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Research Paper

On the symmetric 2-(v, k, λ) designs with a flag-transitive point-imprimitive automorphism group

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The symmetric 2-(v, k, λ) designs with $k > \lambda(\lambda-3)/2$ admitting a flag-transitive point-imprimitive automorphism group are completely classified: they are the known 2-designs with parameters $(16, 6, 2)$, $(45, 12, 3)$, $(15, 8, 4)$ or $(96, 20, 4)$.
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1. Introduction and main result

A 2-(v, k, λ) *design* $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. We say $\mathcal{D}$ is *non-trivial* if $2 < k < v$, and symmetric if $v = b$. All 2-(v, k, λ) designs in this paper are assumed to be non-trivial. An automorphism of $\mathcal{D}$ is a permutation

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of the point set which preserves the block set. The set of all automorphisms of $\mathcal{D}$ with the composition of permutations forms a group, denoted by $\mathrm{Aut}(\mathcal{D})$. For a subgroup G of $\mathrm{Aut}(\mathcal{D})$, G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$, and said to be *point-imprimitive* otherwise. Further, G is said to be *point-quasiprimitive* if each of its non-trivial normal subgroups of G acts transitively on $\mathcal{P}$. Quasiprimitivity is a much weaker property than primitivity (see [30]). In this setting, we also say that $\mathcal{D}$ is either *point-primitive* or *point-imprimitive*, or *point-quasiprimitive*, respectively. A *flag* of $\mathcal{D}$ is a pair (x, B) where x is a point and B is a block containing x. If $G \leqslant \mathrm{Aut}(\mathcal{D})$ acts transitively on the set of flags of $\mathcal{D}$, then we say that G is *flag-transitive* and that $\mathcal{D}$ is a *flag-transitive design*.

In 1987, Davies [8] proved that in a flag-transitive and point-imprimitive 2-(v, k, λ) design, the block size is bounded for a given value of the parameter λ, where $\lambda \geqslant 2$ by a result of Higman-McLaughlin [15] dating back to 1961. In 2005, O'Reilly Regueiro [29] obtained an explicit upper bound. Later that year, Praeger and Zhou [32] improved that upper bound and gave a complete list of feasible parameters. In 2020, Mandić and Šubasić [24] classified the flag-transitive point-imprimitive symmetric 2-designs with $\lambda \leqslant 10$ except for two possible numerical cases. The classification of the flag-transitive point-imprimitive symmetric 2-designs with $\lambda \leqslant 10$ is completed in Theorem 1.2 of the present paper by showing that the two cases remained open in [24] cannot occur.

Recently, Montinaro [27] has classified the symmetric 2-(v, k, λ) designs with $k > \lambda(\lambda - 3)/2$ and such that a block of the 2-design intersects a block of imprimitivity in at least 3 points. In this paper, we complete the work started in [27] by classifying $\mathcal{D}$ with $k > \lambda(\lambda - 3)/2$ regardless the intersection size of a block of $\mathcal{D}$ with a block of imprimitivity. More precisely, our result is the following.

Theorem 1.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive, point-imprimitive automorphism group. If $k > \lambda(\lambda - 3)/2$, then one of the following holds:*

(1) $\mathcal{D}$ is isomorphic to one of the two 2-$(16, 6, 2)$ *designs.*
(2) $\mathcal{D}$ is isomorphic to the 2-$(45, 12, 3)$ *design.*
(3) $\mathcal{D}$ is isomorphic to the 2-$(15, 8, 4)$ *design.*
(4) $\mathcal{D}$ is isomorphic to one of the four 2-$(96, 20, 4)$ *designs.*

In 1945, Hussain [16] and in 1946 Nandi [28] independently proved the existence of three symmetric 2-$(16, 6, 2)$ designs. In 2005, O'Reilly Regueiro [29] proved that exactly two of them are flag-transitive and point-imprimitive. In the same paper O'Reilly Regueiro constructed a 2-$(15, 8, 4)$ design. Such 2-design was proved to be unique by Praeger and Zhou [32] in 2006. One year later, Praeger [31] constructed and proved that there is exactly one flag-transitive and point-imprimitive 2-$(45, 12, 3)$ design. Finally, in 2009, Law, Praeger and Reichard [21] proved there are four flag-transitive and point-imprimitive 2-$(96, 20, 4)$ designs.

As mentioned above, we prove the following result which is the completion of the classification of the flag-transitive 2-designs with $\lambda \leqslant 10$ started in [24].

Theorem 1.2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive, point-imprimitive automorphism group G. If $\lambda \leqslant 10$, then*

(1) $\mathcal{D}$ is isomorphic to one of the two 2-$(16, 6, 2)$ designs.
(2) $\mathcal{D}$ is isomorphic to the 2-$(45, 12, 3)$ design.
(3) $\mathcal{D}$ is isomorphic to the 2-$(15, 8, 4)$ design.
(4) $\mathcal{D}$ is isomorphic to one of the four 2-$(96, 20, 4)$ designs.

1.1. Notation

Throughout the paper any partition of the point set of $\mathcal{D}$ invariant under the point-imprimitive automorphism group G is denoted by Σ, any element of Σ by Δ. The pointwise stabilizers of Σ and Δ are denoted by $G_{(\Sigma)}$ and $G_{(\Delta)}$, respectively. These groups are normal in G and in G_Δ, respectively, hence we may consider the quotient groups $G/G_{(\Sigma)}$ and $G_\Delta/G_{(\Delta)}$, which are denoted by G^Σ and G_Δ^Δ, respectively.

1.2. Outline of the proof

We start by strengthening the classification result obtained in [24]. Indeed, in [24] it is proven that, if $\lambda \leqslant 10$ then $\mathcal{D}$ is known except for two numerical values for the parameters for $\mathcal{D}$. The two exceptions are ruled out here in Theorem 1.2. Subsequently, we focus on the case $\lambda > 10$. In Proposition 2.4 it is shown that G^Σ acts primitively on Σ by using the results contained in [27]. Then, it is proven in Theorem 3.1 that, either G acts point-quasiprimitively on $\mathcal{D}$, or $G_{(\Sigma)} \neq 1$, G^Σ is almost simple and $\mathcal{D}$ has parameters $(2^{a+2}(2^{a-1}-1)^2, 2\,(2^a-1)\left(2^{a-1}-1\right), 2(2^{a-1}-1))$ where $a \geqslant 4$. Afterwards, by combining the O'Nan-Scott theorem for quasiprimitive groups achieved in [30] with an adaptation of the techniques developed by [41], in Theorem 4.1 we show that G^Σ is almost simple also in the quasiprimitive case. Moreover, if L is the preimage in G of $\mathrm{Soc}(G^\Sigma)$ and $\Delta \in \Sigma$, in Proposition 5.3, Corollary 5.4 and Theorem 5.5 it is proven that, either $G_{(\Sigma)} = 1$, L_Δ is contained in a semilinear 1-dimensional group and $|L| \leqslant 4\left|L_\Delta^\Delta\right|^2 \left|\mathrm{Out}(L)\right|^2$, or $G_{(\Sigma)} = 1$, L_Δ is a non-solvable 2-transitive permutation group of degree $|\Delta|$ and $|L| \leqslant |L_\Delta|^2$, or $G_{(\Sigma)} \neq 1$ and a quotient of L_Δ^Σ is isomorphic either to $SL_a(2)$, or to A_7 for $a = 4$. In particular, in each case L_Δ^Σ is a large subgroup of L^Σ. Finally, we use all the above mentioned constraints on L^Σ and on L_Δ^Σ together with the results contained in [3] and [22] to precisely determine the admissible pairs $(L^\Sigma, L_\Delta^\Sigma)$ and from these to prove that there are no examples of $\mathcal{D}$ for $\lambda > 10$. At this point, our classification result follows from Theorem 1.2.

2. Preliminary reductions

It is well known that, if $\mathcal{D}$ is a symmetric 2-(v,k,λ) design, then $r=k$, $b=v$ and $k(k-1)=(v-1)\lambda$ (for instance, see [9]). Moreover, the following fact holds:

Lemma 2.1. *If $\mathcal{D}$ admits a flag-transitive automorphism group G and x is any point of $\mathcal{D}$, then $\left|y^{G_x}\right|\lambda=k\left|B\cap y^{G_x}\right|$ for any point y of $\mathcal{D}$, with $y\neq x$, and for any block B of $\mathcal{D}$ incident with x.*

Proof. Let x, y be points of $\mathcal{D}$, $y\neq x$, and B be any block of $\mathcal{D}$ incident with x. Since (y^{G_x},B^{G_x}) is a tactical configuration by [9, 1.2.6], it follows that $\left|y^{G_x}\right|\lambda=k\left|B\cap y^{G_x}\right|$. $\square$

The following theorem, which is a summary of [27] and some of the results contained in [32], is our starting point.

Theorem 2.2. *Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a symmetric 2-design admitting a flag-transitive, point-imprimitive automorphism group G that leaves invariant a non-trivial partition $\Sigma=\{\Delta_1,...,\Delta_d\}$ of $\mathcal{P}$ such that $|\Delta_i|=c$ for each $i=1,...,d$. Then the following hold:*

I. There is a constant ℓ such that, for each $B\in\mathcal{B}$ and $\Delta_i\in\Sigma$, the size $|B\cap\Delta_i|$ is either 0 or ℓ.

II. There is a constant θ such that, for each $B\in\mathcal{B}$ and $\Delta_i\in\Sigma$ with $|B\cap\Delta_i|>0$, the number of blocks of $\mathcal{D}$ whose intersection set with Δ_i coincides with $B\cap\Delta_i$ is θ.

III. If $\ell=2$, then $G_{\Delta_i}^{\Delta_i}$ acts 2-transitively on Δ_i for each $i=1,...,d$.

IV. If $\ell\geqslant 3$, then $\mathcal{D}_i=\left(\Delta_i,(B\cap\Delta_i)^{G_{\Delta_i}^{\Delta_i}}\right)$ is a flag-transitive non-trivial 2-$(c,\ell,\lambda/\theta)$ design for each $i=1,...,d$.

Moreover, if $k>\lambda(\lambda-3)/2$ then one of the following holds:

V. $\ell=2$ and one of the following holds:

1. *$\mathcal{D}$ is a symmetric 2-$(\lambda^2(\lambda+2),\lambda(\lambda+1),\lambda)$ design and $(c,d)=\left(\lambda+2,\lambda^2\right)$.*
2. *$\mathcal{D}$ is a symmetric 2-$\left(\left(\frac{\lambda+2}{2}\right)\left(\frac{\lambda^2-2\lambda+2}{2}\right),\frac{\lambda^2}{2},\lambda\right)$ design, $(c,d)=\left(\frac{\lambda+2}{2},\frac{\lambda^2-2\lambda+2}{2}\right)$, and either $\lambda\equiv 0 \pmod 4$, or $\lambda=2w^2$, where w is odd, $w\geqslant 3$, and $2(w^2-1)$ is a square.*

VI. $\ell\geqslant 3$ and one of the following holds:

1. *$\mathcal{D}$ is isomorphic to the 2-$(45,12,3)$ design of [31, Construction 4.2].*
2. *$\mathcal{D}$ is isomorphic to one of the four 2-$(96,20,4)$ designs constructed in [21].*

Proof. Apart from III, each case is an immediate consequence of [27,32]. In case III, let $(x, y), (x', y')$ be any two pairs of points of Δ and let B, B' any two blocks of $\mathcal{D}$ containing x, y and x', y', respectively. Then there is α in G mapping the flag (x, B) onto the flag (x', B') since G acts flag-transitively on $\mathcal{D}$. Then $\alpha \in G_\Delta$ since Δ is a block of imprimitivity for G. Therefore, $\{x, y\}^\alpha = (\Delta \cap B)^\alpha = \Delta \cap B' = \{x', y'\}$ since $\ell = 2$, hence $y^\alpha = y'$ since $x^\alpha = x'$. Thus G_Δ, and hence G_Δ^Δ, acts 2-transitively on Δ. $\square$

We are going to focus on case (V) of the previous theorem. Throughout the paper, a 2-design as in case (V.1) or (V.2) of Theorem 2.2 will be simply called a 2-design of type 1 or 2, respectively.

Let $\Delta \in \Sigma$ and $x \in \Delta$. Since $G_{(\Sigma)} \trianglelefteq G_\Delta$ and $G_{(\Delta)} \trianglelefteq G_x$, it is immediate to verify that $(G^\Sigma)_\Delta = (G_\Delta)^\Sigma$ and that $\left(G_\Delta^\Delta\right)_x = (G_x)^\Delta$. Hence, in the sequel $(G^\Sigma)_\Delta$ and $\left(G_\Delta^\Delta\right)_x$ will simply be denoted by G_Δ^Σ and G_x^Δ, respectively. Moreover, the following holds:

$$\frac{G_\Delta^\Sigma}{G_{(\Delta)}^\Sigma} \cong \frac{G_\Delta}{G_{(\Delta)} G_{(\Sigma)}} \cong \frac{G_\Delta^\Delta}{G_{(\Sigma)}^\Delta}. \tag{2.1}$$

A further reduction is the provided by Theorem 1.2, which is proven below.

Proof of Theorem 1.2. This result is proven in [24, Theorem 1] with the following possible exceptions of $(v, k, \lambda, c, d) = (288, 42, 6, 8, 36)$ or $(891, 90, 9, 81, 11)$. Note that $k > \lambda(\lambda - 3)/2$ in both exceptional cases. Actually, the latter does not correspond to any case of Theorem 2.2(V–VI), and hence it cannot occur. The former corresponds to Theorem 2.2(V.1) for $\lambda = 6$. Also, if $\Delta \in \Sigma$ then $G_\Delta^\Delta \cong AGL_1(8)$, $A\Gamma L_1(8)$, A_8, S_8, $PSL_2(7)$ or $PGL_2(7)$ by [18, Lists (A) and (B)] since G_Δ^Δ acts 2-transitively on Δ.

Assume that u divides the order of $G_{(\Sigma)}$, where u is an odd prime, and let ψ be a u-element of $G_{(\Sigma)}$. Then ψ fixes at least a point on each $\Delta \in \Sigma$ since $|\Delta| = 8$. Thus ψ preserves at least two distinct blocks of $\mathcal{D}$ by [20, Theorem 3.1], say B_1 and B_2. Actually, ψ fixes B_1 and B_2 pointwise since any of these intersects each element of Σ in 0 or 2 points and $\psi \in G_{(\Sigma)}$. Hence, ψ fixes at least $2 \cdot 42 - 6$ points of $\mathcal{D}$, but this contradicts [20, Corollary 3.7]. Thus, $\left|G_{(\Sigma)}\right| = 2^i$ with $i \geqslant 0$.

Assume that w divides the order of $G_{(\Delta)}$, where w is an odd prime, $w \geqslant 7$, and let ϕ be a w-element of $G_{(\Delta)}$. Then ϕ fixes the 6 blocks incident with any pair of distinct points of Δ. Therefore, ϕ fixes at least $6 \cdot \binom{8}{2}$ points of $\mathcal{D}$ by [20, Theorem 3.1], and we again reach a contradiction by [20, Corollary 3.7]. Thus, $G_{(\Delta)}$ is a $\{2, 3, 5\}$-group.

Any Sylow 7-subgroup of G is of order 7 since $|\Sigma| = 36$, G_Δ^Δ is one of the groups listed above and the order of $G_{(\Delta)}$ is coprime to 7. Then, by [12, Table B.4] one of the following holds:

(1) $A_9 \trianglelefteq G^\Sigma \leqslant S_9$ and $S_7 \trianglelefteq G_\Delta^\Sigma \leqslant S_7 \times Z_2$;
(2) $PSL_2(8) \trianglelefteq G^\Sigma \leqslant P\Gamma L_2(8)$ and $D_{14} \trianglelefteq G_\Delta^\Sigma \leqslant F_{42}$;

(3) $PSU_3(3) \trianglelefteq G^{\Sigma} \leqslant P\Gamma U_3(3)$ and $PSL_2(7) \trianglelefteq G^{\Sigma}_{\Delta} \leqslant PGL_2(7)$;
(4) $G^{\Sigma} \cong Sp_6(2)$ and $G^{\Sigma}_{\Delta} \cong S_8$.

Since G^{Δ}_{Δ} is one of the 2-transitive groups listed above, (1) is immediately ruled out by (2.1). In (2) the group G_{Δ} is solvable since G^{Σ}_{Δ} is solvable and $G_{(\Sigma)}$ is a 2-group. Thus G^{Δ}_{Δ} is solvable and hence it is isomorphic to $AGL_1(8)$ or $A\Gamma L_1(8)$. Moreover, a quotient group of G^{Δ}_{Δ} must contain a subgroup isomorphic to D_{14} by (2.1) since $G_{(\Delta)}$ is a $\{2,3,5\}$-group, but this is clearly impossible. It follows that only (3) and (4) are admissible.

Let X denote $PSL_2(7)$ or A_8 in cases (3) or (4), respectively. It follows from (2.1) that $G^{\Sigma}_{(\Delta)} = 1$ and $X \leqslant G^{\Delta}_{\Delta}/G^{\Delta}_{(\Sigma)}$ since $G_{(\Delta)}$ is a $\{2,3,5\}$-group and G^{Σ}_{Δ} contains a subgroup isomorphic to X. Thus, $G_{(\Delta)} \leqslant G_{(\Sigma)}$. If $G_{(\Delta)} \neq G_{(\Sigma)}$, then $X \cong Soc(G^{\Delta}_{\Delta}) \leqslant G^{\Delta}_{(\Sigma)}$ since $G^{\Delta}_{(\Sigma)} \trianglelefteq G^{\Delta}_{\Delta}$ and G^{Δ}_{Δ} acts 2-transitively on Δ, whereas $G^{\Delta}_{(\Sigma)}$ is a 2-group. Thus, $G_{(\Delta)} = G_{(\Sigma)} = 1$ since G acts transitively on Σ and $G_{(\Sigma)} \trianglelefteq G$.

Assume that case (3) occurs. Let x and B be a point and a block of $\mathcal{D}$, respectively, and let $\Delta \in \Sigma$ be such that $x \in \Delta$. Then $F_{21} \leqslant G_x \leqslant F_{42}$ by [7] since G acts transitively on the 288 points of $\mathcal{D}$. Actually, $G_x = F_{42}$ since $|G_x| = |G_B|$, G_B acts transitively on B and $k = 42$. This forces $G_{\Delta} \cong PGL_2(7)$ and $G \cong P\Gamma U_3(3)$. Moreover, G_x is the stabilizer of a block of $\mathcal{D}$, that we may assume to be B, since $P\Gamma U_3(3)$ has a unique conjugacy class of subgroups isomorphic to F_{42} by [7]. Hence $G_x = G_B$, and $x \notin B$ since G_B acts transitively on B. Further, $G_B < G_{\Delta}$ since $x \in \Delta$.

Let $B(\Sigma)$ be the set of elements of Σ intersecting B in a non-empty set. Then G_B acts transitively on $B(\Sigma)$ and $|B(\Sigma)| = 21$ since G_B acts transitively on B. Now, G acts on Σ as a rank 3 group and the non-trivial G_{Δ}-orbits on $\Sigma \setminus \{\Delta\}$ have length 14 and 21 by [17, Theorem 1.1]. Therefore, $B(\Sigma) = (\Delta')^{G_B} = (\Delta')^{G_{\Delta}}$, where $\Delta' \in B(\Sigma)$, since $G_B < G_{\Delta}$, G_B acts transitively on $B(\Sigma)$ and $|B(\Sigma)| = 21$. Thus $C(\Sigma) = B(\Sigma)$ if $C \in B^{G_{\Delta}}$, and hence the number η of elements C of $\mathcal{B}$ such that $C(\Sigma) = B(\Sigma)$ is at least $|G_{\Delta} : G_B| = 8$.

The set $\mathcal{R} = \{(C, C') \in \mathcal{B} \times \mathcal{B} : C(\Sigma) = C'(\Sigma)\}$ is an equivalence relation on $\mathcal{B}$. Hence, let $\mathcal{B}^{\Sigma}$ be the quotient set defined by $\mathcal{R}$, and for any block C of $\mathcal{D}$ denote by C^{Σ} the equivalence class containing C. The previous argument shows that $\eta = \left|C^{\Sigma}\right| \geqslant 8$. Finally, consider $\mathcal{I} = \{(\Delta, C^{\Sigma}) \in \Sigma \times \mathcal{B}^{\Sigma} : \Delta \in C(\Sigma)\}$. Then the incidence structure $\mathcal{D}^{\Sigma} = \left(\Sigma, \mathcal{B}^{\Sigma}, \mathcal{I}\right)$ is a 2-$(36, 21, 96/\eta)$ design with $\eta \geqslant 8$ admitting G as flag-transitive automorphism group by [6, Proposition 2.3]. Actually, $\eta = 8, 12$ or 24 since the replication number of $\mathcal{D}^{\Sigma}$ is $\frac{7 \cdot 96}{4\eta}$. If $\eta = 12$ or 24, then $\left|\mathcal{B}^{\Sigma}\right| = 12$ or 24, respectively, but this is impossible since G does not have transitive permutation representations of such degrees by [7]. Thus $\eta = 8$, and hence $\mathcal{D}^{\Sigma}$ is symmetric. However, this case is excluded by [10] since G acts on Σ as a rank 3 group.

Finally, assume that (4) occurs. Then $G \cong Sp_6(2)$, and hence $G_x \cong G_B \cong S_7$, where x and B are a point and a block of $\mathcal{D}$, respectively, by [7]. Actually, G_x is the stabilizer of a block of $\mathcal{D}$, that we may assume to be B, since $Sp_6(2)$ has unique conjugacy class of subgroups isomorphic to S_7 by [7]. Hence $G_x = G_B$, and $x \notin B$ since G_B acts transitively on B.

Let $\Delta \in \Sigma$ be such that $x \in \Delta$. Then G_B preserves Δ since $G_x = G_B$, and $B \cap \Delta = \varnothing$ since G_B acts transitively on the 21 elements of Σ intersecting B in a non-empty set. Therefore, $G_{\Delta,\Delta',B}$ contains a Sylow 5-subgroup of G, where Δ' is any element of Σ such that $\Delta' \cap B \neq \varnothing$. Consequently, $G_{\Delta,\Delta'}$ contains a Sylow 5-subgroup of G. However, this is impossible since G acts 2-transitively on Σ and $|\Sigma| = 36$. This completes the proof. □

2.1. Hypotheses

On the basis of Theorem 1.2, in the sequel we may assume $\mathcal{D}$ is a symmetric 2-(v, k, λ) design of type 1 or 2 with $\lambda > 10$.

Lemma 2.3. *If G preserves a further partition Σ' of the point set of $\mathcal{D}$ in d' blocks of imprimitivity of size c', then $d' = d$, $c' = c$ and one of the following holds:*

(1) $\Sigma = \Sigma'$;
(2) $|\Delta \cap \Delta'| \leqslant 1$ for each $\Delta \in \Sigma$, $\Delta' \in \Sigma'$.

Proof. Suppose there is a G-invariant partition Σ' of the point set of $\mathcal{D}$. Let B any block of $\mathcal{D}$ and let $\Delta' \in \Sigma'$. If $|B \cap \Delta'| \geqslant 3$ then $(v, k, \lambda) = (45, 12, 3), (96, 20, 4)$ by [27, Theorem 1.1], whereas $\lambda > 10$ by our assumptions. Thus, $|B \cap \Delta'| = 2$.

If $\mathcal{D}$ is of different type with respect to Σ and to Σ', then $k = \lambda(\lambda + 1) = \lambda^2/2$, which does not have positive integer solutions. Therefore $\mathcal{D}$ is of the same type with respect to Σ and to Σ', and hence $d' = d$ and $c' = c$.

Let $\Delta \in \Sigma$ such that $\Delta \cap \Delta' \neq \varnothing$. If $|\Delta \cap \Delta'| > 1$ for some $\Delta, \Delta' \in \Sigma$, then $\Delta = \Delta'$ since G induces a 2-transitive group on Δ, and $|\Delta| = |\Delta'|$. Therefore, $\Sigma = \Sigma'$. □

Proposition 2.4. *G^{Σ} acts primitively on Σ. Moreover, one of the following holds:*

(1) $G_{(\Sigma)} \neq 1$ and $\mathrm{Soc}(G_{\Delta}^{\Delta}) \trianglelefteq G_{(\Sigma)}^{\Delta}$;
(2) $G_{(\Sigma)} = 1$ and G acts point-quasiprimitively on $\mathcal{D}$.

Proof. It follows from Lemma 2.3 and [12, Theorem 1.5A] that G^{Σ} acts primitively on Σ.

Assume that $G_{(\Sigma)} \neq 1$. If there is $\Delta' \in \Sigma$ such that $G_{(\Sigma)} \leqslant G_{(\Delta')}$, then $G_{(\Sigma)} \leqslant G_{(\Delta'')}$ for each $\Delta'' \in \Sigma$, and hence $G_{(\Sigma)} = 1$ since $G_{(\Sigma)} \lhd G$ and G acts transitively on Σ, which is a contradiction. Thus, $G_{(\Sigma)} \nleq G_{(\Delta)}$ for each $\Delta \in \Sigma$. Then $1 \neq G_{(\Sigma)}^{\Delta} \trianglelefteq G_{\Delta}^{\Delta}$, and hence $\mathrm{Soc}(G_{\Delta}^{\Delta}) \trianglelefteq G_{(\Sigma)}^{\Delta}$ by [12, Theorem 4.3B] since G_{Δ}^{Δ} is 2-transitive on Δ by Theorem 2.2(III).

Assume that $G_{(\Sigma)} = 1$. Let $N \neq 1$ be any normal subgroup of G. Then $N = G$ since G^{Σ} acts primitively on Σ. Therefore, G acts point-quasiprimitively on $\mathcal{D}$. □

Lemma 2.5. *Let $\gamma \in G$, $\gamma \neq 1$. Then one of the following holds:*

(1) $\mathcal{D}$ is of type 1 and $|\mathrm{Fix}(\gamma)| \leqslant (\lambda+2)\lambda$;
(2) $\mathcal{D}$ is of type 2 and $|\mathrm{Fix}(\gamma)| \leqslant \lambda^2/2 + \sqrt{\lambda^2/2-\lambda}$.

Proof. Let $\Delta \in \Sigma$ and $\gamma \in G$, $\gamma \neq 1$. It follows from by [20, Corollary 3.7] that

$$|\mathrm{Fix}(\gamma)| \leqslant \frac{\lambda}{k-\sqrt{k-\lambda}} \cdot |\Delta| \cdot |\Sigma|\,, \tag{2.2}$$

where $\frac{\lambda}{k-\sqrt{k-\lambda}} \cdot |\Delta| \cdot |\Sigma|$ is either $(\lambda+1)\lambda$ or $\lambda^2/2 + \sqrt{\lambda^2/2-\lambda}$ according to whether $\mathcal{D}$ is of type 1 or 2, respectively. □

Corollary 2.6. *The following hold:*

(1) If $\mathcal{D}$ is of type 1, each prime divisor of $\left|G_{(\Delta)}\right|$ divides $\lambda(\lambda-1)$.
(2) If $\mathcal{D}$ is of type 2, each prime divisor of $\left|G_{(\Delta)}\right|$ divides $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$.

