zeta_9\zeta_{13} - \frac{173918745}{16}\zeta_3^2\zeta_{15} + \frac{17818765965}{128}\zeta_7\zeta_{15} + \frac{1909927305}{16}\zeta_5\zeta_{17} \\ &\quad + \frac{5236585497}{64}\zeta_3\zeta_{19} - \frac{151323893721}{512}\zeta_{23}, \end{aligned}$$

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$$
\begin{aligned}
d_{13} = &-252720\zeta_3^5\zeta_5+3919500\zeta_3^2\zeta_5^3+7076160\zeta_3^3\zeta_5\zeta_7-15151500\zeta_5^3\zeta_7-\frac{80691975}{2}\zeta_3\zeta_5\zeta_7^2\\
&+1592136\zeta_3^4\zeta_9-39864825\zeta_3\zeta_5^2\zeta_9-35638785\zeta_3^2\zeta_7\zeta_9+\frac{3118780665}{32}\zeta_7^2\zeta_9\\
&+96523245\zeta_5\zeta_9^2-\frac{71138925}{2}\zeta_3^2\zeta_5\zeta_{11}+\frac{1550674125}{8}\zeta_5\zeta_7\zeta_{11}+\frac{1400944545}{8}\zeta_3\zeta_9\zeta_{11}\\
&-\frac{21532797}{2}\zeta_3^3\zeta_{13}+\frac{1549708875}{16}\zeta_5^2\zeta_{13}+\frac{2824276455}{16}\zeta_3\zeta_7\zeta_{13}-\frac{149972417595}{128}\zeta_{11}\zeta_{13}\\
&+\frac{1408331925}{8}\zeta_3\zeta_5\zeta_{15}-\frac{145088958105}{128}\zeta_9\zeta_{15}+\frac{314291835}{4}\zeta_3^2\zeta_{17}-\frac{67093614315}{64}\zeta_7\zeta_{17}\\
&-\frac{7140798405}{8}\zeta_5\zeta_{19}-\frac{38877680205}{64}\zeta_3\zeta_{21}+\frac{1130016089475}{512}\zeta_{25},\\
d_{14} = &-34992\zeta_3^7+4592700\zeta_3^4\zeta_5^2-14713125\zeta_3\zeta_5^4+1666980\zeta_3^5\zeta_7-\frac{159013575}{2}\zeta_3^2\zeta_5^2\zeta_7\\
&-\frac{95527215}{4}\zeta_3^3\zeta_7^2+\frac{319787475}{2}\zeta_5^2\zeta_7^2+\frac{752101245}{8}\zeta_3\zeta_7^3-47508930\zeta_3^3\zeta_5\zeta_9\\
&+104792625\zeta_5^3\zeta_9+556189200\zeta_3\zeta_5\zeta_7\zeta_9+\frac{979410285}{8}\zeta_3^2\zeta_9^2-\frac{21984271155}{32}\zeta_7\zeta_9^2\\
&-\frac{21611205}{2}\zeta_3^4\zeta_{11}+\frac{1105144425}{4}\zeta_3\zeta_5^2\zeta_{11}+\frac{3946846365}{16}\zeta_3^2\zeta_7\zeta_{11}-\frac{5522257125}{8}\zeta_7^2\zeta_{11}\\
&-\frac{10943977275}{8}\zeta_5\zeta_9\zeta_{11}-\frac{39833719695}{64}\zeta_3\zeta_{11}^2+\frac{1990403415}{8}\zeta_3^2\zeta_5\zeta_{13}-\frac{44287988745}{32}\zeta_5\zeta_7\zeta_{13}\\
&-\frac{20042115093}{16}\zeta_3\zeta_9\zeta_{13}+\frac{581635463409}{128}\zeta_{13}^2+\frac{1217431215}{16}\zeta_3^3\zeta_{15}-\frac{22305908625}{32}\zeta_5^2\zeta_{15}\\
&-\frac{20343808485}{16}\zeta_3\zeta_7\zeta_{15}+\frac{2297131261425}{256}\zeta_{11}\zeta_{15}-\frac{10210455255}{8}\zeta_3\zeta_5\zeta_{17}\\
&+\frac{68849861535}{8}\zeta_9\zeta_{17}-\frac{36656098479}{64}\zeta_3^2\zeta_{19}+\frac{2024629961445}{256}\zeta_7\zeta_{19}+\frac{214325672925}{32}\zeta_5\zeta_{21}\\
&+\frac{1160149851861}{256}\zeta_3\zeta_{23}-\frac{16950241342125}{1024}\zeta_{27}.
\end{aligned}
\tag{3.29}
$$

4 Resummation of particular transcendental contributions to $\langle\mathcal{W}\rangle^{\text{orb}}$

In an attempt to shed light on the structure of strong coupling limit of $\langle\mathcal{W}\rangle^{\text{orb}}$ one may try to consider separate terms in the transcendental part of (3.1), resum their weak coupling expansion and then expand at strong coupling. As we shall see, this procedure will not give the correct strong-coupling limit of $\langle\mathcal{W}\rangle^{\text{orb}}$: the strong-coupling asymptotics of different functions $W_{\zeta_n^k\dots}$ will be different. That means that all such terms should first be summed up before taking the large λ limit.

As we shall show in appendix B the terms in (3.1), (3.21) which are proportional to the single ζ_{2n+1} have the following coefficient functions

$$
W_{\zeta_{2n+1}} = \frac{2}{N}\left(\frac{\lambda}{8\pi^2}\right)^{n+1}\frac{(-1)^n}{n+1}\frac{2^{3n-1}\,3\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{(n+2)\,\pi\,\Gamma(n)\Gamma(n+1)}\sqrt{\lambda}\,I_1\left(\sqrt{\lambda}\right)+\mathcal{O}\left(\frac{1}{N^3}\right). \tag{4.1}
$$

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Summing all such terms in $\langle \mathcal{W} \rangle^{\rm orb}$ in (3.1), i.e. $W\big|_\zeta = \sum_{n=1}^\infty W_{\zeta_{2n+1}}$, we then get the corresponding contribution to $q^{\rm orb}$ or to Δq in (1.14), (3.26)

$$\Delta q(\lambda)\Big|_\zeta = \sum_{n=1}^\infty (-1)^n \lambda \left(\frac{\lambda}{8\pi^2}\right)^{n+1} \frac{2^{3n-1}\, 3\, \Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi \Gamma(n)\Gamma(n+3)} \zeta_{2n+1} \,. \tag{4.2}$$

We can resum this series by noting that

$$\zeta_{2n+1} \equiv \zeta(2n+1) = \frac{1}{(2n)!}\int_0^\infty dt\; g(t)\; t^{2n}\,, \qquad g(t) = \frac{1}{e^t-1}\,. \tag{4.3}$$

This gives (J_n are Bessel functions)

$$\Delta q(\lambda)\Big|_\zeta = \frac{\lambda^2}{16\pi}\int_0^\infty dt\, g(t)\, f\left(t\sqrt{\lambda}\right)\,, \tag{4.4}$$

$$f(t) = 3\,[J_0(t)]^2 - \frac{12\, J_0(t)\, J_1(t)}{t} - \frac{3\,(t^2-4)\;[J_1(t)]^2}{t^2}\,. \tag{4.5}$$

Using the properties of the Mellin transform[12] we find that the large λ asymptotics of (4.4) is

$$\Delta q(\lambda)\Big|_\zeta \overset{\lambda\gg 1}{=} \frac{\lambda^2}{16\pi^2}\left[-\frac{3}{8} + \lambda^{-1/2} - \frac{9}{4}\,\zeta_3\,\lambda^{-3/2} + \cdots\right]. \tag{4.6}$$

Similarly, we may consider all terms in (3.1) proportional to $\zeta_3\zeta_{2n+1}$ with $n > 1$ (see appendix B). We get the following analog of (4.2)

$$\Delta q(\lambda)\Big|_{\zeta_3\zeta} = -\sum_{n=2}^\infty (-1)^n \frac{3}{\pi}\frac{2^{3n+3}\, n\, \left[\Gamma(n+\frac{3}{2})\right]^2}{\Gamma(2+n)\Gamma(3+n)}\,\lambda\left(\frac{\lambda}{8\pi^2}\right)^{n+3}\zeta_3\,\zeta_{2n+1}\,. \tag{4.7}$$

Summing this series as in (4.3), (4.4) we get

$$\Delta q(\lambda)\Big|_{\zeta_3\zeta} = -\frac{9\zeta_3^2}{4096\pi^8}\lambda^5 + \frac{3\,\zeta_3}{64\pi^5}\lambda^4\int_0^\infty \frac{dt}{e^{2\pi t}-1}\,\hat f\left(t\sqrt{\lambda}\right), \tag{4.8}$$

$$\hat f(t) = -[J_0(t)]^2 + \frac{5J_0(t)J_1(t)}{t} + \frac{(t^2-6)[J_1(t)]^2}{t^2}. \tag{4.9}$$

Expanding at large λ here gives a different asymptotics than in (4.6)

$$\Delta q(\lambda)\Big|_{\zeta_3\zeta} \overset{\lambda\gg 1}{=} -\frac{9\zeta_3^2}{4096\pi^8}\lambda^5 + \frac{3\,\zeta_3}{1024\pi^6}\lambda^4\left[1 - \frac{4\pi^2}{3}\lambda^{-1} + 24\zeta_3\lambda^{-3/2} + \cdots\right] \tag{4.10}$$

As another example one may consider all the terms in W involving only powers of ζ_3. The resulting contribution takes a simple form (see appendix B)

$$\Delta q(\lambda)\Big|_{\sum_n \zeta_3^n} = \sum_{n=1}^\infty \frac{1}{4}(-1)^n 2^{n-1} 3^n\,\lambda\left(\frac{\lambda}{8\pi^2}\right)^{2n}\zeta_3^n = -\frac{3\lambda^3\zeta_3}{8(32\pi^4+3\lambda^2\zeta_3)} \overset{\lambda\gg 1}{=} -\frac{\lambda}{8} + \cdots\,, \tag{4.11}$$

with the large λ asymptotics being again different from (4.6) and (4.10).

[12] Defining the Mellin transform $\widetilde f(s) = \int_0^\infty dx\, x^{s-1}\, f(x)$ and considering the convolution $(f\star g)(x) = \int_0^\infty dt\, f(t\,x)\, g(t)$ we have $\widetilde{(f\star g)}(s) = \widetilde f(s)\,\widetilde g(1-s)$. Let $\alpha < s < \beta$ be the fundamental strip of analyticity of $\widetilde f(s)$. The asymptotic expansion of $f(x)$ for $x\to\infty$ is obtained by looking at the poles of $\widetilde f(s)$ in the region $s \geq \beta$. Then the pole $\frac{1}{(s-s_0)^N}$ in the Mellin transform leads to the term $\frac{(-1)^N}{(N-1)!}\frac{1}{x^{s_0}}\log^{N-1} x$ in the original function.

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Using the general method described in appendix B one is able to generalize (4.11) to the sum of all contributions involving arbitrary powers of ζ_3 and ζ_5 and also of ζ_7

$$\Delta q(\lambda)\big|_{\sum_{n,m}\zeta_3^n\zeta_5^m} = \frac{3\lambda(-2t_3+15t_5-50t_3t_5+450t_5^2+750t_5^3)}{8(1+10t_5)(1+6t_3-40t_5-50t_5^2)}\Big|_{t_3=\zeta_3(\frac{\lambda}{8\pi^2})^2, t_5=\zeta_5(\frac{\lambda}{8\pi^2})^3}$$
$$\overset{\lambda\gg 1}{=} -\frac{9}{16}\lambda+\dots \tag{4.12}$$

$$\Delta q(\lambda)\big|_{\sum_{n,m,k}\zeta_3^n\zeta_5^m\zeta_7^k} = -\lambda\,\frac{N(t_3,t_5,t_7)}{D(t_3,t_5,t_7)}\Big|_{t_3=\zeta_3(\frac{\lambda}{8\pi^2})^2,\ t_5=\zeta_5(\frac{\lambda}{8\pi^2})^3,\ t_7=\zeta_7(\frac{\lambda}{8\pi^2})^4}$$
$$\overset{\lambda\gg 1}{=} -\frac{5}{4}\lambda+\dots\,, \tag{4.13}$$

$$\begin{aligned} N(t_3,t_5,t_7) = -3(&32t_3-240t_5+800t_3t_5-7200t_5^2-12000t_5^3+1680t_7-8400t_3t_7\\ &+135240t_5t_7+25200t_3t_5t_7+252000t_5^2t_7-635040t_7^2\\ &-323400t_3t_7^2-1670900t_5t_7^2+343000t_5^2t_7^2+5227320t_7^3\\ &-720300t_3t_7^3-3344250t_5t_7^3+18727800t_7^4+6302625t_7^5),\\ D(t_3,t_5,t_7) = 4(&4+40t_5-420t_7-735t_7^2)(-8-48t_3+320t_5+400t_5^2-2100t_7\\ &-840t_3t_7-3360t_5t_7+16170t_7^2+5145t_7^3)\,. \end{aligned} \tag{4.14}$$

Comparing (4.11), (4.12) and (4.13) suggests that the strong coupling limit of the sum of monomials involving powers of the first k constants $\zeta_3,\zeta_5,\dots,\zeta_{2k+1}$ should be (cf. (4.11), (4.12), (4.13))

$$\Delta q(\lambda)\big|_{\sum_{n_1,\dots,n_k}\zeta_3^{n_1}\dots\zeta_{2k+1}^{n_k}} \overset{\lambda\gg 1}{=} -\frac{1}{16}(k+1)(2k-1)\lambda+\dots\,, \qquad k=1,2,3,\dots \tag{4.15}$$

Since the coefficient in (4.15) grows with k, summing up such contributions *after* taking the large λ limit would not give a meaningful result.