Proof. Let γ be w-element of $G_{(\Delta)}$, where w is a prime such that $w \nmid \lambda$. Since γ fixes Δ pointwise, γ fixes at least $\mu \geqslant 1$ of the λ blocks of $\mathcal{D}$ incident with any two distinct points of Δ, where $\mu \equiv \lambda \pmod{w}$. Clearly, these μ fixed blocks are pairwise distinct.

If $\mathcal{D}$ is of type 1, then γ fixes at least $\mu\frac{(\lambda+2)(\lambda+1)}{2}$ blocks of $\mathcal{D}$, and hence $|\mathrm{Fix}(\gamma)| \geqslant \mu\frac{(\lambda+2)(\lambda+1)}{2}$. Therefore $\mu = 1$ by Lemma 2.5(1), and the assertion (1) follows.

If $\mathcal{D}$ is of type 2, then γ fixes at least $\mu\frac{\lambda}{4}\left(\frac{\lambda}{2}+1\right)$ blocks of $\mathcal{D}$, and hence $|\mathrm{Fix}(\gamma)| \geqslant \mu\frac{\lambda}{4}\left(\frac{\lambda}{2}+1\right)$. Therefore $\mu \leqslant 3$ by Lemma 2.5(2), and the assertion (2) follows. □

Lemma 2.7. *Let x be any point of $\mathcal{D}$, then $G_{(\Sigma),x}$ lies in a Sylow 2-subgroup of $G_{(\Sigma)}$.*

Proof. Let x be any point of $\mathcal{D}$ and let φ be any w-element of $G_{(\Sigma),x}$, where w is an odd prime. Then φ fixes at least a block B of $\mathcal{D}$ by [20, Theorem 3.1]. Since B intersects each element of Σ in 0 or 2 points, and since φ is a w-element of $G_{(\Sigma)}$, it follows that φ fixes B pointwise. Therefore, φ fixes at least k points of $\mathcal{D}$. Then φ fixes at least k blocks of $\mathcal{D}$ again by [20, Theorem 3.1]. Let B' be further block fixed by φ. We may repeat the previous argument with B' in the role of B thus obtaining φ fixing B' pointwise. Then φ fixes at least $2k-\lambda$ points of $\mathcal{D}$, as $|B \cap B'| = \lambda$. Thus $|\mathrm{Fix}(\varphi)| \geqslant 2k-\lambda$ and hence $|\mathrm{Fix}(\varphi)|$ is greater than or equal to $\lambda(2\lambda+1)$ or $\lambda^2-\lambda$ according to whether $\mathcal{D}$ is of type 1 or 2, respectively. However, this contradicts Lemma 2.5 since $\lambda > 2$. □

Lemma 2.8. *If $G_{(\Sigma)} \neq 1$ and v is odd, then the following hold:*

(1) $G_{(\Sigma)} \cong Soc(G_{\Delta}^{\Delta})$ is an elementary abelian p-group, p an odd prime, acting regularly on Δ;
(2) $G^{\Sigma} \leqslant GL_a(p)$ with $p^a = |\Delta|$ and $\Delta \in \Sigma$.

Proof. Assume that $G_{(\Sigma),x} \neq 1$, where x is any point of $\mathcal{D}$. Then $G_{(\Sigma),x}$ is a Sylow 2-subgroup of $G_{(\Sigma)}$ by Lemma 2.7 since v is odd. Denote $G_{(\Sigma),x}$ simply by S. Then S fixes the same number $t \geqslant 1$ of points in each $\Delta \in \Sigma$ since $|\Delta|$ is odd and S is a Sylow 2-subgroup of $G_{(\Sigma)}$. Therefore, $|\mathrm{Fix}(S)| = t\,|\Sigma|$. Now, if α is any non-trivial element of S then $|\mathrm{Fix}(\alpha)| \geqslant t\,|\Sigma|$, and hence $t = 1$ by Lemma 2.5. Thus S fixes a unique point on each Δ.

Suppose that $|S| \geqslant 4$. Let B be any block of $\mathcal{D}$ preserved by S and let Δ' be such that $|B \cap \Delta'| = 2$. Then there is a subgroup S_0 of S of index at most 2 fixing y and y', where $\{y, y'\} = B \cap \Delta'$. Then $S_0 \leqslant G_{(\Sigma),y} \cap G_{(\Sigma),y'}$, where $G_{(\Sigma),y}$ and $G_{(\Sigma),y'}$ are two distinct Sylow 2-subgroups of $G_{(\Sigma)}$ since each of these fixes a unique point in Δ'. Suppose there is a point z of $\mathcal{D}$ such that $z \in \mathrm{Fix}(G_{(\Sigma),y}) \cap \mathrm{Fix}(G_{(\Sigma),y'})$. The $\left\langle G_{(\Sigma),y}, G_{(\Sigma),y'} \right\rangle \leqslant G_{(\Sigma),z}$ and hence $G_{(\Sigma),y} = G_{(\Sigma),y'} = G_{(\Sigma),z}$ since $G_{(\Sigma),z}$ is a Sylow 2-subgroup of $G_{(\Sigma)}$ by Lemma 2.7. Then $G_{(\Sigma),y}$ fixes also y' in Δ' with $y' \neq y$, and we reach a contradiction since $t = 1$. Thus $\mathrm{Fix}(G_{(\Sigma),y}) \cap \mathrm{Fix}(G_{(\Sigma),y'}) = \varnothing$, and hence $|\mathrm{Fix}(S_0)| \geqslant 2\,|\Sigma|$ since $S_0 \leqslant G_{(\Sigma),y} \cap G_{(\Sigma),y'}$. Now, if we use the above argument this time with $\alpha \in S_0$, $\alpha \neq 1$, we reach a contradiction. Therefore $|S| = 2$. Also, $G_{(\Sigma)} = O(G_{(\Sigma)}).S$ by Proposition 2.4(1), where $O(G_{(\Sigma)})$ is the largest normal subgroup of odd order in the group $G_{(\Sigma)}$, and $|\mathrm{Fix}(S)| = |\Sigma|$.

Let $\Lambda = \left\{ Fix(S)^{\gamma} : \gamma \in G_{(\Sigma)} \right\}$. Since $S = G_{(\Sigma),x}$ is a Sylow 2-subgroup of $G_{(\Sigma)}$, $G_{(\Sigma)} \lhd G$ and G acts point-transitively on $\mathcal{D}$, it follows that Λ is a G-invariant partition of the point set of $\mathcal{D}$ in $|\Delta|$ blocks each of size $|\Sigma|$. Then $|\Sigma| = |\Delta|$ by Lemma 2.3(2), and hence $\lambda = 2$, but this contradicts our assumptions.

Assume that $G_{(\Sigma),x} = 1$. Then $G_{(\Sigma)} = O(G_{(\Sigma)})$. Moreover, $Soc(G_{\Delta}^{\Delta}) \trianglelefteq G_{(\Sigma)}^{\Delta} \cong G_{(\Sigma)}$ by Proposition 2.4(1). Then $G_{(\Sigma)} \cong Soc(G_{\Delta}^{\Delta})$ is an elementary abelian p-group for some odd prime p since $G_{(\Sigma)}$ acts regularly on Δ and $|\Delta|$ is odd. Hence, $G_{(\Sigma)} \trianglelefteq C_G(G_{(\Sigma)}) \trianglelefteq G$. If $C_G(G_{(\Sigma)}) \neq G_{(\Sigma)}$, then $G = C_G(G_{(\Sigma)})$ since G^{Σ} is primitive on Σ by Proposition 2.4. This implies $G_{(\Sigma)} \leqslant Z(G)$ and hence $G_x \leqslant G_{(\Delta)}$ for any $x \in \Delta$. This is a contradiction since G_{Δ}^{Δ} is 2-transitive on Δ. Therefore $C_G(G_{(\Sigma)}) = G_{(\Sigma)} = Soc(G_{\Delta}^{\Delta})$, and hence $G^{\Sigma} \leqslant \mathrm{Aut}(G_{(\Sigma)}) \cong GL_a(p)$. $\square$

3. Further reductions

The aim of this section is to prove the following reduction result:

Theorem 3.1. *One of the following holds:*

(1) $G_{(\Sigma)} = 1$ *and* G *acts point-quasiprimitively on* $\mathcal{D}$.
(2) $G_{(\Sigma)}$ *is a non-trivial self-centralizing elementary abelian* 2*-subgroup of* G *and the following hold:*

  (a) $\mathcal{D}$ *is a symmetric* 2-$(2^{a+2}(2^{a-1}-1)^2, 2\left(2^a-1\right)\left(2^{a-1}-1\right), 2(2^{a-1}-1))$ *design for* $a \geqslant 4$;

(b) Either $G_x^{\Delta} \cong SL_a(2)$ *for* $a \geqslant 4$, *or* $G_x^{\Delta} \cong A_7$ *and* $a = 4$.
(c) G^{Σ} *is almost simple.*

Its proof is structured as follows. Case (1) is an immediate consequence of Proposition 2.4(2), whereas the proof of (2) relies mainly on Corollary 2.6 and the O'Nan Scott theorem applied to G^{Σ} for v even, and on Corollary 2.6 and [4, Theorem 3.1] for v odd.

3.1. Designs of type 1 and point-quasiprimitivity

Lemma 3.2. *Let* $\mathcal{D}$ *be of type 1. If* $G_{(\Sigma)} \neq 1$ *and* v *is even, then the following hold:*

(1) $\mathcal{D}$ *is a symmetric* 2-$(2^{a+2}(2^{a-1}-1)^2, 2\left(2^a-1\right)\left(2^{a-1}-1\right), 2(2^{a-1}-1))$ *design,* $a \geqslant 4$*;*
(2) $G_{(\Sigma)}$ *is a non-trivial self-centralizing elementary abelian* 2*-subgroup of* G*. Also,* $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma} \cong G_x^{\Delta}$ *and one of the following holds:*

(a) $G_x^{\Delta} \cong SL_a(2)$*;*
(b) $G_x^{\Delta} \cong A_7$ *and* $a = 4$.

Proof. Assume that $\mathcal{D}$ is of type 1, $G_{(\Sigma)} \neq 1$ and v is even. Since $Soc(G_{\Delta}^{\Delta}) \trianglelefteq G_{(\Sigma)}^{\Delta}$ by Proposition 2.4(1), it follows that $Soc(G_{\Delta}^{\Delta})_x \trianglelefteq G_{(\Sigma),x}^{\Delta}$, where $x \in \Delta$. Hence, $Soc(G_{\Delta}^{\Delta})_x$ is either trivial, or a 2-group by Lemma 2.7. If G_{Δ}^{Δ} is almost simple, then $Soc(G_{\Delta}^{\Delta})$ acts 2-transitively on Δ with $Soc(G_{\Delta}^{\Delta})_x$ a 2-group. However, this is impossible by [18, List (A)]. Therefore, G_{Δ}^{Δ} is of affine type, and hence $|\Delta| = 2^a$ since $v = \lambda^2(\lambda+2)$ is even. Then $\lambda = 2(2^{a-1}-1)$ and so $|\Sigma| = 2^2(2^{a-1}-1)^2$, where $a \geqslant 4$ since $\lambda > 10$. In particular, $\mathcal{D}$ is a symmetric 2-design with parameters as in (1). Moreover, by [18, List (B)] one of the following holds:

(i) $G_x^{\Delta} \leqslant \Gamma L_1(2^a)$;
(ii) $SL_{a/h}(2^h) \trianglelefteq G_x^{\Delta} \leqslant \Gamma L_{a/h}(2^h)$ with $a/h > 1$;
(iii) $Sp_{a/h}(2^h) \trianglelefteq G_x^{\Delta} \leqslant \Gamma Sp_{a/h}(2^h)$ with $a/h > 1$ and a/h even,
(iv) $G_2(2^{a/6}) \trianglelefteq G_x^{\Delta} \leqslant (Z_{2^{a/6}-1} \times G_2(2^{a/6})) : Z_{a/6}$ with $a \equiv 0 \pmod 6$.
(v) $G_x^{\Delta} \cong A_6$ or A_7 and $|\Delta| = 2^4$.

Since G_x is transitive on the $\lambda(\lambda+1)$ blocks incident with x, it follows that $2\left(2^{a-1}-1\right)\left(2^a-1\right) \mid |G_x|$. Suppose that there is a prime u dividing $\lambda/2 = 2^{a-1}-1$ but not dividing of the order of G_x^{Δ}. Hence, u divides the order of $G_{(\Delta)}$. Let U be a Sylow u-subgroup of $G_{(\Delta)}$, then U is a Sylow u-subgroup of G_x. Further, $G_x = N_{G_x}(U)G_{(\Delta)}$ by the Frattini argument.

Let $y \in Fix(U) \setminus \{x\}$ and let Δ' be the element of Σ containing y. Then $U \leqslant G_y$, and actually $U \leqslant G_{(\Delta')}$ since G_x and G_y are G-conjugate (clearly, $\Delta = \Delta'$ is possible). Thus $Fix(U)$ is a disjoint union of some elements of Σ since this one is a partition of the point set of $\mathcal{D}$, and hence $|\mathrm{Fix}(U)| = t(\lambda+2)$ for some $t \geqslant 1$ since each element in Σ has size $\lambda+2$. Actually, $t \geqslant 2$ since $|\Sigma| = \lambda^2$ and $u \mid \lambda/2$. Also, U is a Sylow u-subgroup of $G_{(\Delta\cup\Delta')}$. Thus, $G_{x,y} = N_{G_{x,y}}(U)G_{(\Delta\cup\Delta')}$ again by the Frattini argument since $G_{(\Delta\cup\Delta')} \lhd G_{x,y}$. Then

$$|G_x : G_{x,y}| = \frac{\left|N_{G_x}(U) : N_{G_{x,y}}(U)\right| \cdot \left|G_{(\Delta)} : G_{(\Delta\cup\Delta')}\right|}{\left|N_{G_{(\Delta)}}(U) : N_{G_{(\Delta\cup\Delta')}}(U)\right|},$$

and hence $\lambda+1 \mid \left|N_{G_x}(U) : N_{G_{x,y}}(U)\right|$ since $\lambda+1 \mid |G_x : G_{x,y}|$ by Lemma 2.1 and $\left(\lambda+1, \left|G_{(\Delta)}\right|\right) = 1$ by Corollary 2.6, being $v = \lambda^2(\lambda+2)$ even. Therefore $\lambda+1 \mid \left|y^{N_{G_x}(U)}\right|$ and $y^{N_{G_x}(U)} \subseteq \mathrm{Fix}(U) \setminus \{x\}$. Since $\mathrm{Fix}(U) \setminus \{x\}$ is a disjoint union of $N_{G_x}(U)$-orbits, it follows that $\lambda+1 \mid |\mathrm{Fix}(U)| - 1$. Therefore, $\lambda+1 \mid t-1$ since $|\mathrm{Fix}(U)| = t(\lambda+2)$. So $|\mathrm{Fix}(\zeta)| \geqslant |\mathrm{Fix}(U)| \geqslant (\lambda+2)^2$ for any non-trivial element $\zeta \in U$ since $t \geqslant 2$, but this contradicts Lemma 2.5(1). Thus, each prime divisor of $2^{a-1}-1$ divides $\left|G_x^{\Delta}\right|$. Clearly, $G_x^{\Delta} \cong SL_7(2)$ for $a=7$. If $a \neq 7$, then $2^{a-1}-1$ admits a primitive prime divisor by [42]. At this point, it is easy to check that the unique admissible cases are $G_x^{\Delta} \cong SL_a(2)$ for $a \geqslant 4$, or $G_x^{\Delta} \cong A_7$ and $a=4$ by using [19, Proposition 5.2.15].

The fact that the two possibilities for G_x^{Δ} are simple groups together with Proposition 2.4(1) imply that either $G_{(\Sigma)}^{\Delta} = G_{\Delta}^{\Delta}$, or $G_{(\Sigma)}^{\Delta} = Soc(G_{\Delta}^{\Delta})$. The former yields $G_{\Delta} = G_{(\Sigma)}G_{(\Delta)}$, and hence $\lambda+1 \mid \left|G_{(\Sigma),x}\right|$ since $\left(\lambda+1, \left|G_{(\Delta)}\right|\right) = 1$ by Corollary 2.6 since v is even. However, this contradicts Lemma 2.7 since $\lambda+1$ is odd. Thus $G_{(\Sigma)}^{\Delta} = Soc(G_{\Delta}^{\Delta})$, and hence $\frac{G_{\Delta}^{\Sigma}}{G_{(\Delta)}^{\Sigma}} \cong \frac{G_{\Delta}^{\Delta}}{G_{(\Sigma)}^{\Delta}} \cong G_x^{\Delta}$ by (2.1). Since $\frac{G_{(\Sigma)}}{G_{(\Sigma)}\cap G_{(\Delta)}} \cong Soc(G_{\Delta}^{\Delta})$ is an elementary abelian 2-group, we have that $\Phi(G_{(\Sigma)}) \leqslant G_{(\Sigma)} \cap G_{(\Delta)}$ for each $\Delta \in \Sigma$. Thus $\Phi(G_{(\Sigma)})$ fixes each point of $\mathcal{D}$, hence $\Phi(G_{(\Sigma)}) = 1$, and so $G_{(\Sigma)}$ is an elementary abelian 2-group.

If $C_G(G_{(\Sigma)}) \neq G_{(\Sigma)}$, then $G = C_G(G_{(\Sigma)})$ since G^{Σ} is primitive on Σ by Proposition 2.4. Thus $G_{(\Sigma)}$ is a subgroup of $Z(G)$ acting transitively on each Δ, and hence $G_x \leqslant G_{(\Delta)}$ for any $x \in \Delta$. However, this is a contradiction since G_{Δ}^{Δ} is 2-transitive on Δ. Therefore $C_G(G_{(\Sigma)}) = G_{(\Sigma)}$, and we obtain (2). $\square$

3.2. Minimal degree of the non-trivial primitive permutation representations of a group

For any group Γ define

$$P(\Gamma) = \min\{c : \Gamma \textit{ has a non-trivial permutation representation of degree } c\}.$$

Note that, $P(\Gamma)$ is the index of the largest proper subgroup of Γ, which necessarily is maximal in Γ, hence $P(\Gamma)$ is the minimal degree of the non-trivial primitive permutation representations of Γ. More information on $P(\Gamma)$ can be found in [19, Section 5.2].

Theorem 3.3. *If $\mathcal{D}$ is of type 1 and v is even, then Theorem 3.1 holds.*

Proof. Assume that $\mathcal{D}$ is of type 1. The assertion follows from Proposition 2.4(2) for $G_{(\Sigma)} = 1$. Hence, assume that $G_{(\Sigma)} \neq 1$. Since G^{Σ} acts primitively on Σ again by Proposition 2.4 and $|\Sigma| = 2^2(2^{a-1}-1)^2$ with $a \geqslant 4$ by Lemma 3.2, it follows from the O'Nan-Scott theorem (e.g. see [12, Theorem 4.1A]) that one of the following holds:

(i) $Soc(G^{\Sigma})$ is non-abelian simple.
(ii) $Soc(G^{\Sigma}) \cong T^2$, where T is a non-abelian simple group such that $|T| = 2^2(2^{a-1}-1)^2$.
(iii) $Soc(G^{\Sigma}) \cong T^2$ and there is a non-abelian almost simple group Q with socle T acting primitively on a set Θ of size $2(2^{a-1}-1)$ such that $\Sigma = \Theta^2$ and $G^{\Sigma} \leqslant Q \wr Z_2$.

Assume that (ii) holds. Then $T \leqslant G^{\Sigma}_{\Delta} \leqslant Aut(T) \times Z_2$, and hence G^{Σ}_{Δ}/T is solvable. Therefore $\left|G^{\Delta}_x\right| \mid |T|$ since $G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \cong G^{\Delta}_x$ with G^{Δ}_x isomorphic either to $SL_a(2)$, or to A_7 for $a = 4$ by Lemma 3.2(2). However, this is impossible since $\left|G^{\Delta}_x\right| \nmid 2^2(2^{a-1}-1)^2$, and (ii) is excluded.

Assume that (iii) holds. Since $G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \cong G^{\Delta}_x$ by Lemma 3.2(2), then $\lambda+1 = 2^a - 1$ divides the order of G^{Σ}_{Δ}. If $a = 6$ then $\lambda = 2 \cdot 31$, $|\Sigma| = 2^2 31^2$ and $G^{\Delta}_x \cong SL_6(2)$. Since $\lambda(\lambda+1) \mid |G_{\Delta}|$, it follows that $31^3 \mid |G|$. On the other hand, $G^{\Sigma} \leqslant Q \wr Z_2$, where $\mathrm{Soc}(Q)$ is isomorphic to one of the groups A_{62} or $PSL_2(61)$ by [12, Table B.2], and hence 31 divides the order of $G_{(\Sigma)}$, which is not the case by Lemma 3.2(2). Thus $a \neq 6$, and hence there is a primitive prime divisor z of $\lambda + 1 = 2^a - 1$ by [42]. If $z \mid |G_{\Delta,\Delta_1}|$ for some $\Delta_1 \in \Sigma \setminus \{\Delta\}$, it follows that there is a z-element φ of G_{Δ,Δ_1} fixing a point x in Δ and x' in Δ_1. Therefore, it fixes at least one of the $\lambda = 2(2^{a-1}-1)$ blocks incident with x and x', say B. Hence, φ fixes a further element in $B \cap \Delta$. Thus, $\varphi \in G_{(\Delta)}$ by [14, Theorem 3.5] since G^{Δ}_{Δ} is of affine type. However, this is impossible by Corollary 2.6(1) since $z \mid \lambda+1$. Then $z \nmid |G_{\Delta,\Delta_1}|$ and hence $z \mid \left|G^{\Sigma}_{\Delta} : G^{\Sigma}_{\Delta,\Delta'}\right|$ for each $\Delta' \in \Sigma \setminus \{\Delta\}$ since $G_{(\Sigma)}$ is a 2-group by Lemma 3.2(2). Actually, $z \mid \left|G^{\Sigma}_{\Delta} : G^{\Sigma}_{(\Delta)} G^{\Sigma}_{\Delta,\Delta'}\right|$ again by Corollary 2.6(1), hence $z \mid \left|G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} : G^{\Sigma}_{(\Delta)} G^{\Sigma}_{\Delta,\Delta'}/G^{\Sigma}_{(\Delta)}\right|$. Thus

$$P(G^{\Delta}_x) \leqslant \left|G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} : G^{\Sigma}_{(\Delta)} G^{\Sigma}_{\Delta,\Delta'}/G^{\Sigma}_{(\Delta)}\right| \leqslant \left|G^{\Sigma}_{\Delta} : G^{\Sigma}_{\Delta,\Delta'}\right|$$

since $G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \cong G^{\Delta}_x$ by Lemma 3.2(2), where $P(G^{\Delta}_x)$ is the minimal degree of the non-trivial primitive permutation representations of G^{Δ}_x. We may choose $\Delta' \in \Sigma \setminus \{\Delta\}$ such that

$$P(G^{\Delta}_x) \leqslant \left|G^{\Sigma}_{\Delta} : G^{\Sigma}_{\Delta,\Delta'}\right| \leqslant 2\frac{2(2^{a-1}-1)}{s-1},$$

where s denotes the rank of Q on Θ since $G^{\Sigma} \leqslant Q \wr S_2$ and $\Sigma = \Theta^2$. If $a \neq 4$, then $P(G^{\Delta}_x) = 2^a - 1$ by [19, Theorem 5.2.2], hence $s = 2$. Then Q acts 2-transitively on the set Θ, with $|\Theta| = 2(2^{a-1}-1)$, and hence $Q \wr Z_2$ acts as a primitive rank 3 group on $\Sigma = \Theta^2$. Moreover, the $(Q \wr Z_2)_{\Delta}$-orbits on $\Sigma \setminus \{\Delta\}$ are two of length $|\Theta| - 1$, one of $(|\Theta|-1)^2$. Each of these orbits is a union of G^{Σ}_{Δ}-orbits, and since the length of each G^{Σ}_{Δ}-orbit distinct from $\{\Delta\}$ is divisible by z, then $z \mid |\Theta|-1$. So $z \mid 2^a-3$, whereas z is a divisor of 2^a-1. Therefore $a=4$, $s=2,3,4$ and $G^{\Delta}_{x} \cong A_m$ with $m=7,8$ since $SL_4(2) \cong A_8$.

Suppose that $s=3$ or 4. Then $\left|G^{\Sigma}_{\Delta} : G^{\Sigma}_{\Delta,\Delta'}\right| \leqslant \frac{28}{s-1} \leqslant 14$, and hence $\left|G^{\Sigma}_{\Delta} : G^{\Sigma}_{\Delta,\Delta'}\right| = m$ by [7]. Then $m \mid |\Sigma|-1$ with $m=7,8$, and we reach a contradiction since $|\Sigma|-1 = 3\cdot 5\cdot 13$. Thus $s=2$, and hence Q acts 2-transitively on Θ. As above, $Q \wr Z_2$ acts as a primitive rank 3 group on $\Sigma=\Theta^2$, and the $(Q\wr Z_2)_{\Delta}$-orbits on $\Sigma\setminus\{\Delta\}$ are two of length 13 and one of length 13^2. None of these lengths is divisible by $z=5$, and so this case is excluded. Thus only (i) occurs, which is the assertion. □

Remark 3.4. If $\mathcal{D}$ is of type 1 with v even and $G_{(\Sigma)} \neq 1$, it follows from Lemma 3.2(2) and Theorem 3.3 that G^{Σ} is an almost simple subgroup of $GL_{a+t}(2)$, where $|\Delta| = 2^a$ with $\Delta \in \Sigma$, $\left|G_{(\Sigma)}\right| = 2^{a+t}$ and $\left|G_{(\Sigma)} \cap G_{(\Delta)}\right| = 2^t$. However, it is not easy to fully exploit the previous property because it is not easy to control the order of $G_{(\Sigma)} \cap G_{(\Delta)}$ in this case. This motivates our choice of an alternative proof in which the embedding of G^{Σ} in $GL_{a+t}(2)$ is partially considered.

3.3. Primitive prime divisors

A divisor s of q^e-1 that is coprime to each q^i-1 for $i<e$ is said to be a *primitive divisor*, and we call the largest primitive divisor $\Phi^*_e(q)$ of q^e-1 the *primitive part* of q^e-1. One should note that $\Phi^*_e(q)$ is strongly related to cyclotomy in that it is equal to the quotient of the cyclotomic number $\Phi_e(q)$ and $(n, \Phi_e(q))$ when $e>2$. Also, $\Phi^*_e(q) > 1$ for $e>2$ and $(q,e) \neq (2,6)$ by Zsigmondy's Theorem (for instance, see [33, P1.7]).