5 Weak coupling expansion for non-symmetric quiver

Let us now discuss the expectation value of the Wilson loops (1.1), (2.5) in the case of the SU(N) × SU(N) quiver for unequal couplings $\lambda_1 \neq \lambda_2$. We shall consider for definiteness $\langle\mathcal{W}\rangle \equiv \langle\mathcal{W}_1\rangle$. Setting

$$\lambda_2 = \rho\,\lambda_1\,, \tag{5.1}$$

the generalization of (3.3) will read

$$\begin{aligned} \log f = 2\sum_{n=1}^{\infty}\left(\frac{\lambda_1}{8\pi^2 N}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{2n+2}(-1)^k\binom{2n+2}{k}\\ \times\left[\frac{1}{2}\sum_a \mathrm{tr}A_1^k\,\mathrm{tr}A_1^{2n+2-k}+\rho^{n+1}\frac{1}{2}\mathrm{tr}A_2^k\,\mathrm{tr}A_2^{2n+2-k}-\rho^{n+1-\frac{k}{2}}\,\mathrm{tr}A_1^k\mathrm{tr}A_2^{2n+2-k}\right]. \end{aligned} \tag{5.2}$$

It is then straightforward to compute the large N expansion of the coefficient functions of the ζ_n-monomial contributions to $\langle\mathcal{W}\rangle$ in the analog of (3.1). The first of them that

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generalizes (3.17) is

$$W_{\zeta_3} = \Big(\frac{\lambda_1}{8\pi^2 N}\Big)^2 \Big[3N^3\,(\rho-1)\,I_2(\sqrt{\lambda_1}) - N\Big(3(\rho-1)\Big(1+\frac{1}{8}\lambda_1\Big)\,I_0(\sqrt{\lambda_1}) - \frac{(192+\lambda_1^2)(\rho-1)-48\lambda_1}{32\sqrt{\lambda_1}} I_1(\sqrt{\lambda_1})\Big) + \mathcal{O}(\frac{1}{N})\Big]. \tag{5.3}$$

The planar (order N) contribution here agrees with the $N\to\infty$ part of the $\mathcal{N}=4$ SYM result in (1.3) expressed in terms of the effective coupling [29, 31, 32] (g^2 there is $\frac{\lambda_1}{16\pi^2}$)[13]

$$\frac{2N}{\sqrt{\lambda}_{\rm eff}} I_1(\sqrt{\lambda}_{\rm eff}) = \frac{2N}{\sqrt{\lambda_1}} I_1\Big(\sqrt{\lambda_1}\Big) + 3N\,\zeta_3\,(\rho-1)\Big(\frac{\lambda_1}{8\pi^2}\Big)^2 I_2\Big(\sqrt{\lambda_1}\Big) + \cdots, \tag{5.4}$$

$$\lambda_{\rm eff} = \lambda_1 + 12\,\zeta_3\,(\rho-1)\,\frac{\lambda_1^3}{(16\pi^2)^2} + \cdots\,. \tag{5.5}$$

Note that the subleading terms in (5.3) proportional to $\rho-1$ are similarly captured by the SYM term if we modify (5.5) as

$$\lambda_{\rm eff} = \lambda_1 + 12\,\zeta_3\,(\rho-1)\Big(1-\frac{1}{N^2}\Big)\,\frac{\lambda_1^3}{(16\pi^2)^2} + \dots \tag{5.6}$$

One can also find the analog of W_{ζ_5} in (3.1), (3.18) and the $\rho-1$ terms there can be generated from the SYM expression by the replacement $\lambda\to\lambda_{\rm eff}$ generalizing (5.6)[14]

$$\begin{aligned}\lambda_{\rm eff} = \lambda_1 &+ 12\,\zeta_3\,(\rho-1)\Big(1-\frac{1}{N^2}\Big)\,\frac{\lambda_1^3}{(16\pi^2)^2} \\ &- \zeta_5\,(\rho-1)\Big[40(\rho+3)\Big(1-\frac{7}{2N^2}\Big)\frac{\lambda_1^4}{(16\pi^2)^3} + \frac{40\pi^2}{3}\Big(1-\frac{2(13+3\rho)}{N^2}\Big)\frac{\lambda_1^5}{(16\pi^2)^4} \\ &+ \frac{64\pi^4}{9}\Big(1+\frac{74+15\rho}{4N^2}\Big)\frac{\lambda_1^6}{(16\pi^2)^5} + \mathcal{O}(\lambda_1^7)\Big] + \dots\,.\end{aligned} \tag{5.7}$$

This suggests that some essential features of the weak coupling expansion of the Wilson loop in the non-symmetric quiver case are already captured by the orbifold case ($\rho=1$) discussed above.

6 Numerical analysis of the quiver matrix model

One may try to compute the Wilson loop (1.1) numerically at finite N and λ by starting with the matrix model representation (2.1), (2.5). While this is a finite dimensional integral, the fact that are interested in the limit $N\gg 1$ makes the numerical integration hard. At the same time, we expect that, in the large N limit, the relevant subset of the integration domain reduces to a neighbourhood of the saddle point solution. This problem is completely analogous to the one in the lattice field theory (where one computes quantum corrections by numerical path integration with $N\sim\hbar^{-1}$) and may thus be dealt with by the same Monte Carlo (MC) methods (see, for instance, [34]).

[13] In the case of the $\lambda_1\neq\lambda_2$ quiver the weak coupling expansion in the planar limit was analysed also in [33] and [24].

[14] The $N\to\infty$ limit of this expression is in agreement with eq. (35) in [29].

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6.1 Orbifold theory

We analysed the Wilson loop expectation value (3.1) in the orbifold case by means of a Metropolis-Hastings Monte Carlo simulation [35] of the integral (2.1), a robust approach that does not require fine tuning.[15]

Given a configuration X of the eigenvalues a_{ai} corresponding to the two SU(N) groups, the matrix integral (2.1) weights each observable $\mathcal{O}(X)$, like the Wilson loop, with a positive number $\exp(-S)$ where $S = S(X)$ corresponds to the total integrand in (2.1) including the Vandermonde factor. A Markov chain obeying detailed balance is built by making a local variation of $X \to X'$ and accepting the new configuration if $S(X') < S(X)$ or, in the case $S(X') > S(X)$, with probability $e^{S(X)-S(X')}$. We tuned the local changes of configuration in order to have an acceptance probability around 50–60% which is a reasonable choice. Iterating this procedure produces a sequence $\{X_n\}$ of configurations distributed according to $\exp(-S)$ and one can measure the quantum expectation value as the ensemble average $\langle\mathcal{O}\rangle = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathcal{O}(X_m)$. The sequence $\{X_n\}$ is correlated and its autocorrelation time has been measured at each data point and taken into account in the estimate of an error in this MC evaluation of $\langle\mathcal{O}\rangle$.[16]

For each value of λ, we ran our code at various values of N and fitted the Wilson loop measurements in order to extract the function $q^{\rm orb}(\lambda)$ (3.23) that governs the leading non-planar correction in (1.10). The procedure is illustrated in figure 1 (left) at the value $\lambda = 1$. figure 1 (right) shows the histogram of measurements of the Wilson loop at $\lambda = 200$, $N = 20$, as a sample point.

To provide non-trivial checks of the numerical code we considered the Wilson loop at $\lambda = 1$ which is a relatively weak coupling. From (3.23), (3.24) we see that for this value the leading ζ_n contributions are negligible and we may assume that the same is true also for higher order contributions. Then

$$q^{\rm orb}(1) = -0.122(2)\,, \tag{6.1}$$

where the error is an estimate of the systematic error determined by including or not the contribution of the ζ_n terms explicitly computed above. The extrapolated slope from the finite N MC simulations at $\lambda = 1$ shown in figure 1 (left) gives

$$\text{MC}: \qquad q^{\rm orb}(1) = -0.12(1)\,, \tag{6.2}$$

which is thus consistent with the analytic estimate (6.1).

[15] For other papers using MC methods in matrix models see, e.g., [36–41] and section 6.5 of [42]. In the explored region of parameter space the Metropolis-Hastings algorithm turned out to be faster than the Hybrid Monte Carlo one, taking into account autocorrelation. The latter algorithm is expected to be preferable at higher N and possibly more efficient for large-scale simulations which are beyond the scope of the present analysis.

[16] As is well known (see, for instance, [43]), denoting by $\langle\cdots\rangle_{\rm MC}$ the average over MC realisations and assuming an exponential autocorrelation for the measurements $\mathcal{O}_n = \mathcal{O}(X_n)$, i.e. $\langle\mathcal{O}_n\mathcal{O}_m\rangle_{\rm MC} = \sigma_{\mathcal{O}}^2\, e^{-|n-m|/\tau_{\mathcal{O}}}$, the variance of the expectation value estimator $\frac{1}{n}\sum_{m=1}^{n}\mathcal{O}(X_m)$ is $\frac{1}{n^2}\sum_{m,k}^{n}\langle\mathcal{O}_m\mathcal{O}_k\rangle_{\rm MC} \sim \frac{2\tau_{\mathcal{O}}+1}{n}\sigma_{\mathcal{O}}^2$ showing that the effective number of decorrelated measurements is roughly $n_{\rm decorr} = n/(2\tau_{\mathcal{O}}+1)$ which is the factor entering the standard deviation of measurements $\sigma = \sigma_{\mathcal{O}}/\sqrt{n_{\rm decorr}}$.

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To compare results at higher values of λ we need to resum the perturbative expansion of $\Delta q(\lambda)$ in (3.23), (3.24), (3.29). We performed a Borel-Padé resummation for values of λ up to 50, see figure 2. The red line there is the perturbative series which is expected to converge for $|\lambda| < \pi^2$ with partial sums blowing up beyond that value.[17] The green line is the [7/6] Padé approximant of the Borel improved series, while the blue line is the Borel transform, which is thus in good agreement with the numerical data points.

At higher values of λ we found similar extrapolations in $1/N^2$. In figure 3 (left) we show the intercept of the extrapolation which is expected to be 1, see (3.20). This is a measure of the systematic error associated with the fit of the N dependence. It increases with λ and we increased the maximal N in order to keep it below the 0.2% level.[18]

The resulting function $q^{\rm orb}(\lambda)$ computed for up to $\lambda = 450$ is shown in figure 3 (right). In the SU(N) $\mathcal{N} = 4$ SYM theory, we know from (1.11) that at strong coupling $q^{\rm SYM}(\lambda) = \frac{1}{96}\lambda^{3/2} + \dots$ which is valid with high accuracy already at $\lambda \gtrsim 20$. In the orbifold theory we find that $q^{\rm orb}(\lambda)$ is negative with a clear bending at large λ suggesting an asymptotic behaviour

$$q^{\rm orb}(\lambda) \sim -\lambda^{\eta}\,, \qquad \eta > 1\,. \tag{6.3}$$

The best fit of the blue data points in figure 3 (right) gives $\eta = 1.49(2)$ where the conservative error estimate includes statistics as well as the systematic effects due to the choice of fitting window. We estimated the latter by dropping some of the data points at smaller values of λ. This exponent is still to be taken with some caution since it is hard to say whether we are already in the asymptotic $\lambda \to \infty$ region but it appears to match the string theory prediction in (1.13) (see also (1.24), (1.25)).

Finally, motivated by the discussion of the possible role of the D3-brane solution of [26] in the SU(N) $\times$ SU(N) orbifold theory (see Introduction), we numerically computed the expectation value $\langle \mathcal{W}_1 \mathcal{W}_2 \rangle$ of the two SU(N) Wilson loops (1.1) and determined (using the same fitting procedure as discussed above) the associated $q_{\mathcal{W}\mathcal{W}}(\lambda)$ function defined as in (1.10)

$$\frac{\langle \mathcal{W}_1 \mathcal{W}_2 \rangle}{\langle \mathcal{W} \rangle_0^2} = 1 + \frac{1}{N^2}\, q_{\mathcal{W}\mathcal{W}}(\lambda) + \mathcal{O}\left(\frac{1}{N^4}\right). \tag{6.4}$$

The corresponding data points are shown in figure 6. They decrease to negative values with rate slower than the one observed in $q^{\rm orb}(\lambda)$. A best fit of the form (1.24) with η fixed at $\frac{3}{2}$ gives $C_{\mathcal{W}\mathcal{W}} = +0.012(2)$ and $a_{1_{\mathcal{W}\mathcal{W}}} = -21(2)$. The coefficient $C_{\mathcal{W}\mathcal{W}}$ has the opposite sign to the one in (1.25) and is close to the SYM value $\frac{1}{96} \approx 0.010$ in (1.11). One possible interpretation of this result is that the "diagonal" correlator $\langle \mathcal{W}_1 \mathcal{W}_2 \rangle$ of the two Wilson loops in the fundamental representation exhibits the (at the leading non-planar order) the

[17] This is the radius of convergence of perturbative expansion in SYM theory in the planar limit. Its origin may be attributed to the form of the single-magnon dispersion relation, which follows from superconformal symmetry [44, 45] and it may also be found using the quantum algebraic curve approach [46]. That such a singularity is also present in the $\mathcal{N} = 2$ theories was first noticed in the mass-deformed $\mathcal{N} = 2^*$ theory case in [47].

[18] Let us note that our numerical analysis is in a region of values of (λ, N) expected to be free from the instanton corrections which are weighted by the typical $\exp(-8\pi^2 N/\lambda)$ factors, at least up to instanton moduli space volume corrections [48].

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strong coupling behaviour which is expected from the D3-brane description, while other terms appearing in (1.27) are less important in the large k limit.

6.2 Non-symmetric quiver

In the case of generic (non-zero) λ_1 and λ_2 the strong-coupling asymptotics of the Wilson loops is given by (2.12). We shall study the functions $p(\lambda, \theta)$ and $q(\lambda, \theta)$ in the ratio (1.28) of $\langle \mathcal{W} \rangle_1$ to the planar SYM result. We begin with the special point $\theta = \frac{\pi}{2}$ or (see (1.29))

$$\lambda_2 = 3\,\lambda_1 \ : \qquad \lambda = \frac{3}{2}\lambda_1, \qquad w\left(\frac{\pi}{2}\right) = 2 - \frac{\pi}{2} = 0.429\ldots\ . \tag{6.5}$$

The numerical results are shown in figure 4. The left panel gives the function $p(\lambda, \frac{\pi}{2})$. As expected, it decreases for large λ towards 1 (this should hold for any θ, see (2.12)) and a good fit is

$$p\left(\lambda, \frac{\pi}{2}\right) = 1.00 + 0.23\,\lambda^{1/2} + 8.2\,\lambda^{-1} + \ldots\ . \tag{6.6}$$

Measurement of the second Wilson loop $\langle \mathcal{W} \rangle_2$ provides the information about the same functions at the complementary value of the angle $\theta' = 2\pi - \theta = \frac{3\pi}{2}$ for which

$$\lambda_1 = 3\,\lambda_2 \ : \qquad \lambda = \frac{3}{2}\lambda_1, \qquad w\left(\frac{3\pi}{2}\right) = 2 + \frac{3\pi}{2} = 6.712\ldots\ . \tag{6.7}$$

The corresponding results are shown in figure 5. The best fit for the $p(\lambda, \frac{3\pi}{2})$ is[19]

$$p\left(\lambda, \frac{3\pi}{2}\right) = 0.99 - 3.2\,\lambda^{1/2} + 3.4\,\lambda^{-1} + \ldots\ . \tag{6.8}$$

The function $q(\lambda; \theta)$ at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ is shown in the right panels of figure 4 and figure 5. Our estimate for the exponent $\eta(\theta)$ in the analog of (6.3) is $\eta(\frac{\pi}{2}) = 1.3(2)$ and $\eta(\frac{3\pi}{2}) = 1.6(2)$. Both values appear to be similar to the one found in the orbifold case ($\theta = \pi$), i.e. $\eta \approx \frac{3}{2}$. It would be desirable to push the MC simulation to larger values of the coupling λ, but that seems to require a dedicated analysis with a substantially increased computational power.