Theorem 3.5. *If $\mathcal{D}$ is of type 1 and v is odd, then G acts point-quasiprimitively on $\mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ is of type 1. Recall that $G_{(\Sigma)}$ is an elementary abelian p-group acting regularly on Δ, p is and odd prime and $G^{\Sigma} \leqslant GL_a(p)$ by Lemma 2.8. Thus $\lambda+2 = |\Delta| = p^a$, $a \geqslant 1$, and hence $\lambda = p^a-2$. Therefore, $(p^a-1)(p^a-2)$ divides $|G_x|$, and hence $|G|$, as $k \mid |G_x|$. Also, $(p^a-1)(p^a-2)^3 \mid \left|G^{\Sigma}\right|$ since $|\Sigma| = (p^a-2)^2$.

If $a=2$ then $(p^2-1)(p^2-2)^3 \mid \left|G^{\Sigma}\right|$ with $G^{\Sigma} \leqslant GL_2(p)$, which is a contradiction. Thus $a>2$, and hence $\Phi^*_a(p) > 1$ by [42] since p is odd. Then G^{Σ} is an irreducible subgroup of $GL_a(p)$ by [14, Theorem 3.5(iv)]. For each divisor m of a the group $\Gamma L_{a/m}(p^m)$ has a natural irreducible action. We may choose m to be maximal such that $G^{\Sigma} \leqslant \Gamma L_{a/m}(p^m)$. Set $K^{\Sigma} = G^{\Sigma} \cap GL_{a/m}(p^m)$. Then $\Phi^*_a(p)\frac{(p^a-2)^3}{((p^a-2)^3,d)} \mid \left|K^{\Sigma}\right|$ by [19, Proposition 5.2.15.(ii)]. Easy computations show that the order of $GL_{a/m}(p^m)$ and hence, $\left|K^{\Sigma}\right|$ is not divisible by $p^a\Phi^*_a(p)\frac{(p^a-2)}{((p^a-2),d)}$ for $(d/m, p^m) = (3,5^2), (4,3), (6,3), (6,5), (9,3)$. So, these cases are excluded. Thus, bearing in mind the maximality of m and the fact that p is odd, [4, Theorem 3.1] implies that K^{Σ} contains a normal subgroup isomorphic to one of the groups $SL_{a/m}(p^m), Sp_{a/m}(p^m), \Omega^-_{a/m}(p^m)$, or $SU_{a/m}(p^{m/2})$ with a/m odd. Since

$\left|G^{\Sigma} : G^{\Sigma}_{\Delta}\right| = \lambda^2$, and $\lambda = p^a - 2$ with p odd, it follows that G^{Σ}_{Δ} is a maximal parabolic subgroup of G^{Σ} by Proposition 2.4 and [34, Theorem 1.6]. Also, $\Phi^*_a(p) \mid \left|G^{\Sigma}_{\Delta}\right|$ since $(\Phi^*_a(p), p^a - 2) = 1$, but this contradicts [14, Theorem 3.5(iv)] applied to G^{Σ}_{Δ}. Thus $G_{(\Sigma)} = 1$, and hence G acts point-quasiprimitively on $\mathcal{D}$ by Proposition 2.4(2). □

3.4. Designs of type 2 and point-quasiprimitivity

Lemma 3.6. *If $\mathcal{D}$ is of type 2 and $\lambda = 2w^2$, where w is odd, $w \geqslant 3$ and $2(w^2 - 1)$ is a square, then G acts point-quasiprimitively on $\mathcal{D}$.*

Proof. Suppose that $\mathcal{D}$ is of type 2 with $\lambda = 2w^2$, where w is odd, $w \geqslant 3$ and $2(w^2 - 1)$ is a square. Then $|\Delta| = \lambda/2 + 1 = w^2 + 1$ is even. If $\mathrm{Soc}(G^{\Delta}_{\Delta})$ is an elementary abelian 2-group, then $w^2 + 1 = 2^s$ for some $s \geqslant 1$. However, it has no integer solutions for [33, B1.1]. Thus $\mathrm{Soc}(G^{\Delta}_{\Delta})$ is non-abelian simple, and hence $\mathrm{Soc}(G^{\Delta}_{\Delta})$ is isomorphic to one of the groups A_{w^2+1}, $PSL_a(s)$ with $w^2 + 1 = \frac{s^a-1}{s-1}$, $a \geqslant 2$ and $(d, s) \neq (2, 2), (2, 3)$, or $PSU_3(w^{2/3})$ by [18, List (A)] since $|\Delta| = w^2 + 1$. Moreover, in the second case one has $w^2 = s\frac{s^{a-1}-1}{s-1}$, and so s is an even power of an odd prime. Therefore $\left(\frac{w}{s^{1/2}}\right)^2 = \frac{s^{a-1}-1}{s-1}$, and hence $a = 2$ by [33, A7.1, A8.1 and B1.1]. Thus $s = w^2$ and $\mathrm{Soc}(G^{\Delta}_{\Delta}) \cong PSL_2(w^2)$.

Note that, $w^2 - 1 \neq 2^t$ with $t \geqslant 1$. Indeed, if it is not so, then $t = w = 3$ by [33, B1.1], and hence $(\lambda, |\Delta|, |\Sigma|) = (18, 10, 145)$ and $\mathrm{Soc}(G^{\Delta}_{\Delta}) \cong A_6$ and $\mathrm{Soc}(G^{\Sigma}) \cong A_{145}$ by [12, Table B.4]. Then $A_{144} \trianglelefteq G^{\Sigma}_{\Delta} \leqslant S_{144}$ but this contradicts (2.1) since $G^{\Delta}_{\Delta} \leqslant P\Gamma L_2(9)$. Thus, in each case $\mathrm{Soc}(G^{\Delta}_{\Delta})$ contains an element of order an odd prime divisor of $w^2 - 1$ fixing at least two points on Δ.

If $G_{(\Sigma)} \neq 1$ then $\mathrm{Soc}(G^{\Delta}_{\Delta}) \trianglelefteq G^{\Delta}_{(\Sigma)}$ by Proposition 2.4(1). Now, let $\eta \in G_{(\Sigma)}$ be an element of order an odd prime divisor of $w^2 - 1$, which exists by the previous argument. Then, for each $\Delta \in \Sigma$ either $\eta \in G_{(\Delta)}$, or η induces an element of $G^{\Delta}_{(\Sigma)}$ fixing at least two points of Δ. Therefore $|\mathrm{Fix}(\eta)| \geqslant 2|\Sigma| = \lambda^2 - 2\lambda + 2$, but this contradicts Lemma 2.5(2) since $\lambda > 10$. Thus $G_{(\Sigma)} = 1$, and hence G acts point-quasiprimitively on $\mathcal{D}$ by Proposition 2.4(2). □

Theorem 3.7. *If $\mathcal{D}$ is of type 2, then G acts point-quasiprimitively on $\mathcal{D}$.*

Proof. Suppose that $\mathcal{D}$ is of type 2. If $\lambda = 2w^2$, where w is odd, $w \geqslant 3$, and $2(w^2 - 1)$ is a square, the assertion follows from Lemma 3.6. Thus, we may assume that $\lambda \equiv 0 \pmod 4$ by Theorem 2.2. Hence, v is odd.

Suppose that $G_{(\Sigma)} \neq 1$. Let $\Delta \in \Sigma$, then there is an odd prime p such that $G_{(\Sigma)} \cong Soc(G^{\Delta}_{\Delta})$ is an elementary abelian p-group acting regularly on Δ and $G^{\Sigma} \leqslant GL_a(p)$ with $p^a = |\Delta|$ by Lemma 2.8 since v is odd. Then $\lambda = 2(p^a - 1)$ and $k = \lambda^2/2 = 2(p^a - 1)^2$. Then $(p^a - 1)^2 \mid |G_B|$, where B is any block of $\mathcal{D}$, since G_B acts transitively on B. Then $(p^a - 1)^2$ divides $\left|G^{\Sigma}\right|$ since $\left|G_{(\Sigma)}\right| = p^a$. Now, we may proceed as in Theorem 3.5 and we see that no admissible groups arise in this case as well. Thus $G_{(\Sigma)} = 1$, and hence G acts point-quasiprimitively on $\mathcal{D}$ by Proposition 2.4(2). □

Proof of Theorem 3.1. The assertion follows from Theorems 3.3 and 3.5 for $\mathcal{D}$ of type 1, from Theorem 3.7 for $\mathcal{D}$ of type 2. $\square$

4. Reduction to the almost simple case

In this section, we analyze the case where G acts point-quasiprimitively on $\mathcal{D}$. In this case $G = G^{\Sigma}$. The main tool used to tackle this case is the O'Nan-Scott theorem for quasiprimitive groups which is proven in [30] and is reported below for reader's convenience. We investigate each of the seven possibilities for G^{Σ} provided in the above mentioned theorem by adapting the techniques developed in [41], and we show that $\mathrm{Soc}(G^{\Sigma})$ is a non-abelian simple group. As we will see, this fact together with the conclusions of Theorem 3.1 yields the following result.

Theorem 4.1. *G^{Σ} is an almost simple group acting primitively on Σ. Moreover, one of the following holds:*

(1) $G_{(\Sigma)} = 1$ and G acts point-quasiprimitively on $\mathcal{D}$.
(2) $G_{(\Sigma)}$ is a self-centralizing elementary abelian 2-subgroup of G. Also, the following hold:

(a) $\mathcal{D}$ is a symmetric 2-$(2^{a+2}(2^{a-1}-1)^2, 2\left(2^a-1\right)\left(2^{a-1}-1\right), 2(2^{a-1}-1))$ design with $a \geqslant 4$;
(b) $G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \cong G^{\Delta}_x$ and either $G^{\Delta}_x \cong SL_a(2)$ with $a \geqslant 4$, or $G^{\Delta}_x \cong A_7$ and $a = 4$.

We only need to prove that G^{Σ} is almost simple, the remaining parts of Theorem 4.1 have been already proven in Theorem 3.1

4.1. Types of quasiprimitive groups

In the sequel we denote $\mathrm{Soc}(G)$ simply by L. Then $L \cong T^h$ with $h \geqslant 1$, where T is a simple group. Let $x \in \mathcal{P}$. By [30, Theorem 1], one of the following holds:

I. *Affine groups.* Here $T \cong Z_p$ for some prime p, and L is the unique minimal normal subgroup of G and is regular on $\mathcal{P}$ of size p^h. The set $\mathcal{P}$ can be identified with $L \cong Z_p^h$ so that $G \leqslant AGL_h(p)$ with L the translation group and $G_x = G \cap GL_h(p)$ acting irreducibly on L. Moreover, G acts primitively on $\mathcal{P}$.
II. *Almost simple groups.* Here $h = 1$, T is a non-abelian simple group, $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$ and $G = TG_x$.
III. In this case $L \cong T^h$ with $h \geqslant 2$ and T is a non-abelian simple group. We distinguish three types:

III(a). *Simple diagonal action.* Define

$$W = \{(\alpha_1, ..., \alpha_h) \cdot \pi : \alpha_i \in \mathrm{Aut}(T), \pi \in S_h, \alpha_i \equiv \alpha_j \pmod{\mathrm{Inn}(T)} \text{ for all } i, j\},$$

where $\pi \in S_h$ just permutes the components α_i naturally. With the usual multiplication, W is a group with socle $L \cong T^h$, and $W = L.(\mathrm{Out}(T) \times S_h)$. The action of W on $\mathcal{P}$ is defined by setting

$$W_x = \{(\alpha, ..., \alpha) \cdot \pi : \alpha \in \mathrm{Aut}(T), \pi \in S_h\}.$$

Thus $W_x \cong \mathrm{Aut}(T) \times S_h$, $L_x \cong T$ and $|\mathcal{P}| = |T|^{h-1}$.

For $1 \leqslant i \leqslant h$ let T_i be the subgroup of L consisting of the h-tuples with 1 in all but the i-th component, so that $T_i \cong T$ and $L \cong T_1 \times \cdots \times T_h$. Put $\mathcal{T} = \{T_1, ..., T_h\}$, so that W acts on $\mathcal{T}$. We say that subgroup G of W is of type III(a) if $L \leqslant G$ and, letting P the permutation group of $G^{\mathcal{T}}$, one of the following holds:

(i) P is transitive on $\mathcal{T}$;
(ii) $h = 2$ and $P = 1$.

We have $G_x \leqslant \mathrm{Aut}(T) \times P$ and $G \leqslant L.(\mathrm{Out}(T) \times P)$. Moreover, in case (i) L is the unique minimal normal subgroup of G and G is primitive on $\mathcal{P}$ if and only if P is primitive on $\mathcal{T}$. In case (ii) G has two minimal normal subgroups T_1 and T_2, both regular on $\mathcal{P}$, and G is primitive on $\mathcal{P}$.

III(b). *Product action.* Let H be a quasiprimitive permutation group on a set Γ of type II or III(a). For $l > 1$, let $W = H \wr S_l$, and take W to act $\Lambda = \Gamma^l$ in its natural product action. Then for $y \in \Gamma$ and $z = (y, ..., y) \in \Lambda$ we have $W_z = H_y \wr S_l$ and $|\Lambda| = |\Gamma|^l$. If K is the socle H, then the socle L of W is K^l.

Now W acts naturally on the l factors in K^l, and we say that a subgroup G of W is of type III(b) if $L \leqslant G$, G acts transitively on these l factors, and one of the following holds:

(i) H is of type II, $K = T$, $h = l$, and L is the unique minimal normal subgroup G; further Λ is a G-invariant partition of $\mathcal{P}$ and, for x in the part $z \in \Lambda$, $L_z = T_y^h < L$ and for some non-trivial normal subgroup R of T_y, L_x is a subdirect product of R^k, that is L_x projects surjectively on each of the direct factors R.
(ii) H is of type III(a), $\mathcal{P} = \Lambda$, $K \cong T^{h/l}$ with $h/l \geqslant 2$, and both G and H have m minimal normal subgroups where $m \leqslant 2$. If $m = 2$ then each of the two minimal normal subgroups of G is regular on $\mathcal{P}$.

III(c). *Twisted wreath action.* Here G is a twisted wreath action $T \wr_\phi P$ defined as follows. Let P have a transitive action on $\{1, ..., h\}$ and let Q be the stabilizer P_1 of the point 1 in this action. We suppose that there is an homomorphism $\phi : Q \to \mathrm{Aut}(T)$ such that $\mathrm{core}_P\left(\phi^{-1}(\mathrm{Inn}(T))\right) = \cap_{x\in P}\phi^{-1}(\mathrm{Inn}(T))^x = \{1\}$. Define

$$L = \left\{ f : P \to T : f(\alpha\beta) = f(\alpha)^{\phi(\beta)} \text{ for all } \alpha \in P,\, \beta \in Q \right\}.$$

Then B is a group under the pointwise multiplication, and $L \cong T^l$. Let P act on L by

$$f(\gamma) = f(\alpha\gamma) \text{ for } \alpha, \gamma \in P.$$

Define $T \wr_\phi P$ to be the semidirect product of L by P with this action, and define the action on $\mathcal{P}$ by setting $G_x = P$. Then $|\mathcal{P}| = |T|^h$, and L is the unique minimal normal subgroup of G and acts regularly on $\mathcal{P}$. We say that G is of type III(c).

Theorem 4.2. *If $\mathcal{D}$ is of type 1, then G is almost simple.*

Proof. Suppose that $\mathcal{D}$ is of type 1. Case (I) is ruled out since G acts imprimitively on $\mathcal{P}$ by our assumptions. Suppose that G is of type III(a) or III(c). Then $(\lambda+2)\lambda^2 = |\mathcal{P}| = |T|^j$ where T is non-abelian simple and $j = h - 1$ or h, where $h \geqslant 2$, respectively. Hence, λ is even.

If $j = 1$, then $h = j + 1 = 2$ and hence G acts primitively on $\mathcal{P}$ (see III(a)), which is a contradiction. Thus $j > 1$. Note that $\left|x^{T_i}\right| = |T|$ since T_i acts semiregularly on $\mathcal{P}$, where T_i is the subgroup of L consisting of the h-tuples with 1 in all but the i-th component, $T_i \cong T$. Moreover, $x^{T_{i_1}} \cap x^{T_{i_2}} = \{x\}$ for each i_1, i_2 such that $i_1 \neq i_2$. Let $w \in x^{T_1} \setminus \{x\}$, then w^{G_x} is the disjoint union of $w^{G_x} \cap x^{T_i}$ for $1 \leqslant i \leqslant h$. Therefore $\left|w^{G_x}\right| = \left|w^{G_x} \cap x^{T_i}\right| h$ since G_x permutes transitively $T_1, ..., T_h$. On the other hand $\lambda + 1 \mid \left|w^{G_x}\right|$ by Lemma 2.1, hence $\lambda + 1 \mid \left|w^{G_x} \cap x^{T_i}\right| h$. Therefore, $\lambda + 1 \leqslant \left|w^{G_x} \cap x^{T_i}\right| h \leqslant |T|\, h$, and hence $(\lambda + 2)\lambda^2 = |T|^j$ implies

$$|T|^j \leqslant (|T|\, h - 1)^2(|T|\, h + 1) \leqslant |T|^3\, h^3,$$

and hence $j = 2$ or 3, or $j = 4$ and $T \cong A_5$, as $j > 1$ and $|T| \geqslant 60$. The latter is ruled out since it does not provide integer solutions for $(\lambda + 2)\lambda^2 = |T|^4$. If $j = 3$ then $|T| > \lambda > |T| - 2$ since $(\lambda + 2)\lambda^2 = |T|^3$, and hence $\lambda = |T| - 1$, but this contradicts λ even. Thus, $j = 2$. If $h = 3$ then G acts primitively on $\mathcal{P}$. Indeed, G is as in III(a) and $Z_3 \leqslant P \leqslant S_3$. Thus $j = h = 2$ and G is as in III(c). Moreover, $\lambda + 1 \mid \left|w^{G_x} \cap x^{T_1}\right|$ for each $w \in x^{T_1} \setminus \{x\}$ since λ is even. Then there is $\theta \geqslant 1$ such that $|T| = \theta(\lambda + 1) + 1$ since $\left|x^{T_1}\right| = |T|$. Then $\lambda^2(\lambda + 2) = |T|^2$ implies

$$(\lambda + 1)\left((\lambda + 1)\lambda - 1\right) = \theta^2(\lambda + 1)^2 + 2\theta(\lambda + 1) \tag{4.1}$$

and hence $\lambda(\lambda + 1) - 1 = \theta^2(\lambda + 1) + 2\theta$. Then

$$\theta = \frac{(\lambda+1)t-1}{2}$$

for some $t \geqslant 1$. If $t \geqslant 2$ then $\theta > \lambda$, and we reach a contradiction, thus $t = 1$ and $\theta = \lambda/2$, which yields a contradiction when substituted in (4.1). Therefore, G is not of type III(a) or III(c).

Suppose that G is of type III(b.ii). Then $G_z \leqslant W_z = H_y \wr S_l$ where $z = (y, y, ..., y)$, and denoted by μ the number of W_z-orbits on Δ, we see that $\mu \geqslant 4$ by [11, Corollary 1.9].

Let $y_1 \in \Delta$, $y_1 \neq y$ such that $\left|y_1^H\right| \leqslant \frac{|\Delta|-1}{\mu-1}$, and let $z_1 = (y_1, y, \ldots, y)$. Since $(H_{y,y_1} \times H_y \times \cdots \times H_y) : S_{l-1} \leqslant W_{z,z_1}$, it results

$$\left|z_1^{G_z}\right| \leqslant \left|z_1^{W_z}\right| \leqslant \frac{\left|H_y\right|^l (l!)}{\left|H_{y,y_1}\right| \left|H_y\right|^{l-1} ((l-1)!)} = l \left|y_1^H\right| \leqslant \frac{l\,(|\Delta|-1)}{\mu-1},$$

and since $\lambda+1 \mid \left|z_1^{G_z}\right|$ by Lemma 2.1, it follows that

$$\lambda + 1 \leqslant \frac{l\,(|\Delta|-1)}{\mu-1}.$$

Then λ is even since $(\lambda+2)\lambda^2 = |\mathcal{P}| = \left(|T|^{h/l-1}\right)^l = |T|^{h-l}$ with $h/l \geqslant 2$ and $l > 1$. Thus

$$|T|^{l(h/l-1)/3} - 1 \leqslant \frac{l\left(|T|^{h/l-1} - 1\right)}{\mu - 1} < \frac{h\left(|T|^{h/l-1} - 1\right)}{3},$$

and hence $l = 2, 3$ since $l > 1$. If $l = 3$ then $|T|^{h/3-1} > \lambda > |T|^{h/3-1} - 2$, and hence $\lambda = |T|^{h/3-1} - 1$ with $h > 3$, whereas λ is even. Thus $l = 2$. Then $\lambda + 1 \mid |T|^{h/2-1} - 1$ by Lemma 2.1 since λ is even and $(\Delta \setminus \{y\} \times \{y\}) \cup (\{y\} \times \Delta \setminus \{y\})$ is union of some non-trivial G_z-orbits, being $G_z \leqslant H_z \wr S_l$. Now, we may apply the final argument used to rule out III(a) with $|T|^{h/2-1}$ in the role of $|T|$ to exclude this case as well.

Finally, assume that G is of type III(b.i). Then $h = l$ and T^h is the unique minimal normal subgroup of G. In this case Σ can be identified with the Cartesian product Γ^h, hence each $\Delta \in \Sigma$ corresponds to a unique h-tuple of elements of Γ. Therefore, it results that $|\Sigma| = \lambda^2 = |\Gamma|^h$. Let $y \in \Gamma$ and $\Delta = (y, ..., y) \in \Sigma$. Then

$$\bigcup_{i=1}^{h} (\{y\} \times \cdots \times \Gamma_i \setminus \{y\} \times \cdots \times \{y\})$$

is a union of some non-trivial G_Δ-orbits and ultimately of some non-trivial G_x-orbits, where $x \in \Delta$. Then $\lambda+1 \mid (\lambda+2)h\,(|\Gamma|-1)$ by Lemma 2.1. Thus $\lambda+1 \mid (\lambda+2)h\left(\lambda^{2/h}-1\right)$, and hence $\lambda + 1 \mid h\left(\lambda^{2/h} - 1\right)$.

If $h = 2$ then $\lambda + 1 \mid 4$, whereas $\lambda > 10$. Thus $h \geqslant 3$ and hence $\lambda^{1/3} \leqslant \lambda^{1-2/h} \leqslant h$. Therefore $5^{h/3} \leqslant |T : T_\Delta|^{h/3} \leqslant h^2$ since T is a non-abelian simple group and $|T : T_\Delta| = \lambda^2$, and so $h \leqslant 7$. Actually, $|T : T_\Delta|^{h/2(1-2/h)} \leqslant h$ implies $h = 3$, and hence $\lambda + 1 \mid 3\left(\lambda^{2/3} - 1\right)$ which is impossible since $\lambda > 10$. This completes the proof. $\square$

Theorem 4.3. *If $\mathcal{D}$ is of type 2, then G is almost simple.*

Proof. If $\mathcal{D}$ is of type 2, then $v \not\equiv 0 \pmod 4$ (see Theorem 2.2). Case (I) is ruled out since G acts imprimitively on $\mathcal{P}$ by our assumptions. Also $v \neq |T|^j$ with T a non-abelian simple group and $j \geqslant 1$, hence G is not of type III(a) or III(c). Assume that G is of type III(b.i). Arguing as in Theorem 4.2 we see that

$$\frac{\lambda^2 - 2\lambda + 2}{2} = |\Sigma| = |\Gamma|^h \text{ and } \lambda + 1 \mid (\lambda/2 + 1)h\left(|\Gamma| - 1\right),$$

where $h \geqslant 2$. It follows that $\lambda + 1 \mid h\left(|\Gamma| - 1\right)$. Since $|\Gamma|^l - 1 = \frac{1}{2}\lambda\left(\lambda - 2\right)$, it follows $\lambda + 1 \mid 3h$ and hence $2 \cdot 60^h \leqslant 2\left|\Gamma\right|^h < 9h^2$, which is impossible for $h \geqslant 2$.

Assume that G is of type III(b.ii). Then

$$\left(\frac{\lambda + 2}{2}\right)\left(\frac{\lambda^2 - 2\lambda + 2}{2}\right) = |\mathcal{P}| = |\Gamma|^l, \tag{4.2}$$

where $|\Gamma| = |T|^{k/l-1}$ with $k/l \geqslant 2$, and from (4.2) we derive that $v \equiv 0 \pmod 4$, a contradiction. This completes the proof. $\square$

Proof of Theorem 4.1. The assertion immediately follows from Theorems 3.1, 4.2 and 4.3. $\square$

5. Reduction to the case $\lambda \leqslant 10$

Let L be the preimage in G of $\mathrm{Soc}(G^\Sigma)$. Hence $L^\Sigma \trianglelefteq G^\Sigma \leqslant \mathrm{Aut}(L^\Sigma)$ with L^Σ non-abelian simple by Theorem 4.1. Moreover, $G = G^\Sigma$ and $L = L^\Sigma$ when $G_{(\Sigma)} = 1$. The first part of this section is devoted to prove that either L_Δ^Δ acts 2-transitively on Δ, where $\Delta \in \Sigma$, or $G_{(\Sigma)} = 1$, $Soc(G_\Delta^\Delta) < L_\Delta^\Delta \leqslant G_\Delta^\Delta \leqslant A\Gamma L_1(u^h)$, where $u^h = |\Delta|$. Then we prove that either $\left|L^\Sigma\right| \leqslant \left|L_\Delta^\Sigma\right|^2$, or $|L| \leqslant 4\left|L_\Delta^\Delta\right|^2 \left|\mathrm{Out}(L)\right|^2$, respectively. As we will see, these constraints on L_Δ^Σ are combined with the information contained in [3] and [22] allows us to completely classify $\mathcal{D}$. The analysis of 2-designs of type 1 and 2 is carried out in separate sections.

Recall that the minimal degree of the non-trivial primitive permutation representations of a non-abelian simple group Γ is denoted by $P(\Gamma)$. It is known that $P(A_\ell) = \ell \geqslant 5$, whereas $P(\Gamma)$ is determined in [7], [19, Theorem 5.2.2] and in [36–38] according to whether Γ is sporadic, classical or exceptional of Lie type, respectively.

The following technical lemma is useful to show that L_{Δ}^{Δ} acts 2-transitively on Δ provided that G_{Δ}^{Δ} is not a semilinear 1-dimensional group.

Lemma 5.1. *If Γ is a non-abelian simple group non-isomorphic to $PSL_2(q)$ and such that $P(\Gamma) < 2\left(|\mathrm{Out}(\Gamma)|+1\right)|\mathrm{Out}(\Gamma)|$, then one of the following holds:*

(1) $\Gamma \cong A_\ell$ *with* $\ell = 7, 8, 9, 10, 11$*;*
(2) $\Gamma \cong M_{11}$*;*
(3) $\Gamma \cong PSL_3(q)$ *with* $q = 4, 7, 8, 16$*;*
(4) $\Gamma \cong PSp_4(3)$*;*
(5) $\Gamma \cong PSU_n(q)$ *with* $(n, q) = (3, 5), (3, 8), (4, 3)$*;*
(6) $\Gamma \cong P\Omega_8^+(3)$*.*

Proof. Assume that $\Gamma \cong A_\ell$ with $\ell \geqslant 5$. Thus, $\ell \neq 5, 6$ since $A_5 \cong PSL_2(5)$ and $A_6 \cong PSL_2(9)$. Thus $7 \leqslant \ell = P(\Gamma) < 12$, and we obtain (1).

If Γ is sporadic then $P(\Gamma) < 12$, and hence $\Gamma \cong M_{11}$ by [7], which is (2).

If Γ is a simple exceptional group of Lie type then $P(\Gamma)$ is provided in [36–38], and it is easy to check that no cases arise.

Finally, suppose that Γ is a simple classical group. Assume that $\Gamma \cong PSL_n(q)$, where $q = p^f$, $f \geqslant 1$, and $n \geqslant 3$. We may also assume that $(n, q) \neq (4, 8)$ since $PSL_4(2) \cong A_8$ was analyzed above. Then $P(\Gamma) = \frac{q^n-1}{q-1}$ by [19, Theorem 5.2.2] since $n \geqslant 3$, and hence

$$\frac{q^n - 1}{q - 1} < 2(2(n, q-1)f + 1) \cdot 2(n, q-1)f. \tag{5.1}$$

Assume that $p^f \geqslant 2f(f+1)$. Then $\frac{q^n-1}{q-1} < 4q(q-1)^2$ and hence $n \leqslant 4$. Actually, $(n, q) = (3, 7)$ by (5.1) since $n \geqslant 3$.

Assume that $p^f < 2f(f+1)$. Then either $p = 2$ and $2 \leqslant f \leqslant 6$, or $p = 3$ and $f = 1, 2$. Hence, $n = 3$ and $q = 4, 8, 16$ by (5.1) since $n \geqslant 3$ and since $PSL_3(2) \cong PSL_2(7)$ cannot occur. Thus, we get (3).

We analyze the remaining classical groups by proceeding as in the $PSL_n(q)$-case. Hence, one obtains $PSp_4(2)'$, $PSp_4(3)$, $PSU_n(q)$ with $(n, q) = (3, 5), (3, 8), (4, 2), (4, 3)$, or $P\Omega_8^+(3)$. Since $PSp_4(2)' \cong PSL_2(9)$ and $PSU_4(2) \cong PSp_4(3)$, $(4, 2)$ is excluded whereas the remaining cases yield (4), (5) and (6), respectively. $\square$

Lemma 5.2. *Let $\Delta \in \Sigma$. If $G_{(\Sigma)} = 1$ and $\mathrm{Soc}(G_{\Delta}^{\Delta}) \cong (Z_u)^h$, where u is an odd prime, then $u^h \neq 3^2, 5^2, 7^2, 11^2$, $19^2, 23^2, 29^2, 59^2$ and 3^4*

Proof. Suppose the $u^h = 3^2, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2$ or 3^4. Then $\lambda = u^h - 2$ or $2(u^h - 1)$ according to whether $\mathcal{D}$ is of type $i = 1$ or 2, respectively. Now, Table 1 contains the admissible pairs $(\lambda, |\Sigma|)_i$ corresponding to type i, where $i = 1, 2$.

Let $(\lambda, |\Sigma|)_i$ be as any of the cases listed in Table 1. If $|\Sigma|$ is a power of a prime, then $A_{|\Sigma|} \leqslant G \leqslant S_{|\Sigma|}$ and $A_{|\Sigma|-1} \leqslant G_\Delta \leqslant S_{|\Sigma|-1}$ with $\Delta \in \Sigma$ by [13, Theorem 1] since

Table 1
Admissible $(\lambda, |\Sigma|)_i$ for $\mathcal{D}$ of type $i = 1, 2$.

u^h	3^2	5^2	7^2	11^2	19^2
$(\lambda, \lvert\Sigma\rvert)_1$	$(7, 7^2)$	$(23, 23^2)$	$(47, 47^2)$	$(7 \cdot 17, 7^2 \cdot 17^2)$	$(17, 17^2)$
$(\lambda, \lvert\Sigma\rvert)_2$	$(16, 113)$	$(48, 1105)$	$(96, 4513)$	$(7 \cdot 103, 53 \cdot 4877)$	$(240, 13^4)$
u^h	23^2	29^2	59^2	3^4	
$(\lambda, \lvert\Sigma\rvert)_1$	$(17 \cdot 31, 17^2 \cdot 31^2)$	$(3^3, 3^6)$	$(3 \cdot 19, 3^2 \cdot 19^2)$	$(79, 79^2)$	
$(\lambda, \lvert\Sigma\rvert)_2$	$(1056, 556513)$	$(41^2, 17 \cdot 82913)$	$(2^4 \cdot 3 \cdot 5 \cdot 29, 101 \cdot 149 \cdot 1609)$	$(160, 12641)$	

$G_{(\Sigma)} = 1$. However, this case is ruled out since $\mathrm{Soc}(G_\Delta^\Delta) \cong (Z_u)^h$ with u an odd prime. Thus $|\Sigma|$ is not a power of a prime, hence only the following cases are admissible:

(1) $u^h = 11^2$ and either $(\lambda, |\Sigma|)_1 = (7 \cdot 17, 7^2 \cdot 17^2)$, or $(\lambda, |\Sigma|)_2 = (7 \cdot 103, 53 \cdot 4877)$;
(2) $u^h = 23^2$ and $(\lambda, |\Sigma|)_1 = (17 \cdot 31, 17^2 \cdot 31^2)$;
(3) $u^h = 29^2$ and $(\lambda, |\Sigma|)_2 = (41^2, 17 \cdot 82913)$;
(4) $u^h = 59^2$ and either $(\lambda, |\Sigma|)_1 = (3 \cdot 19, 3^2 \cdot 19^2)$, or $(\lambda, |\Sigma|)_2 = (2^4 \cdot 3 \cdot 5 \cdot 29, 101 \cdot 149 \cdot 1609)$.

Note that, $|\Sigma|$ is odd in (1)–(4), and G is an almost simple group acting primitively on Σ by Theorem 4.1 since $G_{(\Sigma)} = 1$. Further, G_Δ^Δ is a subgroup of $AGL_2(u)$ acting 2-transitively on Δ by Theorem 2.2(III) since $h = 2$. However, there are no such G by [22, Theorem]. Hence, cases (1)–(4) are ruled out. This completes the proof. □

Proposition 5.3. *Let $\Delta \in \Sigma$, then $Soc(G_\Delta^\Delta) \trianglelefteq L_\Delta^\Delta$ and $G = G_x L$, where x is any point of $\mathcal{D}$. Moreover, one of the following holds:*

(1) $G_{(\Sigma)} = 1$, $\mathrm{Soc}(G_\Delta^\Delta) < L_\Delta^\Delta \leqslant G_\Delta^\Delta \leqslant A\Gamma L_1(u^h)$, where $u^h = |\Delta|$, and $\left|G_\Delta^\Delta : L_\Delta^\Delta\right| \mid |\mathrm{Out}(L)|$.
(2) L_Δ^Δ acts 2-transitively on Δ.

Proof. Since $L_\Delta \trianglelefteq G_\Delta$ and G_Δ^Δ acts primitively on Δ, either $L_\Delta = L_{(\Delta)}$, or $Soc(G_\Delta^\Delta) \trianglelefteq L_\Delta^\Delta$ by [12, Theorem 4.3B]. Moreover, $G = G_\Delta L$ since $L^\Sigma = \mathrm{Soc}(G^\Sigma)$ and G^Σ acts primitively on Σ by Theorem 4.1, and hence $G/L = G_\Delta L/L \cong G_\Delta/L_\Delta$. Also, it results that $G/G_{(\Sigma)}L \cong G^\Sigma/L^\Sigma \leqslant \mathrm{Out}(L^\Sigma)$, which is solvable.

Assume that $G_{(\Sigma)} \neq 1$. Since $(G/L)/(G_{(\Sigma)}L/L) \cong G/G_{(\Sigma)}L$ is solvable and $G_{(\Sigma)}L/L \cong G_{(\Sigma)}/L_{(\Sigma)}$ is a 2-group by Theorem 4.1(2), it follows that G/L is solvable. Therefore, G_Δ/L_Δ and $G_\Delta^\Delta/L_\Delta^\Delta \cong G_\Delta/G_{(\Delta)}L_\Delta$ are solvable. Then $L_\Delta^\Delta = G_\Delta^\Delta$ since $G_\Delta^\Delta \cong E_{2^a} : G_x^\Delta$, where G_x^Δ is either $SL_a(2)$, or $G_x^\Delta \cong A_7$ for $a = 4$ by Theorem 4.1(2.b), and the assertion (2) follows in this case.

Assume that $G_{(\Sigma)} = 1$. Then $G = G^\Sigma$ and $L = L^\Sigma$, and hence G_Δ/L_Δ is isomorphic to a subgroup of $\mathrm{Out}(L)$.

If $L_\Delta = L_{(\Delta)}$ then $G_\Delta^\Delta = G_\Delta/G_{(\Delta)} \cong (G_\Delta/L_\Delta)\,/\,\left(G_{(\Delta)}/L_\Delta\right)$. Therefore G_Δ^Δ is isomorphic to a quotient group of G_Δ/L_Δ, and hence $\mathrm{Out}(L)$ contains a subgroup with a quotient isomorphic to G_{Δ}^{Δ}. Then $|\Delta|(|\Delta|-1) \mid |\mathrm{Out}(L)|$ since G_{Δ}^{Δ} acts 2-transitively on Δ. Therefore

$$P(L) \leqslant |L : L_{\Delta}| = |\Sigma| \leqslant 2\,|\Delta|\,(|\Delta|-1) \leqslant 2\,|\mathrm{Out}(L)|\,, \tag{5.2}$$

where $P(L)$ is the minimal degree of the non-trivial primitive permutation representations of L. Then $L \cong PSL_2(q)$ by Lemma 5.1. If $q > 11$ then $q+1 \leqslant 4(2,q-1)\log q$ which has no solutions. So $q = 4,5,7,8,9,11$, which are ruled out since their corresponding value of $|\Delta|$ does not fulfill $|\Delta|(|\Delta|-1) \mid |\mathrm{Out}(L)|$. Thus $L_{\Delta} = L_{(\Delta)}$ is ruled out. Therefore $Soc(G_{\Delta}^{\Delta}) \trianglelefteq L_{\Delta}^{\Delta}$, and hence L acts transitively on Δ. Then L acts point-transitively on $\mathcal{D}$ since L acts transitively on Σ. Thus, $G = G_x L$ where x is any point of $\mathcal{D}$.

Since assertion (2) immediately follows when $Soc(G_{\Delta}^{\Delta})$ is non-abelian simple, we may assume that $Soc(G_{\Delta}^{\Delta})$ is an elementary abelian u-group for some prime u. Suppose that $L_{\Delta}^{\Delta} = Soc(G_{\Delta}^{\Delta})$. Since $G/L = G_{\Delta}L/L \cong G_{\Delta}/L_{\Delta}$, $G/L \leqslant \mathrm{Out}(L)$ and $G_{\Delta}/G_{(\Delta)}L_{\Delta} \cong G_{\Delta}^{\Delta}/L_{\Delta}^{\Delta}$, it follows that $G_{\Delta}^{\Delta}/L_{\Delta}^{\Delta}$ is isomorphic to a quotient subgroup of $\mathrm{Out}(L)$. Thus $|\Delta|-1 \mid |\mathrm{Out}(L)|$. Moreover,

$$P(L) \leqslant |L : L_{\Delta}| = |\Sigma| < 2\,|\Delta|\,(|\Delta|-1) \leqslant 2\,(|\mathrm{Out}(L)|+1)\,|\mathrm{Out}(L)| \tag{5.3}$$

Then either $L \cong PSL_2(q)$, or L is as in Lemma 5.1. In the latter case L is isomorphic neither to A_ℓ with $7 \leqslant \ell \leqslant 11$, nor to M_{11}, $PSp_4(3)$ or $PSU_3(5)$ since $|\Delta|-1 \leqslant |\mathrm{Out}(L)|$ (we use this weaker constraint rather than $|\Delta|-1 \mid |\mathrm{Out}(L)|$ in order to apply this argument to Theorem 5.5 as well), where $|\Delta|$ is $\lambda+2$ or $\lambda/2+1$ according to whether $\mathcal{D}$ is of type 1 or 2, respectively, and $\lambda > 10$. The unique admissible group among the remaining ones listed in Lemma 5.1 is $L \cong PSU_4(3)$ and $(|\Delta|, |\Sigma|) = (20, 18^2)$ or $(38, 36^2)$ by [7] and [5, Tables 8.3–8.4] since $|\Sigma|$ is not of the form as in Theorem 2.2(V). However, both exceptional cases are ruled out since $|\Delta| > |\mathrm{Out}(PSU_4(3))|$. Thus $L \cong PSL_2(q)$ and hence $|\Delta| \leqslant (2,q-1)f$. Then $f \geqslant 3$ for q odd and $f \geqslant 6$ for q even since $|\Delta|$ is $\lambda+2$ or $\lambda/2+1$ according to whether $\mathcal{D}$ is of type 1 or 2, respectively, and $\lambda > 10$. Thus $P(L) = q+1$, hence (5.3) becomes $p^f + 1 \leqslant 2(2,q-1)f((2,q-1)f+1) \leqslant 4f(2f+1)$ which yields $q = 2^6, 3^3, 3^4$ and $\mathcal{D}$ is of type 2 with $\lambda = 12, 12, 16$, respectively. Thus $|\Sigma| = 61, 61, 113$, respectively, but $L \cong PSL_2(q)$ with $q = 2^6, 3^3, 3^4$ has no such transitive permutation degrees. Thus $Soc(G_{\Delta}^{\Delta}) < L_{\Delta}^{\Delta}$ and $\left|G_{\Delta}^{\Delta} : L_{\Delta}^{\Delta}\right| \mid |\mathrm{Out}(L)|$.

If $G_{\Delta}^{\Delta} \nleqslant A\Gamma L_1(u^h)$, then G_{Δ}^{Δ} contains a normal subgroup isomorphic to one of the groups $SL_n(u^{h/n})$, $Sp_n(u^{h/n})$, $G_2'(2^{h/6})$ with $h \equiv 0 \pmod 6$, A_6 or A_7 for $(u,h) = (2,4)$, or $SL_2(13)$ for $(u,h) = (3,6)$ by [18, List(B)] since $u^h \neq 3^2, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2$ and 3^4 by Lemma 5.2, and L_{Δ}^{Δ} acts 2-transitively on Δ in these cases. This completes the proof. □

Corollary 5.4. *If $G_{(\Sigma)} \neq 1$, then $\lambda = 2(2^{a-1}-1)$, $|\Sigma| = \lambda^2$ and a quotient group of L_{Δ}^{Σ} is isomorphic either to $SL_a(2)$ for $a \geqslant 4$, or to A_7 for $a = 4$.*

Proof. If $G_{(\Sigma)} \neq 1$, then $G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \cong G^{\Delta}_x$ where either $G^{\Delta}_x \cong SL_a(2)$ for $a \geqslant 4$, or $G^{\Delta}_x \cong A_7$ for $a = 4$ by Theorem 4.1. On the other hand, $G_x/L_x \cong G_xL/L = G/L$ is isomorphic to a subgroup of $\mathrm{Out}(L)$ as a consequence of Proposition 5.3. Therefore G_x/L_x, and hence $G_x/G_{(\Delta)}L_x$, is solvable. Since $G_x/G_{(\Delta)}L_x \cong G^{\Delta}_x/L^{\Delta}_x$ and G^{Δ}_x is non-abelian simple, it follows that $L^{\Delta}_x = G^{\Delta}_x$. Thus $L^{\Sigma}_{\Delta} \nleqslant G^{\Sigma}_{(\Delta)}$, and hence $L^{\Sigma}_{\Delta}/\left(L^{\Sigma}_{\Delta} \cap G^{\Sigma}_{(\Delta)}\right) \cong G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \cong G^{\Delta}_x$, which is the assertion. □

5.1. Large subgroups

Let X be a finite group. A proper subgroup Y of X is said to be *large* if the order of Y satisfies the bound $|X| \leqslant |Y|^3$. More information on large subgroups can be found in [3].

Theorem 5.5. *Let $\Delta \in \Sigma$, then L^{Σ}_{Δ} is a large subgroup of L^{Σ}. Moreover, one of the following holds:*

(1) $\left|L^{\Sigma}\right| \leqslant \left|L^{\Sigma}_{\Delta}\right|^2$.
(2) $G_{(\Sigma)} = 1$, $Soc(G^{\Delta}_{\Delta}) < L^{\Delta}_{\Delta} \leqslant G^{\Delta}_{\Delta} \leqslant A\Gamma L_1(u^h)$, *where* $u^h = |\Delta|$. *Furthermore, the following holds:*

   (a) L_{Δ} *does not act* 2*-transitively on* Δ*;*
   (b) $|L| \leqslant 4\left|L^{\Delta}_{\Delta}\right|^2 |\mathrm{Out}(L)|^2$*;*
   (c) $\left|L_{(\Delta)}\right| < 2\,|\mathrm{Out}(L)| < |L_{\Delta}|$.

Proof. Suppose that $G_{(\Sigma)} = 1$. Then $G = G^{\Sigma}$ and $L = L^{\Sigma}$. Assume that L^{Δ}_{Δ} acts 2-transitively on Δ. If $\mathcal{D}$ is of type 1, then $|L : L_{\Delta}| = |\Sigma| \leqslant |\Delta|\,(|\Delta| - 1) \leqslant \left|L^{\Delta}_{\Delta}\right|$, and hence $|L| \leqslant |L_{\Delta}|^2$, which is the assertion (1) in this case.

If $\mathcal{D}$ is of type 2, then $\lambda^2/2 \mid |G_x|$ since G acts flag-transitively on $\mathcal{D}$, and hence $\lambda^2/2 \mid |L_x|\,|\mathrm{Out}(L)|$. On the other hand, $|L_{\Delta}| = \left(\frac{\lambda}{2}+1\right)\frac{\lambda}{2}\,|L_{x,y}|$ since L_{Δ} induces a 2-transitive group on $|\Delta|$. In particular, $|L_x : L_{x,y}| = \frac{\lambda}{2}$ and so $\lambda \mid |L_{x,y}|\,|\mathrm{Out}(L)|$. If $\lambda \mid |\mathrm{Out}(L)|$ then $P(L) \leqslant |L : L_{\Delta}| = \frac{\lambda^2-2\lambda+2}{2} < |\mathrm{Out}(L)|^2$. Then $L \cong PSL_2(q)$ or L is one of the groups listed in Lemma 5.1. Actually, in the latter case only $L \cong PSL_3(q)$ with $(q,\lambda) = (4,12), (16,12), (16,24)$ are admissible since $\lambda \mid |\mathrm{Out}(L)|$ and $\lambda > 10$. Then $|\Sigma| = 61$ or 265, respectively, but none of these divides the order of the corresponding L. So these cases are excluded, and hence $L \cong PSL_2(q)$ with $q = p^f$ and $f \geqslant 6$ since $\lambda \mid |\mathrm{Out}(L)|$, $\lambda > 10$ and $|\mathrm{Out}(L)| = (2, p^f-1)f$. However, $p^f+1 \leqslant |L : L_{\Delta}| \leqslant 2f^2-2f+1$ has no admissible solutions for $f \geqslant 6$. Then $(|\mathrm{Out}(L)|, \lambda) < \lambda$ and hence $|L_{x,y}| \geqslant 2$ since $\lambda \mid |L_{x,y}|\,|\mathrm{Out}(L)|$. Then $|L_{\Delta}| \geqslant \left(\frac{\lambda}{2}+1\right)\lambda > |\Sigma| = |L : L_{\Delta}|$ and we obtain the assertion (1) also in this case.

Assume that L^{Δ}_{Δ} does not act 2-transitively on Δ. Then $Soc(G^{\Delta}_{\Delta}) < L^{\Delta}_{\Delta}$ and G^{Δ}_{Δ} is a 2-transitive subgroup of the semilinear 1-dimensional group by Proposition 5.3. Note that, $|L : L_{\Delta}| = |G : G_{\Delta}| \leqslant 2\left|G^{\Delta}_{\Delta}\right|$, $G_{\Delta}/L_{\Delta} \cong G_{\Delta}L/L = G/L \leqslant \mathrm{Out}(L)$ and

$G_\Delta/G_{(\Delta)}L_\Delta \cong G_\Delta^\Delta/L_\Delta^\Delta$ since $G^\Sigma = G$ and $L = \mathrm{Soc}(G)$ is a non-abelian simple group. Hence, $G_\Delta^\Delta/L_\Delta^\Delta$ is isomorphic to a quotient group of a subgroup of $\mathrm{Out}(L)$. Therefore, $|L : L_\Delta| \leqslant 2\left|L_\Delta^\Delta\right| |\mathrm{Out}(L)|$.

Assume that $\left|L_{(\Delta)}\right| \geqslant 2|\mathrm{Out}(L)|$. Therefore $2\left|L_\Delta^\Delta\right| |\mathrm{Out}(L)| \leqslant |L_\Delta|$, hence $|L : L_\Delta| \leqslant |L_\Delta|$, and we still obtain (1).

Assume that $\left|L_{(\Delta)}\right| < 2|\mathrm{Out}(L)|$. Then $|L| \leqslant 4\left|L_\Delta^\Delta\right|^2 |\mathrm{Out}(L)|^2$. Suppose that $|L_\Delta| \leqslant |\mathrm{Out}(L)|$. Then $2|\Delta| \leqslant |L_\Delta| \leqslant 2|\mathrm{Out}(L)|$ since $\mathrm{Soc}(G_\Delta^\Delta) < L_\Delta^\Delta$. Then $|\Delta| \leqslant |\mathrm{Out}(L)|$, and hence $P(L) \leqslant |L : L_\Delta| = |\Sigma| \leqslant 2|\mathrm{Out}(L)|(|\mathrm{Out}(L)| - 1)$. Then L_Δ is isomorphic either to $PSL_2(q)$ or to one of the groups listed in Lemma 5.1. However, the same argument used in Proposition 5.3 can be also used here to rule out all these groups since $|\Delta| \leqslant |\mathrm{Out}(L)|$. Thus $|L_\Delta| > 2|\mathrm{Out}(L)|$ and hence $|L : L_\Delta| < |L_\Delta|^2$, which means that L_Δ is a large subgroup of L, and we obtain assertions (2a)–(2c).

Suppose that $G_{(\Sigma)} \neq 1$. Then $\lambda = 2(2^{a-1} - 1)$, $|\Sigma| = \lambda^2$ and a quotient group of L_Δ^Σ is isomorphic either to $SL_a(2)$ for $a \geqslant 4$, or to A_7 for $a = 4$ by Corollary 5.4. In either case one has $|\Delta|(|\Delta| - 1) \leqslant \left|L_\Delta^\Sigma\right|$ since $|\Delta| = 2^a$. Since $L_{(\Sigma)} \trianglelefteq L_\Delta < L$ and G^Σ acts primitively on Σ, it follows that $\left|L^\Sigma : L_\Delta^\Sigma\right| = |L : L_\Delta| = |\Sigma| \leqslant |\Delta|(|\Delta| - 1) \leqslant \left|L_\Delta^\Sigma\right|$, and the assertion (1) follows in this case. □

6. Classification of the 2-designs of type 1

In this section we mainly use the constraints on L^Σ provided in Proposition 5.3, Corollary 5.4 and Theorem 5.5 to prove Theorem 6.1 stated below. It is worth noting that, when L^Σ is a Lie type simple group we show that L_Δ^Σ is a large subgroup of L^Σ of order divisible by a suitable primitive prime divisor of $p^\zeta - 1$, where ζ is determined in [19, Proposition 5.2.16]. We combine these constraints on L_Δ^Σ to show that a small number of groups listed in [3] are admissible. These groups are then ruled out by exploiting the 2-transitivity of G_Δ^Δ on Δ.

Theorem 6.1. *If $\mathcal{D}$ is a symmetric 2-$\left((\lambda+2)\lambda^2, (\lambda+1)\lambda, \lambda\right)$ design admitting a flag-transitive and point-imprimitive automorphism group, then $\lambda \leqslant 10$.*

We analyze the cases where L^Σ is sporadic, alternating, a Lie type simple classical or exceptional group separately.

6.1. Novelties

An important tool in carrying out the following analysis is the notion of novelty: a maximal subgroup M of an almost simple group A is a *novel* maximal subgroup (or, simply, a *novelty*) if $M \cap Soc(A)$ is non-maximal in $Soc(A)$. More information on novelties can be found in [5,19,39].

Lemma 6.2. *L^Σ is not sporadic.*

Proof. Either L_{Δ}^{Σ} is maximal in L^{Σ}, or $\mathrm{Out}(L) \cong Z_2$ and G_{Δ}^{Σ} is a novelty. In the latter case, the unique admissible case is $G^{\Sigma} \cong M_{12} : Z_2$ and $G_{\Delta}^{\Sigma} \cong PGL_2(11)$ since $|\Sigma|$ is a square. Hence, $|\Sigma| = 144$ by [39, Table 1] (see also [40] for the cases $L^{\Sigma} \cong Fi_{22}$ or Fi_{24}'). Then $\lambda + 2 = 14$, and hence $G_{(\Sigma)} = 1$ by Theorem 4.1. Since $G_{\Delta} \cong PGL_2(11)$ has no transitive permutation representations of degree 14, the case is ruled out by Proposition 5.3. Therefore, L_{Δ}^{Σ} is maximal in L^{Σ}.

Assume that $L^{\Sigma} \cong M_i$, where $i = 11, 12, 22, 23$ or 24. Since $\left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right| = \lambda^2$, it follows from [19, Table 5.1.C] that, $\lambda^2 = 2^{j_1}3^{j_2}$ for some $j_1, j_2 \geqslant 2$. Then $\lambda = 12$ and either $L^{\Sigma} \cong M_{11}$ and $L_{\Delta}^{\Sigma} \cong F_{55}$, or $L^{\Sigma} \cong M_{12}$ and $L_{\Delta}^{\Sigma} \cong PSL_2(11)$ by [7]. However, these cases are ruled out by Proposition 5.3 since $\lambda + 2$ does not divide the order of L_{Δ}^{Σ}.

Assume that $L^{\Sigma} \cong J_i$, where $i = 1, 2, 3$ or 4. Then λ^2 divides 2^2, $2^6 3^2 5^2$, $2^6 3^4$, or $2^{20} 3^2 11^2$, respectively, by [19, Table 5.1.C]. Thus $i = 2$ and either $\lambda = 10$ and $L_{\Delta}^{\Sigma} \cong PSU_3(3)$, or $\lambda = 60$ and $L_{\Delta}^{\Sigma} \cong PSL_2(7)$ by [7]. However, in these cases L_{Δ}^{Σ} does not have a 2-transitive permutation representation of degree 12 or 62, respectively, and this fact contradicts Proposition 5.3.