Acknowledgments

We would like to thank N. Drukker, S. Giombi, J. Russo, and K. Zarembo for useful discussions and comments on the draft. MB was supported by the INFN grant GSS (Gauge Theories, Strings and Supergravity). AAT was supported by the STFC grant ST/T000791/1.

[19] The small but not negligible deviation of the estimated asymptotic value from 1 suggests that systematic errors should be reduced by $N \to \infty$ extrapolations with larger values of N. This could be related to the much large value of the correcting factor $w(\frac{3\pi}{2})$ as compared to $w(\frac{\pi}{2})$.

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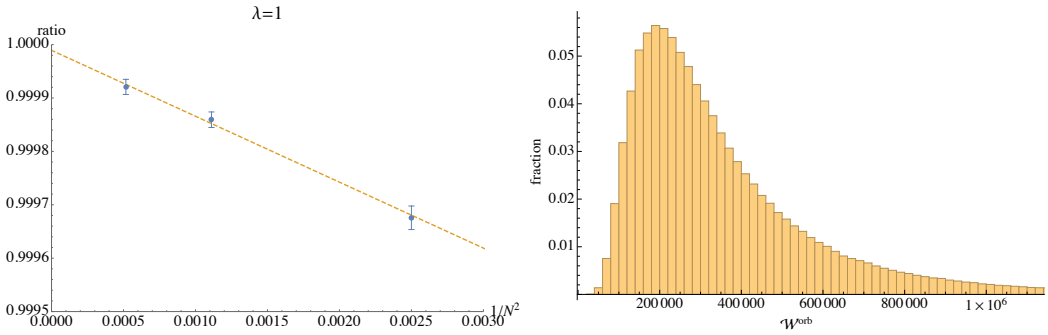

Figure 1. *Left:* fit of the ratio $\frac{\langle\mathcal{W}\rangle^{\rm orb}-\langle\mathcal{W}\rangle^{\rm SYM}}{\langle\mathcal{W}\rangle_0}$ (see (1.3), (3.1)) with a linear function of $1/N^2$ for $\lambda = 1$. The three data points correspond to $N = 20, 30, 44$. It is not necessary to take larger values since the intercept is already very close to the expected value 1. *Right*: histogram of the Monte Carlo measurements of the orbifold Wilson loop from simulation at $\lambda = 200$, $N = 20$. For each (uncorrelated) Monte Carlo step, one records the measured value of $\mathcal{W}^{\rm orb}$ and the plot shows the binned relative frequencies. The best statistical estimator for the quantum expectation value $\langle\mathcal{W}\rangle$ is the mean value of this empirical distribution.

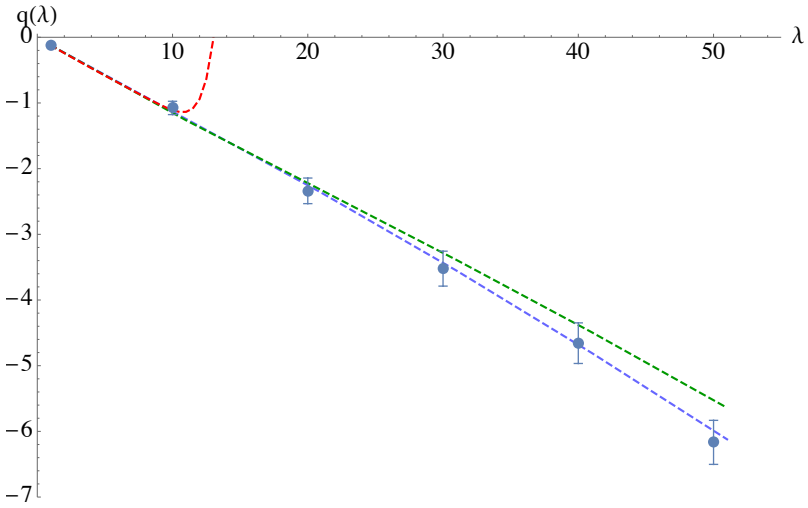

Figure 2. Borel-Padé resummation of the perturbative expansion of $q^{\rm orb}(\lambda)$. The red line is the perturbative expansion (3.24), (3.29) rapidly breaking down around $\lambda = \pi^2$. The green line is its [7/6] Pade' approximant already close to data, while the blue line is its numerical Borel transform.

A Multi-trace SU(N) recursion relations and $\langle\mathcal{W}\,\mathcal{O}\rangle_0$

The correlators $\langle\mathcal{W}\,\mathcal{O}\rangle_0$ in a Gaussian one-matrix model of the Wilson loop operator $\mathcal{W} = \mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}}A}$ and a multi-trace chiral operator $\mathcal{O}$ may be reduced to a differential operator over the coupling constant acting on $\langle\mathcal{W}\rangle_0$ (see (3.12)–(3.15)). This relation is exact at finite N and is achieved by exploiting the SU(N) fusion/fission relations [49] and the associated recursion relations on the expectation values

$$t_{n_1,n_2,\ldots,n_r} \equiv \langle \mathrm{tr}A^{n_1}\mathrm{tr}A^{n_2}\cdots\mathrm{tr}A^{n_r}\rangle_0 \ . \tag{A.1}$$

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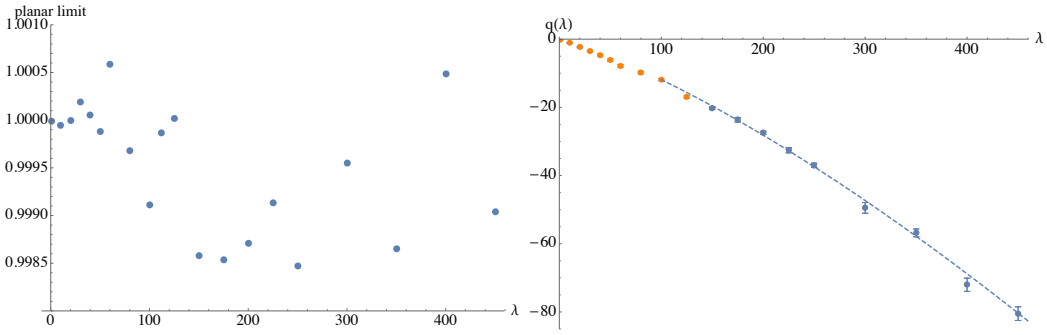

Figure 3. *Left*: intercept in the large N extrapolation which should be equal to 1 due to the planar equivalence with the $\mathcal{N}=4$ SYM. The deviation is a measure of the systematic error which can be reduced at the price of increasing the maximal N used in the simulations and in the extrapolation to $N=\infty$. *Right:* data points for the function $q^{\rm orb}(\lambda)$ defined in (1.10), (3.23). Dashed line is the non-linear fit with the functional form $q^{\rm orb}(\lambda) = C\,\lambda^{\eta}(1+a_1\lambda^{-1/2})$. The fit is performed using data points with $\lambda \geq 100$ which have been determined with a relative error below 3% .