Assume that L^{Σ} is isomorphic to one of the groups HS or McL. By [19, Table 5.1.C] λ^2 divides $2^8 3^2 5^2$ or $2^6 3^6 5^2$, respectively. Then either $L^{\Sigma} \cong HS$, $L_{\Delta}^{\Sigma} \cong M_{22}$ and $\lambda = 10$, or $L^{\Sigma} \cong McL$, $L_{\Delta}^{\Sigma} \cong M_{22}$ and $\lambda = 45$. Both these cases are excluded since they contradict Proposition 5.3.

It is straightforward to check that the remaining cases are ruled out similarly since they do not have a transitive permutation representation of degree λ^2 by [7] and [40]. $\square$

Lemma 6.3. *If $L^{\Sigma} \cong A_{\ell}$, $\ell \geqslant 5$, then L^{Σ} acts primitively on Σ.*

Proof. Assume that $L \cong A_6$. Then $A_6 \cong PSL_2(9) \trianglelefteq G \leqslant P\Gamma L_2(9)$, hence G does not have a primitive permutation representation of degree λ^2 with $\lambda > 10$ by [7]. Thus $\ell \neq 6$ by Theorem 4.1, and hence $\mathrm{Out}(L) \cong Z_2$.

Suppose the contrary of the statement. Then there is a subgroup M of L containing L_{Δ} such that $L_{\Delta}^{\Sigma} < M^{\Sigma} < L^{\Sigma}$ with M^{Σ} maximal in L^{Σ}. Let $x \in \Delta$, then x^M is a union of θ elements of Σ, where $\theta = \left|M^{\Sigma} : L_{\Delta}^{\Sigma}\right|$. Therefore $\left|x^M\right| = \theta(\lambda + 2)$ with $\lambda^2 = \mu\theta$ for some $\mu \geqslant 1$. Then $x^M \setminus \{x\}$ is a union of L_x-orbits since $L_x < L_{\Delta} \leqslant M < L$. Therefore $\frac{\lambda+1}{\eta} \mid \theta(\lambda + 2) - 1$ by Lemma 2.1, where $\eta = (\lambda + 1, 2)$ since $\mathrm{Out}(L) \cong Z_2$. Then $\theta = f\frac{\lambda+1}{\eta} + 1$ for some $f \geqslant 1$, hence

$$\left(f\frac{\lambda+1}{\eta} + 1\right)\mu - 1 = \lambda^2 - 1 \tag{6.1}$$

and so $f\frac{\lambda+1}{\eta}\mu + \mu - 1 = (\lambda - 1)(\lambda + 1)$ which implies $\frac{\lambda+1}{\eta} \mid \mu - 1$. It follows that $\mu = e\frac{\lambda+1}{\eta} + 1$ for some $e \geqslant 1$. Then (6.1) becomes

$$ef\left(\frac{\lambda+1}{\eta}\right)^2 + (e+f)\left(\frac{\lambda+1}{\eta}\right) + 1 = \lambda^2$$

Then $ef < \eta^2 = (\lambda+1,2)^2$, and hence $\eta = 2$ and λ is odd. Therefore $ef = 3$ and $\lambda = 15$, $|\Delta| = 17$ and $|\Sigma| = 225$ since $\lambda > 10$. Moreover, $G_{(\Sigma)} = 1$ by Theorem 4.1. Then $L_\Delta \cong A_{224}$ by [12, Table B.4]. However, this case cannot occur by Proposition 5.3 since A_{224} has no quotient groups with a transitive permutation representations of degree 17. $\square$

Lemma 6.4. *L^Σ is not isomorphic to A_ℓ, $\ell \geqslant 5$.*

Proof. Since L^Σ_Δ is a large maximal subgroup of L^Σ by Theorem 5.5 and Lemma 6.3, and since $|\Sigma| = \lambda^2$ with $\lambda > 10$, only the following cases are admissible by [3, Theorem 2]:

(i) $L^\Sigma_\Delta \cong (S_t \times S_{\ell-t}) \cap A_\ell$ with $1 \leqslant t \leqslant \ell/2$;
(ii) $L^\Sigma_\Delta \cong \left(S_t \wr S_{\ell/t}\right) \cap A_\ell$ with $2 \leqslant t \leqslant \ell/2$.

Suppose that (i) holds. Then $\binom{\ell}{t} = \lambda^2$, and hence $t = 1, 2$, or $t = 3$ and $\ell = 50$ by [1, Chapter 3].

Assume that $t = 1$. Since $L^\Sigma_\Delta \cong A_{\lambda^2-1}$ with $\lambda > 10$, no quotient groups of L^Σ_Δ are isomorphic to $SL_a(2)$ for any $a \geqslant 4$. Also, the minimal degree of the non-trivial transitive permutation representations of A_{λ^2-1} is greater than $\lambda+2$. Thus, this case cannot occur by Proposition 5.3 and Corollary 5.4.

Assume that $t = 2$. Then $L^\Sigma_\Delta \cong (S_2 \times S_{\ell-2}) \cap A_\ell$. Suppose that $G_{(\Sigma)} \neq 1$. Then a quotient group of G^Σ_Δ is isomorphic to $SL_a(2)$ with $a > 4$ by Theorem 4.1 since $\binom{\ell}{2} \neq 14^2$, but this is clearly impossible. Thus $G_{(\Sigma)} = 1$, and hence a quotient group of $L_\Delta \cong (S_2 \times S_{\ell-2}) \cap A_\ell$ must have a 2-transitive permutation representation of degree $\ell - 2 = \lambda + 2$ by Proposition 5.3 since $\lambda > 10$. However, $\binom{\lambda+4}{2} = \lambda^2$ has no integer solutions. Thus, this case is excluded.

Assume $t = 3$ and $\ell = 50$. Then $\lambda + 2 = 142$ since $\lambda > 10$, and hence $G_{(\Sigma)} = 1$ by Theorem 4.1. So $\lambda + 2 \mid |G_\Delta|$, whereas $G_\Delta = (S_3 \times S_{47}) \cap G$, which is a contradiction.

Suppose that (ii) holds. Hence, $L^\Sigma_\Delta \cong \left(S_{\ell/t} \wr S_t\right) \cap A_\ell$ where $s/t, t > 1$. Then $|\Sigma| = \frac{\ell!}{((\ell/t)!)^t t!}$ and $A_{\ell/t} \wr A_t \leqslant L^\Sigma_\Delta \leqslant \left(S_{\ell/t} \wr S_t\right) \cap L$. Easy computations show that $\ell > 25$ since $|\Sigma| = \lambda^2$. Moreover, $\left(A_{\ell/t}\right)^t \trianglelefteq L^\Sigma_\Delta$ and $A_t \leqslant L^\Sigma_\Delta / \left(A_{\ell/t}\right)^t \leqslant (Z_2)^t : S_t$, where $(Z_2)^t$ is a permutation module for A_t. Thus, by [19, Lemma 5.3.4], $L^\Sigma_\Delta / \left(A_{\ell/t}\right)^t$ is isomorphic to one of the following groups:

(1) A_t, S_t;
(2) $A_t \times Z_2, S_t \times Z_2$;
(3) $(Z_2)^{t-1} : A_t, (Z_2)^{t-1} : S_t$;
(4) $(Z_2)^t : A_t, (Z_2)^t : S_t$.

Suppose that $G_{(\Sigma)} \neq 1$. Then a quotient group of L^Σ_Δ is isomorphic either to $SL_a(2)$ for $a \geqslant 4$, or to A_7 for $a = 4$ by Corollary 4.1. Matching such information with (1)–(4)

one obtains $a = 4$, $t = 7, 8$ and $\lambda = 14$. Then $((\ell/t)!)^{t-1} < \frac{\ell!}{((\ell/t)!)^t t!} = 196$ as shown in [25, (3.4)], and hence $\ell = 14, 16$. However, $|\Sigma|$ is not a square for such values of t and ℓ.

Suppose that $G_{(\Sigma)} = 1$. Then $L_\Delta / \left(A_{\ell/t}\right)^t$ is one of the groups in (1)–(4). Assume that L_Δ^Δ does not act 2-transitively on Δ. Then $\mathrm{Soc}(G_\Delta^\Delta) < L_\Delta^\Delta \leqslant G_\Delta^\Delta \leqslant A\Gamma L_1(2^h)$, where $\lambda + 2 = 2^h$ and $\left|G_\Delta^\Delta : L_\Delta^\Delta\right| \leqslant 2$ by Proposition 5.3 since $\ell > 6$. This forces L_Δ^Δ to act 2-transitively on Δ, and we reach a contradiction.

Assume that L_Δ^Δ acts 2-transitively on Δ. If L_Δ^Δ is of affine type then $\lambda + 2 = 2^i$ with $i \leqslant t$. Therefore $((\ell/t)!)^{t-1} < \frac{\ell!}{((\ell/t)!)^t t!} < 2^{2t}$ and hence $\ell/t = 2$, and $t > 12$ since $\ell > 25$. However, this is impossible by [18, List (B)] since $A_t \leqslant L_\Delta / \left(A_{\ell/t}\right)^t$. Thus L_Δ^Δ is almost simple and hence $A_t \trianglelefteq L_\Delta^\Delta \leqslant S_t$ with $t \geqslant 5$ by (1)–(4). Therefore, $t = \lambda + 2 > 12$ and $2^{t-1} \leqslant ((\ell/t)!)^{t-1} < \frac{\ell!}{((\ell/t)!)^t t!} < t^2$, which is a contradiction. This completes the proof. $\square$

Lemma 6.5. *$L^\Sigma \ncong PSL_2(q)$.*

Proof. Assume that $L^\Sigma \cong PSL_2(q)$, $q = p^f$, p prime and $f \geqslant 1$. Then $q \neq 7$ since $PSL_2(7)$ does not have transitive permutation representations of square degree, and $q \neq 9$ by Lemma 6.4 since $PSL_2(9) \cong A_6$. Moreover, if $q = p$, then $p \mid \left|L_\Delta^\Sigma\right|$ since $\left|L^\Sigma : L_\Delta^\Sigma\right| = \lambda^2$. Thus no novelties occur by [5, Table 8.1], hence L_Δ^Σ is maximal in L^Σ. Moreover, no quotients groups of L_Δ^Σ are isomorphic to $SL_a(2)$ with $a \geqslant 4$ or to A_7 and $a = 4$ for $G_{(\Sigma)} \neq 1$. Hence $G_{(\Sigma)} = 1$ by Corollary 5.4. Thus, L_Δ is maximal in L.

Assume that L_Δ is isomorphic to any of the groups A_4, S_4 or A_5. Then $\lambda + 2 \mid |L_\Delta|$, hence $\lambda + 2$ is not a power of a prime since $\lambda > 10$. Thus L_Δ is non-solvable and acts 2-transitively on Δ by Proposition 5.3, hence $L_\Delta \cong A_5$ and $\lambda + 2 = 6$, whereas $\lambda > 10$. Thus, these groups are ruled out.

Assume that L_Δ is isomorphic to $D_{\frac{q\pm 1}{(2,q-1)}}$. Then $\lambda \leqslant (q-1)/(2, q-1)$ since $\lambda + 2 \mid |L_\Delta|$, and hence $q(q \mp 1) = |\Sigma| \leqslant \frac{(q-1)^2}{(2,q-1)^2}$, which has no admissible solutions.

Assume that $PSL_2(q^{1/m}) \trianglelefteq L_\Delta \leqslant PGL_2(q^{1/m})$. Then L_Δ acts 2-transitively on Δ by Proposition 5.3. Moreover, $\lambda + 2 = q^{1/m} + 1$ since $q^{1/m} > 11$, being $\lambda > 10$. Thus $\lambda = q^{1/m} - 1$, and hence L_Δ must contain a Sylow p-subgroup of L, which is a contradiction.

Finally, assume that $L_\Delta \cong E_q : Z_{\frac{q-1}{(q-1,2)}}$. Then $\lambda^2 = q + 1$, which has no solutions by [33, A5.1] since $\lambda > 10$. $\square$

Lemma 6.6. *L^Σ is not isomorphic to one of the groups $PSL_3(4)$, $PSU_4(2)$, $PSL_6(2)$, $PSp_6(2)$, $P\Omega_8^+(2)$, $G_2(2)'$, ${}^2G_2(3)'$ or ${}^2F_4(2)'$.*

Proof. Assume that $L^\Sigma \cong PSL_3(4)$. Then $35 \mid \left|L_\Delta^\Sigma\right|$ since $\left|L^\Sigma : L_\Delta^\Sigma\right| = \lambda^2$, but L^Σ does not contain such a group by [7].

Assume that $L^\Sigma \cong PSU_4(2)$. Then $5 \mid \left|L_\Delta^\Sigma\right|$ since $\left|L^\Sigma : L_\Delta^\Sigma\right| = \lambda^2$, and hence $L_\Delta^\Sigma \cong S_6$ and $\lambda = 6$ by [7], whereas $\lambda > 10$ by our assumptions.

Assume that $L^\Sigma \cong PSL_6(2)$. Thus $2 \cdot 5 \cdot 31 \mid \left|L^\Sigma_\Delta\right|$ since $\left|L^\Sigma : L^\Sigma_\Delta\right| = \lambda^2$, and hence either $L^\Sigma_\Delta \cong SL_5(2)$, or $L^\Sigma_\Delta \cong E_{2^5} : SL_5(2)$ by [5, Tables 8.24 and 8.25]. However, both cases are ruled out since 7 divides $\left|L^\Sigma : L^\Sigma_\Delta\right|$ but 7^2 does not.

Assume that $L^\Sigma \cong PSp_6(2)$. Then $\mathrm{Out}(L^\Sigma) = 1$, $L^\Sigma_\Delta \cong S_8$ and $\lambda = 6$ by [7], whereas $\lambda > 10$.

Assume that $L^\Sigma \cong P\Omega_8^+(2)$. Therefore $21 \mid \left|L^\Sigma_\Delta\right|$ since $\left|L^\Sigma : L^\Sigma_\Delta\right| = \lambda^2$, and hence $\lambda^2 \mid 2^{12} \cdot 3^4 \cdot 5^2$. Then $\left|L^\Sigma : L^\Sigma_\Delta\right|$ must be divisible by one among its primitive degrees 120, 135 or 960 by [7]. Thus 15^2 divides $\left|L^\Sigma : L^\Sigma_\Delta\right|$ in each case, and hence $\lambda = 15j$ for some $j \geqslant 1$. If $G_{(\Sigma)} \neq 1$ then $\lambda = 15j = 2^a - 2$ with $a \geqslant 4$ by Theorem 4.1. Easy computations show that no admissible cases occur. Therefore $G_{(\Sigma)} = 1$, and hence G is a subgroup of $P\Omega_8^+(2).S_3$. Moreover, the order of G is divisible by $(15j)^3(15j+2)(15j+1)$ since G^Δ_Δ acts 2-transitively on Δ, which has size $15j+2$, and since G acts flag-transitively on $\mathcal{D}$ and $k = \lambda(\lambda+1)$. Since no groups occur, this case is excluded.

The case $L^\Sigma \cong G_2(2)'$ is ruled out in Lemma 6.5 since $G_2(2)' \cong PSL_2(8)$. Also, if $L^\Sigma \cong {}^2G_2(3)' \cong PSU_3(3)$ then $L^\Sigma_\Delta \cong PSL_2(7)$ and $|\Sigma| = 36$ by [7]. So $\lambda = 6$, whereas $\lambda > 10$.

Finally, if $L^\Sigma \cong {}^2F_4(2)'$ then $L^\Sigma_\Delta \cong PSL_3(3) : Z_2$ and $|\Sigma| = 1600$ by [7]. Then $\lambda = 40$ and hence $\lambda + 1 = 41$ must divide the order of G by Lemma 2.1. However, this is impossible since $G^\Sigma \leqslant {}^2F_4(2)$ and the order of $G_{(\Sigma)}$ is either 1 or a power of 2 by Theorem 4.1. $\square$

6.2. Primitive prime divisors of the order of a group

Let p be a prime, w a prime distinct from p, and m an integer which is not a power of p. Also let Γ be a group which is not a p-group. Then we define

$$\begin{aligned}
\zeta_p(w) &= \min\{z : z \geqslant 1 \text{ and } p^z \equiv 1 \pmod{w}\} \\
\zeta_p(m) &= \max\{\zeta_p(w) : w \text{ prime}, w \neq p \text{ and } w \mid m\} \\
\zeta_p(X) &= \zeta_p(|X|).
\end{aligned}$$

If L^Σ is isomorphic to a simple group of Lie type over $GF(q)$, $q = p^f$, then $\zeta_p(L^\Sigma)$ is listed in [19, Proposition 5.2.16 and Table 5.2.C]. In the sequel we will denote $\zeta_p(L^\Sigma)$ simply by ζ. It is worth noting that $\Phi^*_\zeta(p) > 1$ by Lemmas 6.5 and 6.6 (the definition of primitive part is given just after Remark 3.4).

Lemma 6.7. *L^Σ_Δ is a large subgroup of L^Σ such that $\left(\Phi^*_\zeta(p), \left|L^\Sigma_\Delta\right|\right) > 1$.*

Proof. Suppose that $\left(\Phi^*_\zeta(p), \left|L^\Sigma_\Delta\right|\right) = 1$. Note that, $\left|G^\Sigma_\Delta : L^\Sigma_\Delta\right| \mid \left|\mathrm{Out}(L^\Sigma)\right|$ since $\left|L^\Sigma : L^\Sigma_\Delta\right| = \left|G^\Sigma : G^\Sigma_\Delta\right|$, as G^Σ acts primitively on Σ. Thus $\left(\Phi^*_\zeta(p), \left|G^\Sigma_\Delta\right|\right) = 1$ since $\left(\Phi^*_\zeta(p), \left|\mathrm{Out}(L^\Sigma)\right|\right) = 1$ by [19, Proposition 5.2.15(ii)]. Therefore $\Phi^*_\zeta(p) \mid \left|G^\Sigma : G^\Sigma_\Delta\right|$, and hence $\Phi_\zeta^*(p) \mid \lambda^2$ since $\left|G^\Sigma : G_\Delta^\Sigma\right| = \lambda^2$. Then $(\Phi_\zeta^*(p), |G_x|) > 1$, where x is any point of $\mathcal{D}$, since $\lambda(\lambda+1) \mid |G_x|$ being G flag-transitive on $\mathcal{D}$. Therefore $(\Phi_\zeta^*(p), |G_\Delta|) > 1$, where Δ is the element of Σ containing x, and hence $(\Phi_\zeta^*(p), \left|G_\Delta^\Sigma\right|) > 1$ since $G_{(\Sigma)}$ is either trivial, or a 2-group by Theorem 3.1, and we reach a contradiction. Thus $\left(\Phi_\zeta^*(p), \left|L_\Delta^\Sigma\right|\right) > 1$. Moreover, L_Δ^Σ is large by Theorem 5.5. $\square$

Lemma 6.8. *L^Σ is not isomorphic to an exceptional simple group of Lie type.*

Proof. Let M be a subgroup of L such that M^Σ is a maximal subgroup of L^Σ containing L_Δ^Σ. Then M^Σ is large since L_Δ^Σ is so by Lemma 5.5. Therefore M^Σ is one of the groups classified in [3, Theorem 5]. Moreover, $\left(\Phi_\zeta^*(p), \left|M^\Sigma\right|\right) > 1$ by Lemma 6.7.

Assume that M^Σ is parabolic. If L^Σ is untwisted then M^Σ can be obtained by deleting the i-th node in the Dynkin diagram of L^Σ, and we see that none of these groups is of order divisible by a prime factor of $\Phi_\zeta^*(p)$. Indeed, for instance, consider the Levi factors of the maximal parabolic subgroups of $L^\Sigma \cong F_4(q)$, $q = p^f$ are of type $B_3(p^f)$, $C_3(p^f)$ or $A_1(p^f) \times A_2(p^f)$, and none of these has order divisible by a prime factor of $\Phi_{12f}^*(p)$. If L^Σ is twisted; that is, L^Σ is centralized by an automorphism γ of the corresponding untwisted group and γ induces a non-trivial symmetry ρ on the Dynkin diagram. In this case the M^Σ exists only when deleting the resulting subset obtained by deleting the i-th node in the Dynkin diagram of corresponding untwisted group is ρ-invariant. The Levi factor of M^Σ is obtained by taking the fixed points of the automorphism γ on the Levi factor of the corresponding untwisted subgroup. Also, in the twisted case the order of any maximal subgroups of L^Σ is not divisible by a prime factor of $\Phi_\zeta^*(p)$. Indeed, for instance, the Levi factors of the maximal parabolic subgroups of ${}^2E_6(q)$, $q = p^f$, are of types ${}^2A_5(q)$, ${}^2D_4(q)$, $A_1(q) \times A_2(q^2)$ and $A_1(q^2) \times A_2(q)$, and none of these has order divisible by a prime factor of $\Phi_{18f}^*(p)$.

Assume that M^Σ is not parabolic. Then (L^Σ, M^Σ) is listed in [3, Table 2]. Since $\left(\Phi_\zeta^*(p), \left|M^\Sigma\right|\right) > 1$, only the groups contained in Table 2 are admissible by [23].

L^Σ is not isomorphic to any of the groups $G_2(3), G_2(5)$ or ${}^2B_2(8)$ since none of these has a transitive permutation representation of square degree by [7]. Also, if $L^\Sigma \cong G_2(4)$ then $L_\Delta^\Sigma \cong PSL_2(13)$ again by [7], and hence $\lambda = 480$. Then $G_{(\Sigma)} = 1$ by Corollary 5.4, and hence $PSL_2(13) \trianglelefteq G_\Delta^\Delta \leqslant PGL_2(13)$. However, $|\Delta| = \lambda + 2 = 482$ does not divide the order of G_Δ^Δ, and we reach a contradiction.

Suppose that $\left|M^\Sigma\right|^2 < \left|L^\Sigma\right|$. Then $G_{(\Sigma)} = 1$, L_Δ^Δ is solvable, $|L| \leqslant 4\left|\mathrm{Out}(L)\right|^2 \left|L_\Delta^\Delta\right|^2$ and $\left|L_{(\Delta)}\right| < 2\left|\mathrm{Out}(L)\right|$ by Theorem 5.5. Thus $|M|^2 < |L| \leqslant 4\left|\mathrm{Out}(L)\right|^2 \left|L_\Delta\right|^2$, and hence $|M : L_\Delta| < 2\left|\mathrm{Out}(L)\right|$. In the remaining admissible groups of Table 2 the last term of the derived series $M^{(\infty)}$ of M is non-abelian simple. Let $P(M^{(\infty)})$ be the minimal degree of the non-trivial primitive permutation representations of $M^{(\infty)}$. If $M^{(\infty)} \nleqslant L_\Delta$, then

$$P(M^{(\infty)}) \leqslant \left|M^{(\infty)} : L_\Delta \cap M^{(\infty)}\right| \leqslant |M : L_\Delta| < 2\left|\mathrm{Out}(L)\right|,$$

Table 2
Admissible (L^{Σ}, M^{Σ}).

L^{Σ}	M^{Σ}	Conditions
$E_7(q)$	$(3, q+1).({}^2E_6(q) \times \frac{q+1}{(3,q+1)}).(3, q+1).2$	
$E_6(q)$	$F_4(q)$	
	$(q^2+q+1 \times {}^3D_4(q)).3$	
$F_4(q)$	${}^3D_4(q).3$	
	${}^2F_4(q)$	
$G_2(q)$	$SU_3(q):2$	
	${}^2G_2(q)$	$q = 3^{2e+1} > 1$
	$G_2(2)$	$q = 5$
	$PSL_2(13)$	$q = 3, 4$
	$2^3.SL_3(2)$	$q = 3$
${}^2B_2(q)$	$13:4$	$q = 8$
${}^3D_4(q)$	$G_2(q)$	

and we reach a contradiction by Lemma 5.1. Therefore $M^{(\infty)} \leqslant L_{\Delta}$, and hence $M^{(\infty)} \leqslant L_{(\Delta)}$ since L_{Δ}^{Δ} is solvable. Then $P(M^{(\infty)}) \leqslant \left|L_{(\Delta)}\right| < 2\left|\mathrm{Out}(L)\right|$ and we again reach a contradiction by Lemma 5.1.

Suppose that $\left|L^{\Sigma}\right| \leqslant \left|M^{\Sigma}\right|^2$. Then one of the following holds by [23]:

(1) $L^{\Sigma} \cong E_7(q)$ and $L_{\Delta}^{\Sigma} = M^{\Sigma} \cong (3, q+1).({}^2E_6(q) \times (q-1)/(3, q+1)).(3, q+1).2$;
(2) $L^{\Sigma} \cong E_6(q)$ and $L_{\Delta}^{\Sigma} = M^{\Sigma} \cong F_4(q)$;
(3) $L^{\Sigma} \cong F_4(q)$ and $L_{\Delta}^{\Sigma} = M^{\Sigma} \cong {}^3D_4(q).Z_3$;
(4) $L^{\Sigma} \cong G_2(q)$ and $L_{\Delta}^{\Sigma} = M^{\Sigma} \cong SU_3(q) : Z_2$.

Then $G_{(\Sigma)} = 1$ by Corollary 5.4. If L_{Δ}^{Δ} does not act 2-transitively on Δ, then L_{Δ}^{Δ} is solvable by Proposition 5.3, hence $\left|L_{(\Delta)}\right| < 2\left|\mathrm{Out}(L)\right|$ by Theorem 5.5. However, this is impossible in cases (1)–(4). Then L_{Δ}^{Δ} acts 2-transitively on Δ, and hence only (4) occurs with $\lambda + 2 = q^3 + 1$. Then $|\Sigma| = (q^3-1)^2$, whereas L_{Δ}^{Σ} is a maximal non-parabolic subgroup of L^{Σ}. So this case is excluded, and the proof is completed. □

Now, it remains to analyze the case where L^{Σ} is a simple classical group.

Proposition 6.9. *L_{Δ}^{Σ} is a large maximal geometric subgroup of L^{Σ}. Moreover, it results that $\left(\Phi_{\zeta}^{*}(p), \left|L_{\Delta}^{\Sigma}\right|\right) > 1$.*

Proof. Recall that L_{Δ}^{Σ} is a large subgroup of L^{Σ} such that $\left(\Phi_{\zeta}^{*}(p), \left|L_{\Delta}^{\Sigma}\right|\right) > 1$ by Lemma 6.7. Let M be a subgroup of L such that M^{Σ} is a maximal subgroup of L^{Σ} containing L_{Δ}^{Σ}, then M^{Σ} is large and $\left(\Phi_{nf}^{*}(p), \left|M^{\Sigma}\right|\right) > 1$. If $\left(L^{\Sigma}, L_{\Delta}^{\Sigma}\right)$ is not $\left(P\Omega_8^{+}(q), G_2(q)\right)$ and $(PSU_4(3), A_7)$ then we may use the same argument of [26, Theorem 7.1], with $L^{\Sigma}, L_{\Delta}^{\Sigma}$ and M^{Σ} in the role of X, X_x and Y, respectively, to prove that M^{Σ} is a geometric subgroup of L^{Σ}.

Assume that $L^{\Sigma} \cong P\Omega_8^+(q)$ and $L^{\Sigma}_{\Delta} \cong G_2(q)$. Then $G_{(\Sigma)} = 1$ by Corollary 5.4. If L^{Δ}_{Δ} does not act 2-transitively on Δ, then $\mathrm{Soc}(G^{\Delta}_{\Delta}) < L^{\Delta}_{\Delta} \leqslant G^{\Delta}_{\Delta} \leqslant A\Gamma L_1(u^h)$, where $u^h = \lambda + 2$. Then $\lambda < f$, where $q = p^f$, and hence $p^{2f} \leqslant |L : L_{\Delta}| < f^2$ a contradiction. Thus L^{Δ}_{Δ} acts 2-transitively on Δ, and hence $q = 2$. However, this is impossible by Lemma 6.6.

Assume that $L^{\Sigma} \cong PSU_4(3)$ and $L^{\Sigma}_{\Delta} \cong A_7$. Then $\lambda = 36$, and hence $G_{(\Sigma)} = 1$ by Corollary 5.4. Therefore, one obtains $G \leqslant P\Gamma U_4(3)$. However, this is impossible by Lemma 2.1 since $\lambda + 1 = 37$ does not divide the order of G. Thus, we have proven that M^{Σ} is a geometric subgroup of L^{Σ} containing L^{Σ}_{Δ}.

If $L^{\Sigma}_{\Delta} \neq M^{\Sigma}$, then G^{Σ}_{Δ} is a novelty, and hence L^{Σ}_{Δ} is listed in [19, Table 3.5.H] for $n \geqslant 13$ and in [5, Section 8] for $3 \leq n \leqslant 12$. Now, the candidates for L^{Σ}_{Δ} must be large subgroups of L^{Σ} and their order must have a factor in common with $\Phi^*_{\zeta}(p)$. For instance, $L^{\Sigma} \cong PSL_n(q)$, $q = p^f$. Then $n > 2$ and $(n, q) \neq (6, 2)$ by Lemmas 6.5 and 6.6. Hence $\zeta = nf$ by [19, Proposition 5.2.16], and the unique admissible case is when L^{Σ}_{Δ} lies in a maximal member of $\mathcal{C}_1(L^{\Sigma})$. However, this is impossible by [14, Theorem 3.5(iv)]. Therefore, no novelties occur when $L^{\Sigma} \cong PSL_n(q)$. The remaining simple groups are analyzed similarly, and it is straightforward to check that no novelties which are compatible with the constraints on L^{Σ}_{Δ}. Thus $L^{\Sigma}_{\Delta} = M^{\Sigma}$, which is the assertion. □

Lemma 6.10. *L^{Σ} is not isomorphic to $PSL_n(q)$.*

Proof. Assume that $L^{\Sigma} \cong PSL_n(q)$. Then $n \geqslant 3$ by Lemma 6.5, and L^{Σ}_{Δ} is a large maximal geometric subgroup of L^{Σ} such that $\left(\Phi^*_{nf}(p), \left|L^{\Sigma}_{\Delta}\right|\right) > 1$ by Proposition 6.9 and by [19, Proposition 5.2.16]. Then $L^{\Sigma}_{\Delta} \notin \mathcal{C}_1(L^{\Sigma})$ by [14, Theorem 3.5(iv)], and hence one of the following holds by [3, Propositions 4.7]:

(i) $\mathrm{Soc}(L^{\Sigma}_{\Delta})$ is one of the groups $PSp'_n(q)$, $PSU_n(q^{1/2})$ and n odd, or $P\Omega^-_n(q)$;
(ii) L^{Σ}_{Δ} is a $\mathcal{C}_3$-group of type $GL_{n/t}(q^t)$, where $t = 2$, or $t = 3$ and either $q = 2, 3$, or $q = 5$ and n is odd.

Assume that (i) holds. Then $G_{(\Sigma)} = 1$ by Corollary 5.4. Thus $G^{\Sigma} = G$, and hence $L^{\Sigma}_{\Delta} = L_{\Delta}$. If L^{Δ}_{Δ} does not act 2-transitively on Δ, then $\mathrm{Soc}(G^{\Delta}_{\Delta}) < L^{\Delta}_{\Delta} \leqslant G^{\Delta}_{\Delta} \leqslant A\Gamma L_1(u^h)$, where $u^h = \lambda + 2$ by Proposition 5.3. Thus L^{Δ}_{Δ} is solvable, and $\lambda + 2 \mid \left|L^{\Delta}_{\Delta}\right|$ since L^{Δ}_{Δ} acts transitively on Δ. Further, $\mathrm{Soc}(L_{\Delta}) \leqslant L_{(\Delta)}$ since $\mathrm{Soc}(L_{\Delta})$ is non-abelian simple. Then $\lambda + 2 \leqslant \left|L^{\Delta}_{\Delta}\right| \leqslant c$, where $c \leqslant 8$, $c \leqslant q - 1$ or $c \leqslant 2$ according as $\mathrm{Soc}(L_{\Delta})$ is isomorphic to one of the groups $PSp'_n(q)$, $PSU_n(q^{1/2})$ and n odd, or $P\Omega^-_n(q)$, respectively, by [19, Propositions 4.8.3(II)–4.8.5(II)]. Therefore, $\mathrm{Soc}(L^{\Sigma}_{\Delta}) \cong PSU_n(q^{1/2})$ and $\lambda \leqslant q - 3$ since $\lambda > 10$ by our assumption. Then $q^2 + q + 1 \leqslant P(L) \leqslant |\Sigma| = \lambda^2 \leqslant (q - 3)^2$ by [19, Proposition 5.2.1] since L acts transitively on Σ and $n \geqslant 3$, a contradiction. Thus, L^{Δ}_{Δ} acts 2-transitively on Δ in (i). Furthermore, $(n, q) \neq (4, 2), (6, 2)$ by Lemmas 6.4 and 6.9, respectively, and $n \geqslant 3$. Then one of the following holds by [19, Propositions 4.8.3–4.8.5]:

(1) $n > 6$, $q = 2$, $L_\Delta \cong Sp_n(2)$ and $\lambda = 2^{2n-1} \pm 2^n - 2$;
(2) $n = 3$, $\mathrm{Soc}(L_\Delta) \cong PSU_3(q^{1/2})$ and $\lambda = q^{3/2} - 1$;
(3) $n = 4$, $\mathrm{Soc}(L_\Delta) \cong P\Omega_4^-(q) \cong PSL_2(q^2)$ and $\lambda = q^2 - 1$.

Cases (2) and (3) are immediately ruled out since $(\lambda, p) = 1$ but L_Δ does not contain a Sylow p-subgroup of L. In case (1), $2^{n(n-1)/2-2}$ must divide the order of a Sylow 2-subgroup of L_Δ which is $2^{n^2/4}$, and we reach a contradiction since $n > 6$.

Assume that (ii) holds. Then $L_\Delta^\Sigma \cong Z_a.PSL_{n/t}(q^t).Z_e.Z_t$, where $a = \frac{(q-1,n/t)(q^t-1)}{(q-1)(q-1,n)}$ and $e = \frac{(q^t-1,n/t)}{(q-1,n/t)}$ by [19, Proposition 4.3.6.(II)], and $t = 2, 3$. Then $G_{(\Sigma)} = 1$ by Corollary 5.4. Recall that $\mathrm{Soc}(G_\Delta^\Delta) \trianglelefteq L_\Delta^\Delta$ by Proposition 5.3.

If G_Δ^Δ is of affine type, then $L_\Delta^\Delta \leqslant Z_e.Z_t$ for $n > t$. Thus $\lambda + 2 \leqslant e \leqslant n/t$, and hence

$$\frac{q^n - 1}{q - 1} = P(L) \leqslant |L : L_\Delta| = |\Sigma| \leqslant n^2/2 \tag{6.2}$$

by [19, Theorem 5.2.2] since $n \geqslant 3$ and $(n, q) \neq (4, 2)$. However, (6.2) has no admissible solutions. Thus $n = t = 3$ since $t = 2, 3$ and $n \geqslant 3$. Moreover, $L_\Delta \cong Z_{\frac{q^2+q+1}{(3,q-1)}}.Z_3$ and hence $|\Sigma| = \frac{1}{3}q^3(q+1)(q-1)^2$. On the other hand, since $\lambda + 2$ is a power of prime and divides the order of L_Δ, it follows that $\lambda \leqslant q^2 + q - 1$ and $|\Sigma| \leqslant (q^2 + q - 1)^2$, and we reach a contradiction.

If G_Δ^Δ is almost simple, then $t < n$ and L_Δ^Δ acts 2-transitively on Δ by Proposition 5.3. Thus $L_\Delta^\Delta \cong PSL_{n/t}(q^t).Z_e.Z_t$ and hence either $(n/t, q^t) = (2, 9)$ and $\lambda + 2 = 6$, or $\lambda + 2 = \frac{q^n-1}{q^t-1}$ since $t = 2, 3$. The former contradicts $\lambda > 10$, the latter yields $\lambda = \frac{q^n-2q^t+1}{q^t-1}$ and hence $(\lambda, p) = 1$. Then L_Δ^Σ must contain a Sylow p-subgroup of L^Σ, which is a contradiction. □

Lemma 6.11. *L^Σ is not isomorphic to $PSU_n(q)$.*

Proof. Assume that $L^\Sigma \cong PSU_n(q)$. Then $n \geqslant 3$ by Lemma 6.5. Moreover, L_Δ^Σ is a large maximal geometric subgroup of L^Σ such that $\left(\Phi_\zeta^*(p), \left|L_\Delta^\Sigma\right|\right) > 1$, where ζ is either nf or $(n-1)f$ according to whether n is even or odd, respectively, by Proposition 6.9 and [19, Proposition 5.2.16]. Then $L_\Delta^\Sigma \notin \mathcal{C}_1(L^\Sigma)$ for n odd by [14, Theorem 3.5(iv)], and hence one of the following holds by [3, Propositions 4.17]:

(i) L_Δ^Σ is a $\mathcal{C}_1$-subgroup of L^Σ and n is even;
(ii) L_Δ^Σ is a $\mathcal{C}_3$-subgroup of L^Σ of type $GU_{n/3}(3^3)$ and n is odd.

Assume that (i) holds. Then $L_\Delta^\Sigma \cong Z_{\frac{q+1}{(q+1,n)}}.PSU_{n-1}(q).Z_{(q+1,n-1)}$ is the stabilizer of a non-isotropic point of $PG_{n-1}(q^2)$ by [19, Propositions 4.1.4.(II)-4.1.18.(II)]. Also, $G_{(\Sigma)} = 1$ by Corollary 5.4. If L_Δ^Δ does not act 2-transitively on Δ, then $\mathrm{Soc}(G_\Delta^\Delta) < L_\Delta^\Delta \leqslant G_\Delta^\Delta \leqslant A\Gamma L_1(u^h)$ with $u^h = \lambda + 2$ by Proposition 5.3. Then $\lambda \leqslant (q+1, n-1) - 2$ by [19, Proposition 4.1.4.(II)], and hence $q^{n-1} \leqslant |L : L_{\Delta}| < q$, which is impossible for $n \geqslant 4$. Then L_{Δ}^{Δ} acts 2-transitively on Δ, and hence $n = 4$ and $\lambda^2 = q^3(q-1)$. So $q-1$ is a square, which is impossible by [33, B1.1].

Assume that (ii) holds. Then $L_{\Delta}^{\Sigma} \cong Z_7.PSU_{n/3}(3^3).Z_{(n/3,7)}.Z_3$ by [19, Proposition 4.3.6(II)]. Hence, $G_{(\Sigma)} = 1$ by Corollary 5.4. Also, L_{Δ}^{Δ} is forced to act 2-transitively on Δ by Proposition 5.3 since $\lambda > 10$. Therefore, $n = 9$ and $\lambda = 3^9 - 2$. However, $|L : L_{\Delta}| \neq \lambda^2$ in this case, which is ruled out. □

Lemma 6.12. *L^{Σ} is not isomorphic to $PSp_n(q)'$.*

Proof. Assume that $L^{\Sigma} \cong PSp_n(q)'$. Then $n \geqslant 4$ by Lemma 6.5 since n is even. Also $(n, q) \neq (4, 2)$ by Lemma 4.2 since $PSp_4(2)' \cong A_6$. Thus $L^{\Sigma} \cong PSp_n(q)$. Moreover, L_{Δ}^{Σ} is a large maximal geometric subgroup of L^{Σ} such that $\left(\Phi_{nf}^*(p), \left|L_{\Delta}^{\Sigma}\right|\right) > 1$ by Proposition 6.9 and [19, Proposition 5.2.16]. Then $L_{\Delta}^{\Sigma} \notin \mathcal{C}_1(L^{\Sigma})$ by [14, Theorem 3.5(iv)], and hence one of the following holds by [3, Propositions 4.22]:

(i) L_{Δ}^{Σ} is a $\mathcal{C}_8$-subgroup of L^{Σ};
(ii) L_{Δ}^{Σ} is a $\mathcal{C}_3$-subgroup of L^{Σ} of type $Sp_{n/2}(q^2)$, $Sp_{n/3}(q^3)$ or $GU_{n/2}(q)$;
(iii) $\left(L^{\Sigma}, L_{\Delta}^{\Sigma}\right)$ is either $(PSp_4(7), 2^4.O_4^-(2))$, or $(PSp_4(3), 2^4.\Omega_4^-(2))$.

Assume that Case (i) holds. Then $L_{\Delta}^{\Sigma} \cong O_n^{\varepsilon}(q)$, with $\varepsilon = \pm$ and q even, by [19, Proposition 4.8.6.(II)]. Then $\lambda^2 = \frac{q^{n/2}}{2}(q^{n/2}+\varepsilon)$ since $\lambda^2 = \left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right|$, and hence $q^{n/2}+\varepsilon$ is a square. Then $(n, q) = (6, 2)$ by [33, B1.1] since $n \geqslant 4$. However, $L^{\Sigma} \cong PSp_6(2)$ cannot occur by Lemma 6.6.

Assume the Case (ii) holds. Then L_{Δ}^{Σ} is isomorphic to $PSp_{n/t}(q^t).Z_t$ with $t = 2, 3$ or to $Z_{(q+1)/2}.PGU_{n/2}(q).Z_2$ with q odd by [19, Propositions 4.3.7(II) and 4.3.10.(II)]. Also, in both cases it results $G_{(\Sigma)} = 1$ by Corollary 5.4.

If L_{Δ}^{Δ} does not act 2-transitively on Σ, then $Z_{(q+1)/2}.PSU_{n/2}(q) \leqslant L_{(\Delta)}$ and $\left|L_{\Delta}^{\Delta}\right| \mid 4f(n/2, q+1)$ since $\lambda > 10$. By [3, Corollary 4.3(ii)–(iii)] we obtain

$$p^{f\frac{n^2+2n}{4}} \leqslant \left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right| \leqslant 16f^2(n/2, q+1)^2,$$

which has no solutions for q odd and $n \geqslant 4$.

If L_{Δ}^{Δ} acts 2-transitively on Σ. Then L_{Δ}^{Σ} is isomorphic to $PSp_2(q^t).Z_t$ with $t = 2, 3$ or to $Z_{(q+1)/2}.PGU_3(q).Z_2$ with q odd. Then either $\lambda = q^t - 1$ with $t = 2, 3$, or $\lambda = q^3 - 1$, respectively. In each case $(\lambda, q) = 1$, and hence L_{Δ}^{Σ} must contain a Sylow p-subgroup of L^{Σ}, which is a contradiction.

Finally, Case (iii) cannot occur since $\left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right|$ is a non-square integer. □

Lemma 6.13. *L^{Σ} is not isomorphic to simple classical group.*

Proof. In order to prove the assertion we only need to tackle the case $L^{\Sigma} \cong P\Omega_n^{\varepsilon}(q)$, where $\varepsilon \in \{\circ, \pm\}$ since the remaining classical groups are ruled out in Lemmas 6.10, 6.11 and 6.12. Hence, assume that $L^{\Sigma} \cong P\Omega_n^{\varepsilon}(q)$, where $\varepsilon \in \{\circ, \pm\}$. Then $n \neq 4$ by Lemma 6.5. Moreover, $n \neq 5, 6$ by Lemmas 6.10, 6.11 and 6.12, since $P\Omega_5^{\circ}(q) \cong PSp_4(q)$ with q odd, $P\Omega_6^{+}(q) \cong PSL_4(q)$ and $P\Omega_6^{-}(q) \cong PSL_6(q)$. Thus, $n \geqslant 7$. Further, $(n, q, \varepsilon) \neq (8, 2, +)$ by Lemma 6.6. Finally, L_{Δ}^{Σ} is a large maximal geometric subgroup of L^{Σ} such that $\left(\Phi_{\zeta}^{*}(p), \left|L_{\Delta}^{\Sigma}\right|\right) > 1$ by Proposition 6.9, where ζ is either nf, $(n-1)f$ or $(n-2)f$ by [19, Proposition 5.2.16] according to whether ε is $-, \circ$ or $+$, respectively. Thus, one of the following holds by [3, Proposition 4.23]:

(i) L_{Δ}^{Σ} is $\mathcal{C}_1$-subgroup of L^{Σ};
(ii) Either $(n, q) = (7, 3), (7, 5)$, or $(n, q, \varepsilon) = (8, 3, +)$ and L_{Δ}^{Σ} is of type $O_1(q) \wr S_n$;
(iii) L_{Δ}^{Σ} is a $\mathcal{C}_3$-subgroup of L^{Σ}. Moreover, its type is either $O_{n/2}^{\varepsilon'}(q^2)$ with $(\varepsilon, \varepsilon') = (-, -)$ and $n/2$ even or $(\varepsilon, \varepsilon') = (+, \circ)$ and $n/2$ odd, or $GU_{n/2}(q)$ with $\varepsilon = -$ and $n/2$ odd or $\varepsilon = +$ and $n/2$ even;
(iv) $L^{\Sigma} \cong P\Omega_8^{+}(3)$ and $L_{\Delta}^{\Sigma} \cong 2^6.\Omega_6^{+}(2)$.

Assume that (i) holds. Then one of the following cases occurs by [19, Propositions 4.1.6(II), 4.1.7(II) and 4.1.20(II)]:

(1) L_{Δ}^{Σ} is the stabilizer in L^{Σ} of a non-singular point of $PG_{n-1}(q)$:

  (a) $\varepsilon = \circ$ and $L_{\Delta}^{\Sigma} \cong \Omega_{n-1}^{-}(q).Z_2$
  (b) $\varepsilon = +$ and $L_{\Delta}^{\Sigma} \cong \Omega_{n-1}(q)$ with $q \equiv 1 \pmod 4$, or $q \equiv 3 \pmod 4$ and $n/2$ even;
  (c) $\varepsilon = +$ and $L_{\Delta}^{\Sigma} \cong \Omega_{n-1}(q).Z_2$ with $q \equiv 3 \pmod 4$ and $n/2$ odd;
  (d) $\varepsilon = +$ and $L_{\Delta}^{\Sigma} \cong Sp_{n-2}(q)$ with q even.

(2) $\varepsilon = +$ and L_{Δ}^{Σ} is the stabilizer in L^{Σ} of a non-singular line of type "$-$" of $PG_{n-1}(q)$:

  (a) $L_{\Delta}^{\Sigma} \cong \left(Z_{\frac{q+1}{(q+1,2)}} \times \Omega_{n-2}^{-}(q)\right).Z_2$ with q even or $q \equiv 1 \pmod 4$;
  (b) $L_{\Delta}^{\Sigma} \cong \left(Z_{\frac{q+1}{2}} \times \Omega_{n-2}^{-}(q)\right).[4]$ with $q \equiv 3 \pmod 4$ and $n/2$ odd;
  (c) $L_{\Delta}^{\Sigma} \cong Z_2.\left(Z_{\frac{q+1}{4}} \times P\Omega_{n-2}^{-}(q)\right).[4]$ with $q \equiv 3 \pmod 4$ and $n/2$ even.

In each case $G_{(\Sigma)} = 1$ by Corollary 5.4 since $n \geqslant 7$. Hence, $L = L^{\Sigma}$ and $G = G^{\Sigma}$. Moreover, L_{Δ}^{Δ} acts 2-transitively on Δ by Proposition 5.3 since $\lambda > 10$. Then $\varepsilon = +$, $n \geqslant 8$, $q = 2$, $L_{\Delta}^{\Sigma} \cong Sp_{n-2}(2)$ and $\lambda = 2^{2(n/2-1)} \pm 2^{n/2-2} - 2$. On the other hand,

$$\left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right| = \lambda^2 = 2^{n/2-1}\left(2^{n/2} - 1\right)$$

and hence $2^2(2^{2(n/2-1)-1} \pm 2^{n/2-1} - 1)^2 = 2^{n/2-1}\left(2^{n/2} - 1\right)$, which has no admissible integer solutions for $n \geqslant 8$.

Assume that (iii) holds. The possibilities for L_{Δ}^{Σ} are provided in [19, Propositions 4.3.16(II), 4.3.18(II) and 4.3.20(II)]. Thus, $G_{(\Sigma)} = 1$ by Corollary 5.4. If L_{Δ}^{Δ} does not act 2-transitively on Δ, then $\mathrm{Soc}(G_{\Delta}^{\Delta}) < L_{\Delta}^{\Delta} \leqslant G_{\Delta}^{\Delta} \leqslant A\Gamma L_1(u^h)$ with $u^h = \lambda + 2$ by Proposition 5.3. Thus, L_{Δ} is forced to be of type $GU_{n/2}(q)$ with $\left|L_{\Delta}^{\Delta}\right| \mid (n/2, 2, q)(q + 1, n/2)$ since $\lambda > 10$. Then $\lambda \leqslant n - 2$ and so $q^{\frac{1}{8}n(n+2)} \leqslant |L : L_{\Delta}| \leqslant (n-2)^2$, which has no solutions for $n \geqslant 8$. Therefore L_{Δ}^{Δ} acts 2-transitively on Δ, and hence $L \cong P\Omega_8^-(q)$ and $L_{\Delta} \cong P\Omega_4^-(q).Z_4 \cong PSL_2(q^4).Z_4$ since $n \geqslant 8$. Thus, $\lambda = q^4 - 2$. If q is odd then L_{Δ} must contain a Sylow p-subgroup of L since $|L : L_{\Delta}| = \lambda^2$, which is not the case. So, q is even and $|\Sigma| = q^{12}(q^6 - 1)(q^2 - 1)$, which is different from $(q^4 - 2)^2$.

Finally, it is easy to check that $\left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right|$ is a non-square in (ii) and (iv), hence these ones are ruled out. This completes the proof. □

Proof of Theorem 6.1. Since L^{Σ} is almost simple by Theorem 4.1, the assertion follows from Lemmas 6.2, 6.4, 6.8 or 6.13. □

7. Classification of the 2-designs of type 2

In this section, we assume that $\mathcal{D}$ is of type 2. Recall that $G_{(\Sigma)} = 1$ and G is an almost simple group acting point-quasiprimitively on $\mathcal{D}$ by Theorem 4.1. Thus, $G = G^{\Sigma}$ and $L = L^{\Sigma}$ where $L = \mathrm{Soc}(G)$. Further, constraints for L are provided in Proposition 5.3 and Theorem 5.5 which are then combined with the results contained in [3,22]. An important restriction is provided in Lemma 7.5 where it is proven that, if L is Lie type simple group, either L_{Δ} lies in a maximal parabolic subgroup of L, or L_{Δ}^{Δ} is a non-solvable group acting 2-transitively on Δ. We use all this information to prove the following result.

Theorem 7.1. *If $\mathcal{D}$ is a symmetric 2-$\left(\left(\frac{\lambda+2}{2}\right)\left(\frac{\lambda^2-2\lambda+2}{2}\right), \frac{\lambda^2}{2}, \lambda\right)$ design admitting a flag-transitive and point-imprimitive automorphism group, then $\lambda \leqslant 10$.*

We analyze the cases where L^{Σ} is sporadic, alternating, exceptional of Lie type or classical separately.

Recall by Theorem 2.2 (VI.2)) that, when $\mathcal{D}$ is of type 2, either $\lambda \equiv 0 \pmod 4$ and hence $|\Delta| = \lambda/2 + 1$ is odd, or $\lambda = 2w^2$ with w odd, $w \geqslant 3$, $2(w^2 - 1)$ a square and $|\Delta| = w^2 + 1$. In both cases it results that $|\Delta| \not\equiv 0 \pmod 4$.

A preliminary filter in the study of the 2-designs of type 2 is the following lemma.

Lemma 7.2. *If $\mathcal{D}$ is of type 2, then the following hold:*

(1) $|\Sigma|$ is odd and $2\,|\Sigma| - 1$ is a square.
(2) If u is any prime divisor of $|\Sigma|$, then $u \equiv 1 \pmod 4$.

(3) If $\lambda = 2w^2$, w odd, $w \geqslant 3$ and such that $2(w^2-1)$ is a square, then $\mathrm{Soc}(G_\Delta^\Delta)$ *is isomorphic to one of the groups A_{w^2+1}, $PSL_2(w^2)$, or $PSU_3(w^{2/3})$ and w^2 is the cube of an integer.*

Proof. $|\Sigma|$ is clearly odd. If $\lambda = 2w^2$, where w is odd, $w \geqslant 3$, and $2(w^2-1) = x^2$ then $2w^4 - 2w^2 t + (1 - |\Sigma|) = 0$, whereas if $\lambda = 4t$ for some $t \geqslant 1$, then $16t^2 - 8t + (2 - 2|\Sigma|) = 0$. In both cases $y^2 = 2|\Sigma| - 1$ for some positive integer y. Therefore, $|\Sigma| = \frac{y^2+1}{2}$ is odd. Moreover, if u is any prime divisor of $|\Sigma|$ then $y^2 \equiv -1 \pmod{u}$ and hence $u \equiv 1 \pmod{4}$. Thus, we obtain (1) and (2).

Finally, (3) follows from the first part of the proof of Lemma 3.6. □

Lemma 7.3. *L is not isomorphic to a sporadic group.*

Proof. Assume that L is sporadic. Then the possibilities for G, G_x with x any point of $\mathcal{D}$, and $|\Sigma| = |G : G_x|$ are provided in [2]. It is easy to see that $2|\Sigma| - 1$ is never square when $|\Sigma|$ is any of such degrees. Thus, L cannot be a sporadic simple group by Lemma 7.2(2). □

Lemma 7.4. *L is not isomorphic to A_s with $s \geqslant 5$.*

Proof. Assume that $L \cong A_s$, where $s \geqslant 5$. Then one of the following holds by [22]:

(1) $G_\Delta \cong (S_t \times S_{s-t}) \cap G$, $1 \leqslant t < s/2$;
(2) $G_\Delta \cong (S_{s/t} \wr S_t) \cap G$, $s/t, t > 1$;
(3) $G_\Delta \cong A_7$ and $|\Sigma| = 15$.