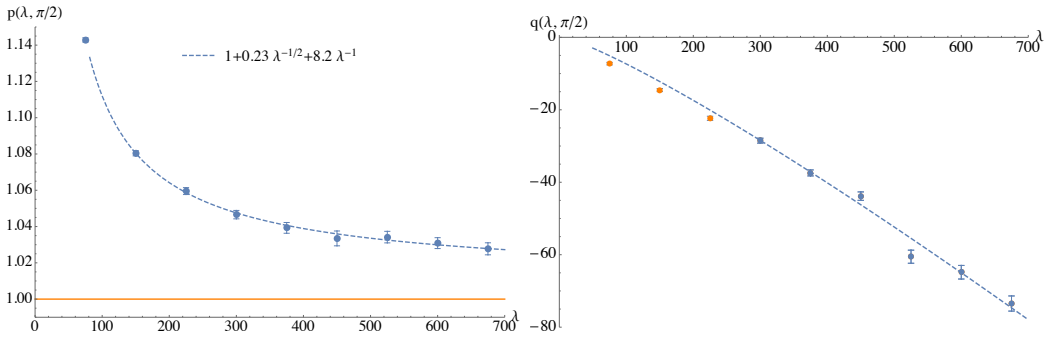

Figure 4. Functions $p(\lambda, \frac{\pi}{2})$ (left) and $q(\lambda, \frac{\pi}{2})$ (right) for the quiver at the point $\lambda_2 = 3\lambda_1$, with $\lambda = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} = \frac{3}{2}\,\lambda_1$.

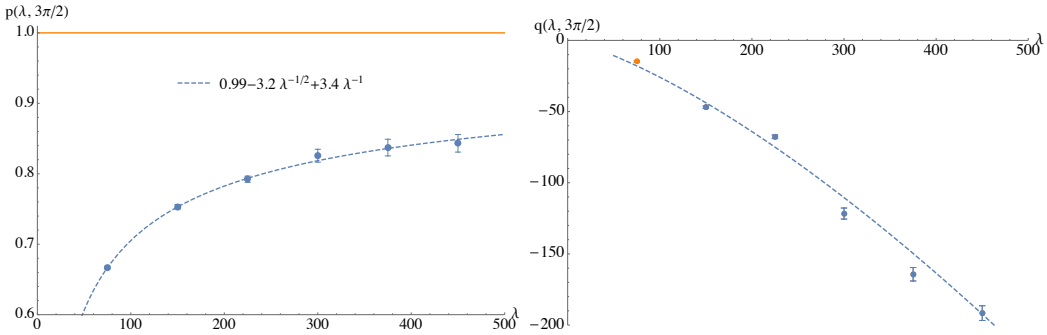

Figure 5. Functions $p(\lambda, \frac{3\pi}{2})$ (left) and $q(\lambda, \frac{3\pi}{2})$ (right) for the quiver at the point $\lambda_1 = 3\lambda_2$, with $\lambda = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} = \frac{3}{2}\,\lambda_2$. The angle $\theta = \frac{3\pi}{2}$ corresponds to the Wilson loop for the second SU(N) factor.

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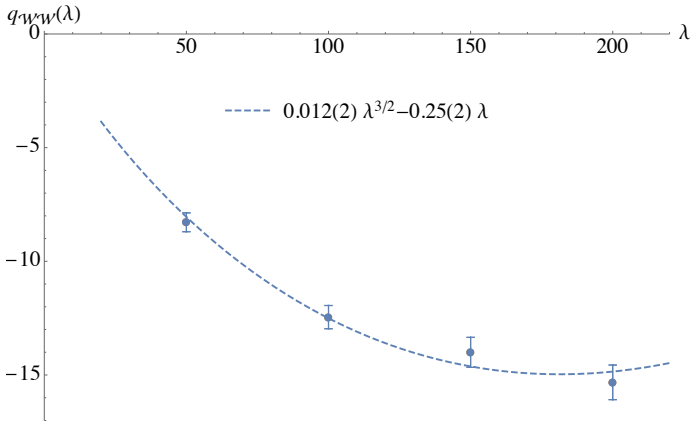

Figure 6. Data for the $q_{\mathcal{W}\mathcal{W}}(\lambda)$ function in (6.4) controlling the $1/N^2$ correction to $\langle \mathcal{W}_1 \mathcal{W}_2 \rangle$ in the orbifold theory.

Let us consider as an example $\langle \mathcal{W} : \mathrm{tr}A^6 : \rangle_0$. We find ($g = \sqrt{\frac{\lambda}{N}}$)

$$
\begin{aligned}
: \mathrm{tr}A^6 : &= \mathrm{tr}A^6 - 3N\mathrm{tr}A^4 + \frac{15}{4}N(\mathrm{tr}A)^2 - 3\mathrm{tr}A\mathrm{tr}A^3 - \frac{3}{2}(\mathrm{tr}A)^2 \\
&\quad + \frac{15}{4}\left(N^2+1\right)\mathrm{tr}A^2 - \frac{5}{8}\left(2N^2+N^4\right), \\
\left\langle \mathcal{W} : \mathrm{tr}A^6 : \right\rangle_0 &= \sum_{k=0}^{\infty} \frac{g^k}{2^{\frac{k}{2}}\, k!}\Big(t_{k,6} - 3N t_{k,4} + \frac{15}{4}\left(N^2+1\right) t_{k,2} \\
&\qquad - \frac{5}{8}\left(2N^2+N^4\right) t_k + \frac{15}{4} N t_{k,1,1} - 3\, t_{k,1,3} - \frac{3}{2} t_{k,2,2} \Big). \qquad \text{(A.2)}
\end{aligned}
$$

Doing Wick contractions leads to a combination of "single-trace" terms that can be traded for ∂_g differential operators acting on $\langle \mathcal{W} \rangle_0$ and we finally obtain

$$
\begin{aligned}
\left\langle \mathcal{W} : \mathrm{tr}A^6 : \right\rangle_0 &= \sum_{k=0}^{\infty} \frac{g^k}{2^{\frac{k}{2}}\, k!}\left[\frac{k}{2} t_{k+4} - kN t_{k+2} - \frac{3}{8} k \left(-1+2k-N^2\right) t_k + \frac{3}{16} k(k-1) N t_{k-2} \right] \\
&= \sum_{k=0}^{\infty} \frac{g^k}{2^{\frac{k}{2}}\, k!}\left[\frac{2k(k-1)(k-2)(k-3)(k-4)}{g^4} - \frac{3}{8} k \left(-1+2k-N^2\right) \right. \\
&\qquad \left. - \frac{2Nk(k-1)(k-2)}{g^2} + \frac{3g^2 N}{32} \right] t_k \\
&= \left[2g\partial_g^5 + \frac{3}{8}\left(N^2+1\right) g\partial_g - \frac{3}{4}(g\partial_g)^2 - 2N\, g\partial_g^3 + \frac{3N}{32} g^2 \right] \langle \mathcal{W} \rangle \ . \qquad \text{(A.3)}
\end{aligned}
$$

This procedure can be easily coded in symbolic manipulation programs.