Assume that (1) holds. Then $|\Sigma| = \binom{s}{t}$ and $A_t \times A_{s-t} \trianglelefteq L_\Delta \leqslant (S_t \times S_{s-t}) \cap L$. Suppose that $t \geqslant 5$. Then $s - t > t \geqslant 5$. Hence, both A_t and A_{s-t} are simple groups. Moreover, L_Δ^Δ is either A_t or A_{s-t} by Proposition 5.3 since $\lambda > 10$. If $L_\Delta^\Delta \cong A_t$ then either $|\Delta| = t$, or $t = 6$ and $|\Delta| = 10$, or $t = 7, 8$ and $|\Delta| = 15$ since $t \geqslant 5$. Actually, $t = 6$ and $|\Delta| = 10$ imply $\lambda = 2w^2 = 18$ and $|\Sigma| = 145$, which is not of the form $\binom{s}{6}$, and hence it cannot occur. Also, if $t = 7, 8$ and $|\Delta| = 15$ then $\binom{s}{t} = |\Sigma| = 365$, and we reach a contradiction. Thus, $\lambda = 2(t-1)$, and hence $|\Sigma| = 2t^2 - 6t + 5$. Therefore, we have

$$2^t < \left(\frac{s}{t}\right)^t \leqslant \binom{s}{t} = 2t^2 - 6t + 5,$$

which is impossible for $t \geqslant 5$. We reach the same contradiction for $L_\Delta^\Delta \cong A_{s-t}$.

Assume that $1 \leqslant t \leqslant 4$. If L_Δ^Δ is non-solvable, then $s - t \geqslant 5$ and $L_\Delta^\Delta \cong A_{s-t}$ and the previous argument rules out this case. Thus L_Δ^Δ is solvable, and hence $|\Delta| = 3$ by Proposition 5.3 since $L_\Delta \leqslant (S_t \times S_{s-t}) \cap L$ and $|\Delta|$ is odd by Lemma 7.2(3). Then $\lambda = 4$, whereas $\lambda > 10$ by our assumptions.

Assume that (2) holds. Then $|\Sigma| = \frac{s!}{((s/t)!)^t (t!)}$ and $A_{s/t} \wr A_t \leqslant L_\Delta \leqslant \left(S_{s/t} \wr S_t\right) \cap L$. Moreover, $\left(A_{s/t}\right)^t \trianglelefteq L_\Delta$ and $A_t \leqslant L_\Delta / \left(A_{s/t}\right)^t \leqslant (Z_2)^t : S_t$, where the action of A_t on $(Z_2)^t$ and on its permutation module are equivalent. Thus $L_\Delta / \left(A_{s/t}\right)^t$ is isomorphic to one of the groups A_t, S_t, $Z_2 \times A_t$, $Z_2 \times S_t$, $(Z_2)^{t-1}(2) : A_t$, $(Z_2)^{t-1}(2) : S_t$, $(Z_2)^t : A_t$ or $(Z_2)^t : S_t$ by [19, Lemma 5.3.4]. If $t \leqslant 4$ then L_Δ^Δ is solvable, and hence $\lambda/2+1 = |\Delta| = 3$ by Proposition 5.3 since $|\Delta|$ is odd by Lemma 7.2(3). However, this is impossible since $\lambda > 10$. Thus $t \geqslant 5$. Also, L_Δ^Δ is non-solvable otherwise we reach a contradiction as above. Then L_Δ^Δ acts 2-transitively on Δ by Proposition 5.3. Therefore $L_\Delta^\Delta \cong A_t$, and hence either $|\Delta| = t$ for $t \geqslant 5$, or $t = 6$ and $|\Delta| = 10$, or $t = 7, 8$ and $|\Delta| = 15$. On the other hand, $2^{t-1} \leqslant ((s/t)!)^{t-1} < \frac{s!}{((s/t)!)^t (t!)} < 2t^2$ as shown in [25, (34)]. Thus $(t, s) = (3, 6), (5, 10), (7, 14)$, and hence $|\Sigma| = 15$, 945, 135135, which are ruled out since they violate Lemma 7.2(1). Then either $t = 6$, $|\Delta| = 15$ and $|\Sigma| = 41$, or $t = 7, 8$, $|\Delta| = 15$ and $|\Sigma| = 365$. However, both cases cannot occur since $\frac{(2t)!}{(2)^t (t!)} > |\Sigma|$.

Finally, (3) is excluded by Lemma 7.2(1) since $2|\Sigma| - 1$ is not a square. $\square$

Lemma 7.5. *Let L be a simple group of Lie type. Then the following hold:*

(1) If either $p \mid \lambda - \mu$ for some $\mu \in \{0, 1, 2\}$, or $p \mid \lambda - 3$ and $p \neq 5$, then L_Δ lies in a maximal parabolic subgroup of L.
(2) If L_Δ does not lie in a maximal parabolic subgroup of L, then L_Δ^Δ is a non-solvable group acting 2-transitively on Δ.

Proof. Since $|\Sigma| = \frac{\lambda^2 - 2\lambda + 2}{2}$ it is immediate to see that $(|\Sigma|, \lambda - \mu) = 1$ for either $\mu = 0, 1, 2$, or $\mu = 3$ and $p \neq 5$. In these cases L_Δ contains a Sylow p-subgroup of L, and hence L_Δ lies in a maximal parabolic subgroup of [34, Theorem 1.6] since L is a non-abelian simple group acting transitively on Σ, and (1) holds.

Suppose to the contrary that L_Δ does not lie in a maximal parabolic subgroup of L and that L_Δ^Δ is solvable. Then $p \mid |\Sigma|$ by [34, Theorem 1.6]. Also $p \neq 2, 3$ by Lemma 7.2(1)–(2), and $\mathrm{Soc}(G_\Delta^\Delta) < L_\Delta^\Delta < G_\Delta^\Delta \leqslant A\Gamma L_1(u^h)$, where $\frac{\lambda}{2} + 1 = |\Delta| = u^h$ for some prime u by Proposition 5.3. Also, u is odd since either $\lambda \equiv 0 \pmod 4$, or $\lambda = 2w^2$ with w odd and $w \geq 3$ by Theorem 2.2. Then $\lambda = 2(u^h - 1)$ with $h > 1$ since $\lambda > 10$.

If $p = 5$ divides $\lambda - 3$, then $u = p$ since $(|\Sigma|, |\Delta|) = (|\Sigma|, \lambda - 3) = (|\Delta|, \lambda - 3) \mid 5$. Thus $|\Sigma| = 2 \cdot 5^{2h} - 6 \cdot 5^h + 5$, and $5^2 \nmid |\Sigma|$ since $h > 1$. Hence, L_Δ contains a subgroup of index 5 of a Sylow 5-subgroup of L. It is a straightforward check that none of the groups listed in [22] fulfills the previous constraint. So this case is excluded.

If $p \nmid \lambda - \mu$ for each $\mu \in \{0, 1, 2, 3\}$. Then $\left(p, \left|L_{(\Delta)}\right|\right) = 1$ by Corollary 2.6(2) since $p \neq 2, 3$. Thus $|L_\Delta|_p = \left|L_\Delta^\Delta\right|_p$, and hence $p \mid \frac{\lambda}{2}\left(\frac{\lambda}{2} + 1\right) h$ since $L_\Delta^\Delta < G_\Delta^\Delta \leqslant A\Gamma L_1(u^h)$. Actually, $p \nmid \frac{\lambda}{2}$ by our assumption. If $p \mid \frac{\lambda}{2} + 1$, then $p = 5$ divides $\lambda - 3$ since $(|\Sigma|, |\Delta|) = (|\Sigma|, \lambda - 3) = (|\Delta|, \lambda - 3) \mid 5$, which we saw being impossible. Therefore, $|L_\Delta|_p \mid h$ and any Sylow p-subgroup of L_Δ is cyclic since $L_\Delta^\Delta < G_\Delta^\Delta \leqslant A\Gamma L_1(u^h)$. However, this is impossible by [22]. Thus L_{Δ}^{Δ} is non-solvable, and hence L_{Δ}^{Δ} acts 2-transitively on Δ by Proposition 5.3. $\square$

Lemma 7.6. *L is not a simple exceptional group of Lie type.*

Proof. Assume that L_{Δ} is parabolic. Then $L \cong E_6(q)$ and $L_{\Delta} \cong [q^{16}].z.(P\Omega_{10}^{+}(q) \times (q-1)/ez).z$, where $z = (2, q-1)$ e $e = (3, q-1)$ by [22, Table 1]. If L_{Δ}^{Δ} is solvable, then the order of L_{Δ}^{Δ} must divide $(q-1)/e$. So does $|\Delta|$, and hence

$$\frac{q^9-1}{q-1} \cdot (q^8+q^4+1) = |\Sigma| \leqslant 2\,|\Delta|^2 \leqslant 2\frac{(q-1)^2}{e^2},$$

which is clearly impossible. Thus L_{Δ}^{Δ} is non-solvable acting 2-transitively on Δ by Proposition 5.3, and this is impossible too.

Assume that L_{Δ} is not parabolic. Then L_{Δ}^{Δ} is non-solvable acting 2-transitively on Δ by Lemma 7.5 with $|\Delta|$ odd. Also, $|L| \leqslant |L_{\Delta}|^2$ by Theorem 5.5. All these constraints together with [22, Table 1] lead to the following admissible cases:

(1) $L \cong E_7$ and $\mathrm{Soc}(L_{\Delta}^{\Delta}) \cong PSL_2(q)$;
(2) $L \cong {}^3D_4(q)$ and $\mathrm{Soc}(L_{\Delta}^{\Delta})$ is isomorphic to one of the groups $PSL_2(q^j)$ with $j = 1$ or 3, $PSL_3(q)$ or $PSU_3(q)$;
(3) $L \cong {}^2G_2(q)$, $q = 3^{2m+1}$, $m \geqslant 1$, and $\mathrm{Soc}(L_{\Delta}^{\Delta}) \cong PSL_2(q)$.

The admissible values for $|\Delta|$ are $q+1, q^2+1, q^2+q+1, q^2+1$, hence $q \mid \lambda$ in any case since $|\Delta| = \lambda/2+1$. Therefore q is coprime to $|\Sigma| = \frac{\lambda^2-2\lambda+2}{2}$, and hence L_{Δ} must contain a Sylow p-subgroup of L, which is not the case. This completes the proof. $\square$

Lemma 7.7. *Let L be a simple group. Then the following cases are admissible:*

(1) q is even and L_{Δ} lies in a maximal parabolic subgroup of L;
(2) q is odd and one of the following holds:

(a) L_{Δ} lies in maximal member of $\mathcal{C}_1(L) \cup \mathcal{C}_2(L)$, with L_{Δ} lying in a maximal parabolic subgroup of L of type either P_i or $P_{m,m-i}$ only for $L \cong PSL_n(q)$.
(b) $L \cong PSL_2(q)$ and L_{Δ} is isomorphic to one of the groups $D_{q\pm1}$, A_4, S_4, A_5 or $PGL_2(q^{1/2})$.

Proof. G is one of the groups classified by [22] since G acts primitively on Σ and the size of this one is odd. Actually, $L \cong X(q)$, where $X(q)$ denotes any simple classical group by Lemmas 7.3, 7.4 and 7.6.

Assume that $G_{\Delta} = N_G(X(q_0))$ with $q = q_0^s$ and q, s odd by [22]. Then $L_{\Delta} = X(q_0)$ is maximal in L by [19, Tables H-I] for $n \geqslant 13$ and [5, Section 8.2] for $2 \leqslant n \leqslant 12$. Then $s = 3$ and $L \cong PSL_m^{\epsilon}(q)$, where $\epsilon = \pm$ by [3, Propositions 4.7, 4.17, 4.22 and 4.23] since

L_Δ is a large subgroup of L by Theorem 5.5. Then $L_\Delta \cong \frac{j}{(q-\epsilon,n)}.PGL^{\epsilon}_n(q^{1/3})$, where $j = \frac{q-\epsilon}{(q^{1/3}-\epsilon,\frac{q-\epsilon}{(q-\epsilon,n)})}$, by [19, Proposition 4.5.3(II)].

Assume that L^{Δ}_{Δ} does not act 2-transitively on Δ. Then L^{Δ}_{Δ} is solvable and $\lambda/2+1 \leqslant (n, q^{1/3}-\varepsilon)$ by Proposition 5.3. Moreover, it results that $\lambda \equiv 0 \pmod 4$ by Lemma 7.2(3). Therefore, $q^{n(8n+3)/18} \leqslant |L : L_\Delta| < 2(n, q^{1/3}-\varepsilon)^2$, which is impossible for $n \geqslant 2$. Thus L^{Δ}_{Δ} acts 2-transitively on Δ and hence $n = 2, 3$. Then $\lambda = 2q^{1/3}$, $2(q^{2/3}+q^{1/3})$ or $2q$ according to whether $n = 2$, $(n, \varepsilon) = (3, +)$ or $(n, \varepsilon) = (3, -)$, respectively. Therefore, $|\Sigma|$ is coprime to q and we reach a contradiction by [34, Theorem 1.6] since L^{Δ} is a non-parabolic subgroup of L.

Assume that $L \cong \Omega_7(q)$ and $L_\Delta \cong \Omega_7(2)$. Then $q = 3, 5$ since L_Δ is a large subgroup of L, and hence $|\Sigma| = 3159$ or 157421875, respectively. However, both contradict Lemma 7.2(1) since $2|\Sigma|-1$ is not a square.

Assume that $L \cong P\Omega^{+}_8(q)$, where q is a prime and $q \equiv \pm 3 \pmod 8$, and either $L_\Delta \cong \Omega^{+}_8(2)$, or $L_\Delta \cong 2^3 \cdot 2^6 \cdot PSL_3(2)$. In the former case $q = 3, 5$ since L_Δ is a Large subgroup of L, and hence $|\Sigma| = 28431$ or 51162109375. However, both contradict Lemma 7.2(1). Then $L_\Delta \cong 2^3 \cdot 2^6 \cdot PSL_3(2)$, and hence $|\Sigma| = 57572775$, but $2|\Sigma|-1$ is not a square.

Finally, the case $L \cong PSU_3(5)$ and $L_\Delta \cong P\Sigma L_2(9)$ implies $|\Sigma| = 175$, and we again reach a contradiction by Lemma 7.2(1). $\square$

Lemma 7.8. *L is not isomorphic to $PSL_2(q)$.*

Proof. Assume that $L \cong PSL_2(q)$ and L_Δ is isomorphic to one of the groups $D_{q\pm1}$, A_4, S_4, A_5 or $PGL_2(q^{1/2})$. If $L_\Delta \cong D_{q\pm1}$ then $|\Sigma| = \frac{q(q\mp1)}{2}$ and hence $2|\Sigma|-1 = q^2+q+1$ or $(q-1)^2+(q-1)+1$ must be square by Lemma 7.2(1). However this is impossible by [33, A7.1].

If $L_\Delta \cong PGL_2(q^{1/2})$ then $|\Sigma| = q^{1/2}(q+1)/2$, and hence $2|\Sigma|-1 = q^{3/2}+q^{1/2}-1$. Moreover, L^{Δ}_{Δ} acts 2-transitively on Δ. If it is not so, then $\lambda/2+1 \leqslant (2, q^{1/2}-1)$ as a consequence of Proposition 5.3, whereas $\lambda > 10$. Thus, either $|\Delta| = q^{1/2}+1$, or $|\Delta| = q^{1/2}$ and $q^{1/2} = 7, 11$ since either $\lambda \equiv 0 \pmod 4$ and hence $|\Delta| = \lambda/2+1$ is odd, or $\lambda = 2w^2$, where w is odd, $w \geqslant 3$, $2(w^2-1)$ is a square and $|\Delta| = w^2+1$. The two numerical cases are ruled out since they violate Lemma 7.2(1), whereas the former yields $\lambda = 2q^{1/2}$ with q odd. Then $|\Sigma|$ is coprime to q and hence L_Δ must contain a Sylow p-subgroup of L, which is not the case.

Finally, assume that $L_\Delta \cong A_4$, S_4 or A_5. In the first two cases λ must be divisible by 4 by Lemma 7.2(3), hence $\lambda/2+1 = |\Delta| = 3$ since $|\Delta|$ is odd, and so $\lambda = 4$, whereas $\lambda > 10$. Thus $L_\Delta \cong A_5$. The previous argument can be applied to exclude the case $\lambda \equiv 0 \pmod 4$. Therefore $\lambda = 18$, $|\Delta| = 10$ and $145 = |\Delta| = \frac{q(q^2-1)}{120}$ which has no integer solutions. $\square$

Lemma 7.9. *L is not isomorphic to $PSL_n(q)$, $n \geqslant 2$.*

Proof. Assume that $L \cong PSL_n(q)$. Then $n \geqslant 3$ by Lemma 7.8, and hence $M \in \mathcal{C}_1(L) \cup \mathcal{C}_2(L)$ by Lemma 7.7.

Assume that L_Δ lies in a maximal parabolic subgroup M of L. If L_Δ is not of type $P_{h,n-h}$ then $L_\Delta = M$ by [19, Table 3.5.H] for $n \geqslant 13$ and by [5, Section 8.2] for $3 \leqslant n \leqslant 12$. Also, L_Δ is as in [19, Proposition 4.1.17.(II)] and $|\Sigma| = \begin{bmatrix} n \\ h \end{bmatrix}_q$, the Gaussian number, where $h \leqslant n/2$.

Assume that L_Δ^Δ is solvable. Then $\left|L_\Delta^\Delta\right| \mid q-1$ by [19, Proposition 4.17.(II)], and hence

$$\frac{1}{2}q^{h(n-h)} \leqslant \begin{bmatrix} n \\ h \end{bmatrix}_q = |\Sigma| < 2q^2.$$

Then either $n = 3$ and $h = 1$, or $(n, h, q) = (4, 1, 2), (4, 2, 2)$ since $|\Sigma|$. The numerical cases are immediately ruled by Lemma 7.2(1). Therefore $n = 3$, $|\Sigma| = q^2 + q + 1$ and hence $X^2 = q^2 + (q+1)^2$, where $X^2 = 2\,|\Sigma| - 1$ again by Lemma 7.2(1). Thus $(q, q+1, X)$ is primitive solution of the Pythagorean equation and hence it is of the form as in [33, P3.1]. Easy computations show that $q = 3$. Then $|\Sigma| = 13$ and hence $\lambda = 6$, whereas $\lambda > 10$.

Assume that L_Δ^Δ is non-solvable. Then L_Δ^Δ acts 2-transitively on Δ by Proposition 5.3. Then $\mathrm{Soc}(L_\Delta^\Delta)$ is isomorphic to $PSL_x(q)$, where $x \in \{h, n-h\}$ and $x \geqslant 2$, by [18, List (B)] and again by [19, Proposition 4.17.(II)]. Note that $(x, q) \neq (2, 5), (2, 9)$ since $\lambda > 10$. Moreover, $(x, q) \neq (2, 7)$. Indeed, if it is not so, then $\lambda = 12$ and hence $|\Sigma| = 121$, which is impossible by [12, Table B.4] since $L \cong PSL_n(7)$. Thus $|\Delta| = \frac{q^x-1}{q-1}$, $\lambda = 2q\frac{q^{x-1}-1}{q-1}$ and hence

$$8q^{2x-2} \geqslant 2q^2\left(\frac{q^{x-1}-1}{q-1}\right)^2 - 2q\frac{q^{x-1}-1}{q-1} + 1 = |\Sigma| = \begin{bmatrix} n \\ x \end{bmatrix}_q \geqslant \frac{q^{\frac{1}{2}x(2n-x+1)}}{2q^{\frac{1}{2}x(x+1)}} = \frac{1}{2}q^{x(n-x)} \tag{7.1}$$

and so $2^{x(n-x)-2x+2} \leqslant q^{h(n-h)-2x+2} \leqslant 16$. Then $x(n-2-x) \leqslant 2$, and hence $x = h = 2$ and $n = 5$ since $x \geqslant 2$ and $h \leqslant n/2$. Also, $q = 2, 3$, and hence $|\Sigma| = 155$ or 1210, respectively, but both values of $|\Sigma|$ contradict Lemma 7.2(1).

Assume that L_Δ is of type $P_{h,n-h}$, where $h < n/2$. If L_Δ^Δ is solvable, then $\left|L_\Delta^\Delta\right| \mid (q-1)^2$ by [19, Proposition 4.1.22.(II)]. Also, $|\Sigma| \geqslant \begin{bmatrix} n \\ h \end{bmatrix}_q$ since L_Δ lies in a maximal parabolic subgroup of type P_h. Hence,

$$\frac{1}{2}q^{h(n-h)} \leqslant \begin{bmatrix} n \\ h \end{bmatrix}_q \leqslant |\Sigma| < 2(q-1)^4.$$

Then $n \leqslant 5$ and $h = 1$ since $h < n/2$. We actually obtain $n = 3$ since $|\Sigma| = \frac{(q^n-1)(q^{n-1}-1)}{(q-1)^2}$. Hence, $2q^3 + (2q+1)^2 = 2\,|\Sigma| - 1 = X^2$ for some positive odd integer X by Lemma 7.2(1). Then $2q^3 = (X - 2q - 1)(X + 2q + 1)$ and hence $q = 2^t$, $t \geqslant 1$. Then $2^s + 2^{t+1} + 1 = X = 2^{3t+1-s} - 2^{t+1} - 1$ for some integer s such that $0 \leqslant s \leqslant 3t$. Thus $2^{3t+1-s} = 2^s + 2^{2(t+1)} + 2$. If $s > 1$ then $s = 3t$, which is clearly impossible. Then $s = 1$ and hence $2^{3t} = 2^{2(t+1)} + 4$, which has no integer solutions for $t \geqslant 1$. Thus, L_Δ^Δ is a non-solvable group acting 2-transitively on Δ by Proposition 5.3. Then L_{Δ}^{Δ} is isomorphic to $PSL_x(q)$, where $x \in \{h, m-h\}$ and $x \geqslant 2$, by [19, Proposition 4.1.22.II] and [18, List (B)]. Then the same conclusion of (7.1) holds since $|\Sigma| \geqslant \left[\begin{smallmatrix} m \\ x \end{smallmatrix}\right]_q$. Thus $n = 5$, $x = 2$ and $q = 2, 3$. Then $|\Sigma| = 1085$ or 7865, respectively, but both values of $|\Sigma|$ contradict Lemma 7.2(1).

Assume that L_Δ lies in a maximal member of $\mathcal{C}_2(L)$. Then q is odd by [22], and L_Δ is of type $GL_{n/t}(q) \wr S_t$, where either $t = 2$, or $t = 3$ and either $q \in \{5, 9\}$ and n odd, or $(n, q) = (3, 11)$ by [3, Proposition 4.7]. Moreover,

$$L_\Delta \cong \left[\frac{(q-1)^{t-1}(q-1, n/t)}{(q-1, n)}\right].PSL_{n/t}(q)^t.\left[(q-1, n/t)^{t-1}\right].S_t \tag{7.2}$$

by [19, Proposition 4.2.9.(II)]. In addition, L_{Δ}^{Δ} is non-solvable and acts 2-transitively on Δ by Lemma 7.5. Then $t < n$ by (7.2) and $|L| < |L_\Delta|^2$ by Theorem 5.5. Hence,

$$q^{n^2-2} < \frac{(q-1)^{2t-2}(q-1, n/t)^{2t}}{(q-1, n)^2} \cdot (q^{2n^2/t-2t}) \cdot (t!)^2 \tag{7.3}$$

by (7.2) and [3, Corollary 4.1.(i)]. If $t = 3$ then $q^{n^2-2} < 36q^{2n^2/3+2}$ and hence $q^{n^2/3-4} < 36$, which has no solutions for $q \geqslant 3$ and $n \geqslant 6$. Thus $t = 2$, and hence $\mathrm{Soc}(L_{\Delta}^{\Delta}) \cong PSL_{n/2}(q)$ since L_{Δ}^{Δ} is non-solvable and acts 2-transitively on Δ. Then either $|\Delta| = 5$ and $(n, q) = (4, 9)$, or $|\Delta| = q$ and $(n, q) = (4, 5), (4, 7), (4, 11)$, or $|\Delta| = \frac{q^{n/2}-1}{q-1}$. In each case one has $q^{5n} \leqslant q^{n(3n-2)/2} \leqslant |L : L_\Delta| = |\Sigma| \leqslant 2|\Delta|^2 < 2q^n$, which is a contradiction. This completes the proof. □

Lemma 7.10. *L is not isomorphic to $PSU_n(q)$.*

Proof. Assume that $L \cong PSU_n(q)$. Then $n \geqslant 3$ by Lemma 7.8 since $PSU_2(q) \cong PSL_2(q)$. Moreover, one of the following holds by Lemma 7.7 and by [3, Proposition 4.17]:

(i) q is even and L_Δ lies in a maximal parabolic subgroup of L.
(ii) q is odd and L_Δ is a maximal $\mathcal{C}_1$-subgroup of L of type $GU_t(q) \perp GU_{n-t}(q)$;
(iii) q is odd, L_Δ is a maximal $\mathcal{C}_2$-subgroup of L of type $GU_{n/t}(q) \wr S_t$ and one of the following holds:
   (a) $t = 2$.
   (b) $t = 3$ and $(q, z) = (5, 3), (13, 1)$, where $z = (n, q + 1)$.
   (c) $t = n = 4$ and $q = 5$.

Suppose that (i) holds. Then L_Δ is a maximal parabolic subgroup of L by [19, Table 3.5.H] for $n \geqslant 13$ and [5, Section 8.2] for $3 \leqslant n \leqslant 12$. If L_{Δ}^{Δ} does not act 2-transitively on Δ, then L_{Δ}^{Δ} is solvable by Proposition 5.3. Thus $\left|L_{\Delta}^{\Delta}\right| \mid q^2-1$ by [19, Proposition 4.18.(II)], and hence

$$(q-1)q^{n^2-3} < |L| < 4\,|Out(L)|^2\left|L_{\Delta}^{\Delta}\right|^2 = 16f^2(n,q+1)^2(q^2-1)^2$$

by Theorem 5.5(2b). Thus $q^{n^2-6} < 16n^2(q+1)$, and hence $n=3$ and $q=4,8$ since q is even. So, $|\Sigma|=65$ and 513, respectively. However, both cases contradict Lemma 7.2(1). Thus L_{Δ}^{Δ} acts 2-transitively on Δ by Proposition 5.3.

Assume that L_{Δ}^{Δ} is non-solvable. Then either $t \geqslant 2$, $PSL_t(q^2) \trianglelefteq L_{\Delta}^{\Delta}$ and $|\Delta| = \frac{q^{2t}-1}{q-1}$, or $n=4$, $t=1$, $PSL_2(q) \trianglelefteq L_{\Delta}^{\Delta}$ and $|\Delta|=q+1$, or $n-2t=3$, $PSU_3(q) \trianglelefteq L_{\Delta}^{\Delta}$ and $|\Delta|=q^3+1$ by [19, Proposition 4.1.18(II)] and by [18, List (B)] since q is even. On the other hand, by [19, Proposition 4.1.18.(II)] and [3, Lemma 4.1], one obtains

$$|\Sigma| = \frac{\prod_{i=4}^{2t+3}\left(q^i-(-1)^i\right)}{\prod_{j=1}^{t}\left(q^{2j}-1\right)} > \frac{q^{(2t+3)(t+2)-3}}{q^3+1}\cdot\frac{1}{(q^2-1)(q^4-1)q^{t(t+1)-6}} > q^{t^2+6t}. \tag{7.4}$$

If $|\Delta| = \frac{q^{2t}-1}{q-1}$, then

$$2q^{4t-2} > 2\left(q\frac{q^{2t-1}-1}{q-1}\right)^2 - 2\left(q\frac{q^{2t-1}-1}{q-1}\right)+1 = |\Sigma| > q^{t^2+6t}, \tag{7.5}$$

and we reach a contradiction.

In the remaining cases, we have $\lambda=2q^i$ and $|\Sigma|=2q^{2i}-2q^i+1$ with $i=1,2$. Both cases lead to $q^{t^2+6t} < |\Sigma| < 8q^4$ and hence to a contradiction.

Assume that L_{Δ}^{Δ} is solvable. As above $\left|L_{\Delta}^{\Delta}\right| \mid q^2-1$, hence $|\Delta| \mid q^2-1$ by Proposition 5.3. Then $q^{t^2+6t} \leqslant |\Sigma| \leqslant 2\,|\Delta|^2 \leqslant 2(q^2-1)^2$ by (7.4), which is clearly impossible for $t \leqslant 2$.

Note that, L_{Δ} is clearly non-parabolic in the remaining cases. Thus $p \mid |\Sigma|$, and hence $q \geqslant 5$ by Lemma 7.2(3) since q is odd.

Suppose that (ii) holds. Then L_{Δ}^{Δ} is non-solvable and acts 2-transitively on Δ by Lemma 7.5(2). Then either $t=3$ or $n-t=3$, and in both cases $L_{\Delta} \cong PSU_3(q)$ by [19, Proposition 4.1.4.(II)]. Then $\lambda/2+1=q^3+1$ and so $\lambda=2q^3$. Then $|\Sigma|=2q^6-2q^3+1$, whereas $p \mid |\Sigma|$.

Suppose that (iii) holds. Then

$$L_{\Delta} \cong \left[\frac{(q+1)^{t-1}(q+1,n/t)}{(q+1,n)}\right].PSU_{n/t}(q)^t.\left[(q+1,n/t)^{t-1}\right].S_t \tag{7.6}$$

by [19, Proposition 4.2.9.(ii)]. Case (iii.c) implies $|\Sigma| = 5687500$ but this contradicts Lemma 7.2(1). So, it does not occur. Thus $t=2$ or 3. Also L_{Δ}^{Δ} is non-solvable and acts 2-transitively on Δ by Lemma 7.5(2). Then $t<n$ by (7.6) since $t=2,3$, and

$$(q-1)q^{n^2-3} < |L| < |L_{\Delta}|^2 = \frac{(q+1)^{2t-2}(q+1,n/t)^{2t}\,(t!)^2}{(q+1,n)^2}q^{n^2/t-t} \tag{7.7}$$

by Theorem 5.5 and [3, Corollary 4.3(ii)]. If $t = 3$ then $n \geqslant 6$ since $t < n$, and hence (7.7) implies $(q-1)q^{2n^2/3} < 4(q+1)^4 n^4/9$, which has no admissible solutions for $q \geqslant 5$. Then $t = 2$, $n \geqslant 4$ and $q \geqslant 5$. Also, (7.7) implies $(q-1)q^{n^2/2-1} < 4(q+1)^2 n^2$, and again no admissible solutions arise. This completes the proof. □

Lemma 7.11. *If L is not isomorphic to $PSp_n(q)'$.*

Proof. Assume that $L \cong PSp_n(q)'$. Then $n \geqslant 4$ by Lemma 7.8 and $(n,q) \neq (4,2)$ by Lemma 7.4 since $PSp_4(2)' \cong A_6$. Thus $L \cong PSp_n(q)$. By Lemma 7.7, [22] and [3, Proposition 4.22] one of the following holds:

(i) L_Δ lies in a maximal parabolic subgroup of L and q is even.
(ii) L_Δ is a maximal $\mathcal{C}_1$-subgroup of L of type $Sp_i(q) \perp Sp_{n-i}(q)$ with q odd
(iii) L_Δ is a maximal $\mathcal{C}_2$-subgroup of L of type $Sp_{n/t}(q) \wr S_t$, where $t = 2, 3$, or $(n,t) = (8,4)$, or $(n,t) = (10,5)$ and $q = 3$.

Suppose that (i) holds. Then L_Δ is a maximal parabolic subgroup of L by [19, Table 3.5.H] for $n \geqslant 13$ and by [5, Section 8.2] for $4 \leqslant n \leqslant 12$. Thus

$$|\Sigma| = \prod_{i=0}^{t-1} \frac{q^{n-2i}-1}{q^{i+1}-1} > \frac{1}{2} q^{(n-1)t-3t(t-1)/2}$$

If L_Δ^Δ is solvable then $\left|L_\Delta^\Delta\right| \mid (q-1,t)$ by [19, Proposition 4.1.19(II)] and hence $|\Delta| \mid (q-1,t)$ by Proposition 5.3. Therefore $q^{(n-1)t-3t(t-1)/2/2} < |\Sigma| < 2(q-1,t)^2$, which is clearly impossible.

If L_Δ^Δ is non-solvable then L_Δ^Δ acts 2-transitively on Δ by Proposition 5.3. Hence, by [18, List (B)], one of the following holds:

(I) $\mathrm{Soc}(L_\Delta^\Delta) \cong PSL_t(q)$, either $t \geqslant 2$, or $t = 1$, $n = 4$ and $|\Delta| = \frac{q^t-1}{q-1}$;
(II) $\mathrm{Soc}(L_\Delta^\Delta) \cong PSp_{n-t}(2)$, $n - t \geqslant 6$ and $|\Delta| = 2^{2(n-t)-1} \pm 2^{n-t-1}$.

Then (I) is ruled out since it implies $q^{(n-1)t-3t(t-1)/2/2} < |\Sigma| < 2q^{2t}$, which is impossible; (II) is ruled out since it contradicts $|\Delta| \not\equiv 0 \pmod 4$.

Suppose that (ii) holds. Then $L_\Delta \cong Sp_i(q) \circ Sp_{n-i}(q)$ by [19, Proposition 4.1.3.(II)]. Then $L_\Delta^\Delta \cong PSp_j(q)$ with $j \in \{i, n-i\}$ by Lemma 7.5. Hence, one of the following holds (recall that q is odd):

(1) $j = 2$ and $|\Delta| = 6$ for $q = 9$, $|\Delta| = q$ and $q = 5, 7, 11$, or $|\Delta| = q + 1$;
(2) $j \geqslant 6$ and $|\Delta| = 2^{j-1} \pm 2^{j/2-1}$.

Actually, in (1) $q = 5$ and $|\Delta| = 5$ cannot occur since $\lambda > 10$, and $q \neq 7, 9, 11$ by Lemma 7.2(2). Also (2) is ruled out since it contradicts $|\Delta| \not\equiv 0 \pmod 4$. Thus $|\Delta| = q+1$

and $\lambda = 2q$, hence $|\Sigma|$ is coprime to q. So, L_Δ must contain a Sylow q-subgroup of L, which is a contradiction.

Suppose that (iii) holds. Then $q \mid |\Sigma|$, and hence $q \neq 3$ by Lemma 7.2(2). Thus either $t = 2, 3$, or $(n, t) = (8, 4)$. Also, L_Δ^Δ is non-solvable and acts 2-transitively on Δ by Lemma 7.5(2).

If $t = 2$, then either $n = 4$, $L_\Delta^\Delta \cong PSL_2(q)$ and either $|\Delta| = 5$ for $q = 5$, or $|\Delta| = 6$ for $q = 9$, $|\Delta| = q$ for $q = 7, 11$, or $|\Delta| = q + 1$, or $n \geqslant 12$, $L_\Delta^\Delta \cong PSL_{n/2}(2)$ and $|\Delta| = 2^{n-1} \pm 2^{n/2-1}$. However, all these cases are excluded by the same argument previously used.

If $t = 3, 4$, by Theorem 5.5 and by [19, Proposition 4.2.10.(II)] and [3, Corollary 4.3(iii)], we have

$$q^{\frac{n(n+1)}{2}}/4 < |L| < |L_\Delta|^2 < 2^{2(t-1)} q^{n(n/t+1)} (t!)^2, \tag{7.8}$$

which implies $q^{\frac{n(n+1)}{2} - \frac{n(n+t)}{t}} < 2^{2t}(t!)^2$, and hence $t = 3$, $n = 6$ and $q = 5$ or 13 since $q \equiv 1 \pmod 4$ by Lemma 7.2(2). Therefore $|\Sigma| = 44078125$ or 3929239732405, but both cases contradict Lemma 7.2(1). $\square$

Lemma 7.12. *L is not isomorphic to a simple classical group.*

Proof. In order to complete the proof we need to tackle the case $L \cong P\Omega_n^\varepsilon(q)$ with $\varepsilon \in \{\pm, \circ\}$ since the other simple groups are analyzed in Lemmas 7.9, 7.10 and 7.11. Since L is non-abelian simple, $n > 2$ and $(n, \varepsilon) \neq (4, +)$. Also (n, ϵ) is neither $(3, \circ)$ for q odd, nor $(4, -), (6, +)$ by Lemma 7.9 since in these cases L is isomorphic to $PSL_2(q)$, $PSL_2(q^2)$ or $PSL_4(q)$, respectively. Finally, (n, ϵ) is neither $(4, -)$, nor $(5, \circ)$ for q odd, otherwise L would be isomorphic to $PSU_4(q)$ or $PSp_4(q)$, respectively, which are excluded in Lemmas 7.10 and 7.11, respectively. Thus, $n \geqslant 7$. By [22] and [3, Proposition 4.23] one of the following holds:

(1) Either q is even and L_Δ lies in a maximal parabolic subgroup, or q is odd and L_Δ is the stabilizer in L of a non-degenerate subspace of $PG_{n-1}(q)$.
(2) L_Δ is a $\mathcal{C}_2$-subgroup of L of type $O_{n/t}^{\varepsilon'}(q) \wr S_t$, where q is odd, and either $t = 2$, or $n = t = 7$ and $q = 5$, or $7 \leqslant n = t \leqslant 13$ and $q = 3$.

Assume that q is even and that L_Δ lies in a maximal parabolic subgroup M of type P_m. Thus $\varepsilon = \pm$ and hence $n \geqslant 8$. If $(\varepsilon, m) \neq (+, n/2 - 1)$, then $L_\Delta = M$ by [19, Table 3.5.H] for $n \geqslant 13$ and by [5, Section 8.2] for $7 \leqslant n \leqslant 12$. Nevertheless, in each case we have that $L_\Delta \cong [q^y] : GL_m(q) \times \Omega_{n-2m}^\varepsilon(q)$, where $y = nm - \frac{m}{2}(3m - 1)$, by [19, Proposition 4.1.20.II]. Therefore,

$$|\Sigma| = \begin{bmatrix} \frac{n-1+\varepsilon}{2} \\ m \end{bmatrix}_q \prod_{i=0}^{m-1} \left(q^{\frac{n-1-\varepsilon}{2} - i} + 1 \right) > q^{m\left(\frac{n-1+\varepsilon-2m}{2} + \frac{n-\varepsilon-m}{2}\right)} = q^{\left(n - \frac{3}{2}m - \frac{1}{2}\right)m} \tag{7.9}$$

by (7.1) (see also [35, Exercise 11.3]).

Assume that L_{Δ}^{Δ} is solvable. Then $\left|L_{\Delta}^{\Delta}\right| \mid q-1$, and hence $|\Delta| \mid q-1$. Then $q^{(n-\frac{3}{2}m-\frac{1}{2})m} < |\Sigma| \leqslant 2\,|\Delta|^2 = 2(q-1)^2$, which is impossible for $n \geqslant 8$.

Assume that L_{Δ}^{Δ} is non-solvable. Then L_{Δ}^{Δ} acts 2-transitively on Δ by Proposition 5.3, hence one of the following holds by [18]:

(I) $\mathrm{Soc}(L_{\Delta}^{\Delta})$ is isomorphic to $PSL_m(q)$, $m \geqslant 2$, and either $|\Delta| = \frac{q^m-1}{q-1}$, or $|\Delta| = 8$ for $(m,q) = (4,2)$.
(II) $\mathrm{Soc}(L_{\Delta}^{\Delta}) \cong \Omega_4^-(q) \cong PSL_2(q^2)$, $\varepsilon = -$, $n = 2m+4$ and $|\Delta| = q^2+1$.
(III) $\mathrm{Soc}(L_{\Delta}^{\Delta}) \cong PSL_2(q)$, $\varepsilon = +$, $n = 2m+4$ and $|\Delta| = q+1$.
(IV) $\mathrm{Soc}(L_{\Delta}^{\Delta}) \cong PSL_4(q)$, $\varepsilon = +$, $n = 2m+6$ and either $|\Delta| = \frac{q^4-1}{q-1}$, or $|\Delta| = 8$ for $q = 2$.

Assume that (I) or (IV) holds. Then $|\Delta| \neq 8$ since $|\Delta|$ is not divisible by 4. Then $|\Delta| = \frac{q^e-1}{q-1}$, where either $e = m$, or $e = 4$ and $n = 2m+6$ and $\varepsilon = +$. Furthermore, $m \geqslant 2$ in both cases since $n \geqslant 8$. Now, $|\Delta| = \lambda/2+1$ and $|\Sigma| = (\lambda^2 - 2\lambda + 2)/2$ imply

$$|\Sigma| = 2\left(q\frac{q^{e-1}-1}{q-1}\right)^2 - 2\left(q\frac{q^{e-1}-1}{q-1}\right) + 1,$$

and so $q^{(n-\frac{3}{2}m-\frac{1}{2})m} < |\Sigma| < 2q^{2e-2}$.

If $e = m$ and $q^{(n-\frac{3}{2}m-\frac{1}{2})m-2m+2} < 2$ then $n \leqslant 10 - \frac{8}{m}$, and hence $(n,m,\varepsilon) = (8,4,+)$ since n is even, $n \geqslant 8$ and $m \leqslant n/2$. Then

$$2q^6 + 4q^5 + 6q^4 + 2q^3 - 2q + 1 = |\Sigma| = \left(q^2+1\right)\left(q^3+1\right)(q+1)$$

If $e = 4$, $n = 2m+6$ and $\varepsilon = +$, then $m = 1$ and so

$$2q^6 + 4q^5 + 6q^4 + 2q^3 - 2q + 1 = |\Sigma| = (q^4-1)(q^3+1),$$

which has not integer solutions.

Assume that case (II) or (III) holds. Then $n = 2m+4$ and $|\Delta| = q^j+1$ with $j = 2,1$, respectively. Then $|\Sigma| < 8q^{2j}$. On the other hand, $|\Sigma| > q^{(n-\frac{3}{2}m-\frac{1}{2})m}$ by (7.9) since $n \geqslant m/2$. Therefore $q = 2$, $n = 8$ and either $m = 1$ or $m = 4$. However, both cases contradict $n = 2m+4$. This excludes case (1) for q even.

In the remaining cases, namely (1) and (2) for q odd, it results that $p \mid |\Sigma|$ by [34, Theorem 1.6]. Also p is odd, and $p \neq 3$ by Lemma 7.2(2). Therefore, in the sequel we may assume that $q \geqslant 5$. Then either L_{Δ} is maximal in L, or $L \cong P\Omega_n^+(5)$, L_{Δ} is a $\mathcal{C}_2$-subgroup of L of type $O_2^+(5) \wr S_{n/2}$ and G_{Δ} is a novelty by [19, Table 3.5.H–I] or [5, Section 8.2] according to whether $n \geqslant 13$ or $7 \leqslant n \leqslant 12$, respectively. In the latter case n is forced to be 4 by (2), whereas $n \geqslant 8$. Therefore, L_{Δ} is maximal in L.

Assume that L_{Δ} is the stabilizer in L of a non-degenerate subspace of $PG_{n-1}(q)$, q odd. Since L_{Δ}^{Δ} is non-solvable and acts 2-transitively on Δ by Lemma 7.5, one of the following holds by [19, Propositions 4.1.6.(II)] and [18] and since q is odd, $q \geqslant 5$ and $n \geqslant 7$:

(i) L_Δ preserves a non-degenerate 4-subspace of type $+$ and $\mathrm{Soc}(L_\Delta^\Delta) \cong PSL_2(q)$;
(ii) L_Δ preserves a non-degenerate 4-subspace of type $-$ and $\mathrm{Soc}(L_\Delta^\Delta) \cong PSL_2(q^2)$;
(iii) L_Δ preserves a non-degenerate 6-subspace of type $-$ and $\mathrm{Soc}(L_\Delta^\Delta) \cong PSL_4(q)$.

Assume that (i) or (ii) holds. Then either $|\Delta| = q^j + 1$, where $j = 1$ or 2, respectively, or $j = 1$ and $|\Delta| = q$ for $q = 5, 7, 11$ or $|\Delta| = 6$ for $q = 9$, or $j = 2$ and $|\Delta| = 6$ for $q = 3$. The first case implies $\lambda = 2q^j$, therefore $|\Sigma|$ is coprime to q, whereas L_Δ must contain a Sylow q-subgroup of L, which is not the case. Also $q \neq 3, 7, 9, 11$ by Lemma 7.2(2). Finally, $|\Delta| = q = 5$ cannot occur since $\lambda > 10$.

Assume that (iii) holds. Then $|\Delta| = \frac{q^4-1}{q-1}$ since q is odd. Then $\lambda = 2q\frac{q^3-1}{q-1}$, hence $|\Sigma|$ is coprime to q, whereas L_Δ must contain a Sylow q-subgroup of L, which is not the case. This excludes (1).

Finally, assume that (2) holds. Hence, q is odd. Furthermore, $q \mid |\Sigma|$ since L_Δ is not parabolic. Then $p \equiv 1 \pmod 4$ since by Lemma 7.7(2), and hence $p \neq 3, 7, 11$. Therefore, either $t = 2$, or $t = n = 7$ and $q = 5$.

Assume that $t = 2$. Then $n/2 > 1$ since $n \geqslant 8$, and hence L_Δ cannot be any of the groups listed in [19, Proposition 4.2.15(II)]. If $n/2$ is odd, then $L_\Delta \cong \Omega_{n/2}(q)^2.2.2$ by [19, Proposition 4.2.14(II)]. Then $n = 10$ and $Soc(L_\Delta^\Delta) \cong \Omega_5(q) \cong PSp_4(q)$ since L_Δ induces a 2-transitive non-solvable group on Δ by Lemma 7.5(2). Thus $\frac{\lambda+2}{2} = \frac{q^4-1}{q-1}$, and hence $\lambda = 2q\frac{q^3-1}{q-1}$. Then q does not divide $|\Sigma| = \frac{\lambda^2-2\lambda+2}{2}$ since q is odd, a contradiction. Thus, $n/2$ is even. Then $\varepsilon = (\varepsilon')^2$ and L_Δ is isomorphic to one of the groups $2.(P\Omega_{n/2}^{\varepsilon'}(q))^2.4.2$, $(2 \times \Omega_{n/2}^{\varepsilon'}(q)^2.2).2$, or $\Omega_{n/2}^{\varepsilon'}(q)^2.2.2$ by [19, Proposition 4.2.11(II)] since q is odd. Then either $n = 8$ and $L_\Delta^\Delta \cong PSL_2(q^i)$, $i = 1, 2$, or $n = 12$, $\varepsilon' = +$ and $L_\Delta^\Delta \cong PSL_6(q)$ since L_Δ induces a 2-transitive non-solvable group on Δ by Lemma 7.5(2). Thus $\frac{\lambda+2}{2}$ is either 5, or $q^i + 1$ with $i = 1, 2$, or $\frac{q^6-1}{q-1}$ since $p \neq 3, 7, 11$. Actually, $\frac{\lambda+2}{2} \neq 5$ since $\lambda > 10$ by our assumptions. Therefore, either $\lambda = q^i$, $i = 1, 2$, or $\lambda = 2q\frac{q^5-1}{q-1}$. Then q does not divide $|\Sigma| = \frac{\lambda^2-2\lambda+2}{2}$ since q is odd, a contradiction.

Assume that $t = n = 7$ and $q = 5$. Then $L_\Delta \cong 2^6.A_7$ by [19, Proposition 4.2.15(II)]. So $|\Sigma| = 29752734375$, which contradicts Lemma 7.2(1), and hence (2) is ruled out. This completes the proof. $\square$

Proof of Theorem 7.1. Since L^Σ is almost simple by Theorem 4.1, the assertion follows from Lemmas 7.3, 7.4, 7.6 and 7.12. $\square$

Proof of Theorem 1.1. Since $\lambda \leqslant 10$ by Theorems 6.1 and 7.1, the assertion follows from Theorem 1.2. $\square$

Data availability

No data was used for the research described in the article.

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References

[1] M. Aigner, G.M. Ziegler, Proofs from the Book, 5th ed., Springer-Verlag, Berlin, New York, 2014.
[2] M. Aschbacher, Overgroups of Sylow subgroups in sporadic groups, Mem. Am. Math. Soc. 343 (1986).
[3] S.H. Alavi, T.C. Burness, Large subgroups of simple groups, J. Algebra 421 (2015) 187–233.
[4] J. Bamberg, T. Penttila, Overgroups of cyclic Sylow subgroups of linear groups, Commun. Algebra 36 (2008) 2503–2543.
[5] J. Bray, D.F. Holt, C.M. Roney-Dougal, The Maximal Subgroups of the Low-Dimensional Finite Classical Groups, with a foreword by Martin Liebeck, London Mathematical Society Lecture Note Series, vol. 407, Cambridge University Press, Cambridge, 2013.
[6] A.R. Camina, P.H. Zieschang, On the normal structure of the flag transitive automorphism groups of 2-designs, J. Lond. Math. Soc. (2) 41 (1990) 555–564.
[7] J.H. Conway, R.T. Curtis, R.A. Parker, R.A. Wilson, An Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[8] H. Davies, Flag-transitivity and primitivity, Discrete Math. 63 (1987) 91–93.
[9] P. Dembowski, Finite Geometries, Springer, Berlin, Heidelberg, New York, 1968.
[10] U. Dempwolff, Primitive rank 3 groups on symmetric designs, Des. Codes Cryptogr. 22 (2001) 191–207.
[11] A. Devillers, M. Giudici, C.H. Li, G. Pearce, C.E. Praeger, On imprimitive rank 3 permutation groups, J. Lond. Math. Soc. 84 (2011) 649–669.
[12] J.D. Dixon, B. Mortimer, Permutation Groups, Springer Verlag, New York, 1966.
[13] R.M. Guralnick, Subgroups of prime power index in a simple group, J. Algebra 81 (1983) 304–311.
[14] C. Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order, Geom. Dedic. 2 (1974) 425–460.
[15] D.G. Higman, J.E. McLaughlin, Geometric ABA-groups, Ill. J. Math. 5 (1961) 382–397.
[16] Q.M. Hussain, On the totality of the solutions for the symmetrical incomplete block designs $\lambda = 2$, $k = 5$ or 6, Sankhya 7 (1945) 204–208.
[17] W.M. Kantor, R.A. Liebler, The rank 3 permutation representations of the finite classical groups, Trans. Am. Math. Soc. 271 (1982) 1–71.
[18] W.M. Kantor, Homogenous designs and geometric lattices, J. Comb. Theory, Ser. A 38 (1985) 66–74.
[19] P. Kleidman, M.W. Liebeck, The Subgroup Structure of the Finite Classical Groups, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990.
[20] E.S. Lander, Symmetric Designs: An Algebraic Approach, London Mathematical Society Lecture Note Series, vol. 74, Cambridge University Press, Cambridge, 1983.
[21] M. Law, C.E. Praeger, S. Reichard, Flag-transitive symmetric 2-$(96, 20, 4)$-designs, J. Comb. Theory, Ser. A 116 (2019) 1009–1022.
[22] M.W. Liebeck, J. Saxl, The primitive permutation groups of odd degree, J. Lond. Math. Soc. 31 (1985) 250–264.
[23] M.W. Liebeck, J. Saxl, G.M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, Proc. Lond. Math. Soc. 65 (1992) 297–325.
[24] J. Mandić, A. Šubasić, Flag-transitive and point-imprimitive symmetric designs with $\lambda \leqslant 10$, J. Comb. Theory, Ser. A 189 (2022) 105620.
[25] A. Montinaro, E. Francot, On flag-transitive 2-k^2, k, λ designs with $\lambda \mid k$, J. Comb. Des. 30 (2022) 653–670.

[26] A. Montinaro, Classification of the non-trivial 2-(k^2, k, λ) designs, with $\lambda \mid k$, admitting a flag-transitive almost simple automorphism group, J. Comb. Theory, Ser. A 195 (2023) 105710.
[27] A. Montinaro, Flag-transitive, point-imprimitive symmetric 2-(v, k, λ) designs with $k > \lambda(\lambda - 3)/2$, arXiv:2203.09261.
[28] H.K. Nandi, Enumerations of nonisomorphic solutions of balanced incomplete block designs, Sankhya 7 (1946) 305–312.
[29] E. O'Relly Reguerio, On primitivity and reduction for flag-transitive symmetric designs, J. Comb. Theory, Ser. A 109 (2005) 135–148.
[30] C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graph, J. Lond. Math. Soc. 47 (1993) 227–239.
[31] C.E. Praeger, The flag-transitive symmetric designs with 45 points, blocks of size 12, and 3 blocks on every point pair, Des. Codes Cryptogr. 44 (2007) 115–132.
[32] C.E. Praeger, S. Zhou, Imprimitive flag-transitive symmetric designs, J. Comb. Theory, Ser. A 113 (2006) 1381–1395.
[33] P. Ribenboim, Catalan's Conjecture: Are 8 and 9 the Only Consecutive Powers?, Academic Press, Inc., Boston, 1994.
[34] G.M. Seitz, Flag-transitive subgroups of Chevalley group, Ann. Math. 97 (1973) 27–56.
[35] D.E. Taylor, The Geometry of the Classical Groups, Heldermann Verlag, 1992.
[36] A.V. Vasilyev, Minimal permutation representations of finite simple exceptional groups of types G_2 and F_4, Algebra Log. 35 (1996) 663–684.
[37] A.V. Vasilyev, Minimal permutation representations of finite simple exceptional groups of types E_6, E_7 and E_8, Algebra Log. 36 (1997) 518–530.
[38] A.V. Vasilyev, Minimal permutation representations of finite simple exceptional groups of twisted type, Algebra Log. 37 (1998) 17–35.
[39] R.A. Wilson, Maximal subgroups of automorphism groups of simple groups, J. Lond. Math. Soc. 32 (1985) 460–466.
[40] R.A. Wilson, Maximal subgroups of sporadic groups, in: R. Guralnick, G. Hiss, K. Lux, P. Huu Tiep (Eds.), Finite Simple Groups: Thirty Years of the Atlas and Beyond, in: Contemporary Mathematics, vol. 694, 2017, pp. 58–72.
[41] Z. Zhang, J. Chen, S. Zhou, On symmetric designs with flag-transitive and point-quasiprimitive automorphism groups, J. Comb. Des. 32 (2004) 107–126.
[42] K. Zsigmondy, Zur Theorie der Potenzreste, Monatshefte Math. Phys. 3 (1892) 265–284.