B Coefficient functions of ζ-terms in $\langle \mathcal{W} \rangle^{\mathbf{orb}}$

Here we shall provide some details of the weak-coupling computation of the coefficient functions W_ζ in (3.1). Generalizing the calculation in (3.27), the contribution proportional

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to a single ζ_{2n+1} to the expectation value $\langle f\rangle_0$ is given by

$$\begin{aligned}\langle f\rangle_0 &= 1+2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2 N}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{2n+2}(-1)^k\binom{2n+2}{k}\left\langle \mathrm{tr}A_1^k\,\mathrm{tr}A_1^{2n+2-k}\right\rangle_{0,\mathrm{c}}+\mathcal{O}\left(\zeta^2\right)\\ &= 1+2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2 N}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{n+1}\binom{2n+2}{2k}\left\langle \mathrm{tr}A_1^{2k}\,\mathrm{tr}A_1^{2(n-k+1)}\right\rangle_{0,\mathrm{c}}\\ &\quad -2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2 N}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{n}\binom{2n+2}{2k+1}\left\langle \mathrm{tr}A_1^{2k+1}\,\mathrm{tr}A_1^{2n-2k+1}\right\rangle_{0,\mathrm{c}}+O\left(\zeta^2\right).\end{aligned} \tag{B.1}$$

Using that the connected correlators are given by [23]

$$\begin{aligned}\left\langle \mathrm{tr}A^{2k_1}\mathrm{tr}A^{2k_2}\right\rangle_{0,\mathrm{c}} &= N^{k_1+k_2}\frac{2^{k_1+k_2}\,\Gamma\left(k_1+\frac{1}{2}\right)\Gamma\left(k_2+\frac{1}{2}\right)}{\pi\,(k_1+k_2)\,\Gamma(k_1)\Gamma(k_2)}+\mathcal{O}\left(N^{k_1+k_2-2}\right),\\ \left\langle \mathrm{tr}A^{2k_1+1}\mathrm{tr}A^{2k_2+1}\right\rangle_{0,\mathrm{c}} &= N^{k_1+k_2+1}\frac{2^{k_1+k_2+1}\,k_1\,k_2\,\Gamma\left(k_1+\frac{3}{2}\right)\Gamma\left(k_2+\frac{3}{2}\right)}{\pi\,(k_1+k_2+1)\,\Gamma(k_1+2)\Gamma(k_2+2)}+\mathcal{O}\left(N^{k_1+k_2-1}\right),\end{aligned} \tag{B.2}$$

we get

$$\begin{aligned}\langle f\rangle_0 &\overset{N\to\infty}{=} 1+2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{n+1}\binom{2n+2}{2k}\frac{2^{n+1}\,\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(n+1-k+\frac{1}{2}\right)}{\pi\,(n+1)\,\Gamma(k)\Gamma(n+1-k)}\\ &\quad -2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\sum_{k=0}^{n}\binom{2n+2}{2k+1}\frac{2^{n+1}\,k(n-k)\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(n-k+\frac{3}{2}\right)}{\pi\,(n+1)\,\Gamma(k+2)\Gamma(n-k+2)}+\mathcal{O}(\zeta^2)\\ &= 1+2\sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}\frac{3}{\pi}\frac{2^{3n+1}\Gamma(n+\frac{1}{2})\Gamma\left(n+\frac{3}{2}\right)}{\pi\Gamma(n)\Gamma(n+3)}+\mathcal{O}\left(\zeta^2\right)\ .\end{aligned} \tag{B.3}$$

This leads to (4.2) after using (3.26).

To prove the relation (4.11) for the contribution to Δq of the sum of terms proportional to powers of ζ_3 one may start with the following U(N) 2-matrix (A,B) model with the ζ_3 term in the exponent representing the corresponding contribution coming from f in (2.1), (3.3)[20]

$$\mathcal{Z}=\lim_{N\to\infty}\mathcal{Z}_N\,,\qquad \mathcal{Z}_N(\xi)=\int[dAdB]\;e^{-\mathrm{tr}A^2-\mathrm{tr}B^2-\frac{\xi}{N^2}(\mathrm{tr}A^2-\mathrm{tr}B^2)^2},\qquad \xi\equiv 3\,\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2. \tag{B.4}$$

Then according to (3.26),

$$\Delta q(\lambda)\Big|_{\sum_n\zeta_3^n}=\frac{\lambda^2}{8}\frac{d}{d\lambda}\log\mathcal{Z}\ . \tag{B.5}$$

Since the integrand in $\mathcal{Z}_N$ depends only on $\mathrm{tr}A^2$ and $\mathrm{tr}B^2$, introducing the radial coordinates r_A, r_B we get (ignoring irrelevant constant factor)

$$\mathcal{Z}_N(x)=\int_0^{\infty}dr_A\,dr_B\,r_A^{N-1+N(N-1)}\,r_B^{N-1+N(N-1)}\;e^{-r_A^2-r_B^2-\frac{\xi}{N^2}(r_A^2-r_B^2)^2}\,. \tag{B.6}$$

[20] In this special case there will be no difference between U(N) and SU(N) cases.

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The large N limit is found from a saddle point of the effective action $S_{\rm eff} = (N^2-1)\log(r_A r_B) - \frac{\xi}{N^2}(r_A^2 - r_B^2)^2 - r_A^2 - r_B^2$. Choosing the symmetric saddle with $r_A = r_B = \sqrt{\frac{N^2-1}{2}}$ and integrating over the fluctuations gives

$$\mathcal{Z}(\xi) = \frac{1}{\sqrt{1+2\xi}} \ . \tag{B.7}$$

As a result, using (B.5) we find the strong-coupling asymptotics in (4.11). An alternative more rigorous and general approach is based on observing that $\mathcal{Z}$ in (B.4) may be represented as

$$\begin{aligned} \mathcal{Z}(\xi) &= \lim_{N\to\infty} e^{-\xi(\partial_x^2+\partial_y^2-2\partial_{xy}^2)} Z(x)Z(y)\Big|_{x=y=0} , \\ Z(x) &\equiv \int [dA]\, e^{-{\rm tr}A^2+\frac{x}{N}{\rm tr}A^2} = \left(1-\frac{x}{N}\right)^{-\frac{N^2-1}{2}} = e^{-\frac{N}{2}x+\frac{1}{4}x^2+\mathcal{O}(1/N)} \ . \end{aligned} \tag{B.8}$$

As a result, we get again (B.7).

Similar approach can be used to derive (4.15) for the contribution of terms proportional to products of $\zeta_3, \zeta_5, \ldots, \zeta_{2k+1}$. For example, let us consider the $\zeta_3\zeta_5$ terms. The new interaction term in the exponent in the analog of (B.4) will be

$$\begin{aligned} \Delta S_{\zeta_5} = -\frac{\eta}{N^3}\Big[& 2({\rm tr}A^3)^2 + 2({\rm tr}B^3)^2 - 3{\rm tr}A^2{\rm tr}A^4 - 3{\rm tr}B^2{\rm tr}B^4 \\ & {\rm tr}A^4{\rm tr}B^2 + 3{\rm tr}A^2{\rm tr}B^4 - 4{\rm tr}A^3{\rm tr}B^3\Big] , \qquad \eta = -\frac{10}{3}\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^3 . \end{aligned} \tag{B.9}$$

In this case instead of (B.8) we will need to consider

$$\begin{aligned} \mathcal{Z}(\xi,\eta) = \lim_{N\to\infty} \exp\Big[& -\xi(\partial_{x_1}^2+\partial_{y_1}^2-2\partial_{x_1y_1}^2) - \eta(2\partial_{x_2}^2+2\partial_{y_2}^2-3\partial_{x_1x_3}^2-3\partial_{y_1y_3}^2 \\ & +3\partial_{x_3y_1}^2+3\partial_{x_1y_3}^2-4\partial_{x_2y_2}^2)\Big] Z(x_1,x_2,x_3)\ Z(y_1,y_2,y_3)\Big|_{x_i=y_i=0} , \end{aligned} \tag{B.10}$$

$$Z(x_1,x_2,x_3) = \int [dA]\, e^{-{\rm tr}A^2+\frac{x_1}{N}{\rm tr}A^2+\frac{x_2}{N^{3/2}}{\rm tr}A^3+\frac{x_3}{N^2}{\rm tr}A^4} \ . \tag{B.11}$$

Expanding (B.11), taking log and sending $N\to\infty$ we find

$$Z(x_1,x_2,x_3) = \exp\left[\frac{N}{2}(x_1+x_3)+\frac{1}{4}x_1^2+\frac{3}{16}x_2^2+x_1x_3+\frac{9}{8}x_3^2+\mathcal{O}(1/N)\right] \ . \tag{B.12}$$

Using this in (B.10) gives

$$\begin{aligned} \mathcal{Z}(\xi,\eta) = 1 & -3\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + 15\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^3 + \frac{27\zeta_3^2}{2}\left(\frac{\lambda}{8\pi^2}\right)^4 - 165\zeta_3\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^5 \\ & + \left(-\frac{135\zeta_3^3}{2}+\frac{1125\zeta_5^2}{2}\right)\left(\frac{\lambda}{8\pi^2}\right)^6 + \frac{2565}{2}\zeta_3^2\zeta_5\left(\frac{\lambda}{8\pi^2}\right)^7 + \ldots \end{aligned} \tag{B.13}$$

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As a result

$$\Delta q(\lambda)\Big|_{\zeta_3,\zeta_5} = \frac{\lambda^2}{8}\frac{d}{d\lambda}\log\mathcal{Z} = -\frac{3\zeta_3}{4}\lambda\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{45\zeta_5}{8}\lambda\left(\frac{\lambda}{8\pi^2}\right)^3 + \frac{9\zeta(3)^2}{2}\lambda\left(\frac{\lambda}{8\pi^2}\right)^4 - 75\zeta_3\zeta_5\lambda\left(\frac{\lambda}{8\pi^2}\right)^5 + \left(-27\zeta_3^3 + \frac{675\zeta_5^2}{2}\right)\lambda\left(\frac{\lambda}{8\pi^2}\right)^6 + 630\zeta_3^2\zeta_5\lambda\left(\frac{\lambda}{8\pi^2}\right)^7 + \cdots , \tag{B.14}$$

which agrees with the results given in the main text.

It is interesting to note that (B.10) can be computed in a closed form generalizing (B.7)

$$\mathcal{Z}(\xi,\eta) = \left[1 + 2\xi - 9\eta + 6\xi\eta - \frac{81}{2}\eta^2 - \frac{27}{2}\eta^3\right]^{-1/2} . \tag{B.15}$$

Applying $\frac{\lambda^2}{8}\frac{d}{d\lambda}$ to the log of (B.15) as in (B.14) then gives the exact form of $\Delta q(\lambda)\Big|_{\zeta_3,\zeta_5}$.

C Wilson loop in SU(N) "orientifold" $\mathcal{N} = 2$ superconformal theory

It is possible to give a similar discussion of the large N expansion of the Wilson loop $\langle \mathcal{W} \rangle$ and the free energy in a particular $\mathcal{N} = 2$ superconformal gauge theory involving in addition to the SU(N) $\mathcal{N} = 2$ vector multiplet also two hypermultiplets — in rank-2 symmetric and antisymmetric SU(N) representations. This theory admits a regular 't Hooft large N limit and thus is similar to the quiver theory discussed above. It should be dual to the type IIB superstring on a particular orientifold $\mathrm{AdS}_5 \times S^5/(\mathbb{Z}_2^{\rm orient} \times \mathbb{Z}_2^{\rm orb})$ (see [50]).

This theory is one of the five cases of $\mathcal{N} = 2$ superconformal theories admitting a gauge group SU(N) with generic N [51]. The corresponding BPS circular Wilson loop is again equal to the $\mathcal{N} = 4$ SYM one at the planar level.[21] Here we shall focus on the weak-coupling expansion of the first subleading $1/N^2$ correction, i.e. of the corresponding function $q(\lambda)$ defined as in (1.10).

From the supersymmetric localization, the free energy and the Wilson loop expectation value $\langle \mathcal{W} \rangle^{\rm orient}$ in this theory are described by the Hermitian one-matrix model of the similar structure as in (2.1) where instead of (3.3) now we have [52]

$$\log f = 2\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\lambda}{8\pi^2 N}\right)^{n+1}\frac{\zeta(2n+1)}{n+1}\sum_{k=1}^{n-1}\binom{2n+2}{2k+1}\mathrm{tr}A^{2k+1}\mathrm{tr}A^{2n-2k+1} . \tag{C.1}$$

One can then organise the expansion of $\langle \mathcal{W} \rangle^{\rm orient}$ in powers of monomials of ζ_{2n+1}-constants as in (3.1). One finds that, as in the orbifold theory, at the leading non-planar level all appearing ζ_{2n+1}-monomials are multiplied by $I_1(\sqrt{\lambda})$ times a power of λ (cf. (3.21)).

[21] This planar equivalence extends to classes of "even" observables, while "odd" sectors display deviations from SYM case already at the planar level [23].

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Explicitly, for Δq defined as in (3.23), i.e. $\Delta q = q^{\text{orient}} - q^{\text{SYM}}$, we find

$$\begin{aligned}\frac{1}{8\pi^2}\Delta q(\lambda) = &-\frac{15\zeta_5}{4}\left(\frac{\lambda}{8\pi^2}\right)^4+\frac{105\zeta_7}{2}\left(\frac{\lambda}{8\pi^2}\right)^5-\frac{2205\zeta_9}{4}\left(\frac{\lambda}{8\pi^2}\right)^6+\left(\frac{75\zeta_5^2}{2}+\frac{10395\zeta_{11}}{2}\right)\left(\frac{\lambda}{8\pi^2}\right)^7\\ &+\left(-\frac{3675}{4}\zeta_5\zeta_7-\frac{1486485\zeta_{13}}{32}\right)\left(\frac{\lambda}{8\pi^2}\right)^8+\left(\frac{22785\zeta_7^2}{4}+8505\zeta_5\zeta_9+\frac{6441435\zeta_{15}}{16}\right)\left(\frac{\lambda}{8\pi^2}\right)^9\\ &+\left(-375\zeta_5^3-\frac{853335\zeta_7\zeta_9}{8}-\frac{571725\zeta_5\zeta_{11}}{8}-\frac{109504395\zeta_{17}}{32}\right)\left(\frac{\lambda}{8\pi^2}\right)^{10}\\ &+\left(13125\zeta_5^2\zeta_7+504630\zeta_9^2+\frac{3620925\zeta_7\zeta_{11}}{4}+\frac{4601025\zeta_5\zeta_{13}}{8}+\frac{459349605\zeta_{19}}{16}\right)\left(\frac{\lambda}{8\pi^2}\right)^{11}+\dots .\end{aligned} \tag{C.2}$$

Like in (1.15), (1.16) there is again a relation between Δq and the large N limit of the difference of the orientifold and SU(N) SYM free energies[22]

$$\Delta q(\lambda) = -\frac{\lambda^2}{4}\frac{d}{d\lambda}\Delta F(\lambda)\,, \qquad \Delta F(\lambda) = \lim_{N\to\infty}\left[F^{\text{orient}}(\lambda;N) - F^{\text{SYM}}(\lambda;N)\right]. \tag{C.3}$$

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