Uniform Mixed Equilibria in Network Congestion Games with Link Failures

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Abstract

Motivated by possible applications in fault-tolerant routing, we introduce the notion of uniform mixed equilibria in network congestion games with adversarial link failures, where players need to route traffic from a source to a destination node. Given an integer $\rho \geq 1$, a ρ-uniform mixed strategy is a mixed strategy in which a player plays exactly ρ edge disjoint paths with uniform probabilities, so that a ρ-uniform mixed equilibrium is a tuple of ρ-uniform mixed strategies, one for each player, in which no player can lower her cost by deviating to another ρ-uniform mixed strategy. For games with weighted players and affine latency functions, we show existence of ρ-uniform mixed equilibria and provide a tight characterization of their price of anarchy. For games with unweighted players, instead, we extend the existential guarantee to any class of latency functions and, restricted to games with affine latencies, we derive a tight characterization of both the prices of anarchy and stability.

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1 Introduction

Consider n users who need to send an unsplittable amount of traffic from a source to a destination in a network which is subject to adversarial link failures. In particular, each user u_i is coupled with an adversary a_i who, upon knowledge of the mixed strategy adopted by u_i to route her traffic, chooses which links to corrupt. Thus, every pair (u_i, a_i) is engaged in a two-player Stackelberg game in which u_i is the leader, a_i is the follower, and both are interested in the probability that the path selected by u_i as a realization of her mixed strategy fails: u_i wants to minimize it, while a_i aims at its maximization[1]. To make things more

[1] We stress that a_i is only aware of the mixed strategy chosen by u_i and not of its final realization.

interesting, all users also play an atomic congestion game among themselves, in which each of them wants to minimize the expected latency of the chosen mixed strategy. We assume that a user's priority is to route her traffic at any cost, so that she will be interested in selecting the mixed strategy of minimum expected latency among those minimizing the failure probability (of its realization). It is not difficult to see that u_i minimizes this probability if and only if she assigns uniform probabilities to the maximum number of edge disjoint paths connecting her source-destination pair.

Motivated by the above scenario, we introduce the notion of uniform mixed equilibrium for (network) congestion games (with adversarial link failures). Formally, given an integer $\rho \geq 1$, a ρ-uniform mixed strategy is a mixed strategy in which exactly ρ edge disjoint paths are chosen with uniform probabilities. Thus, given an n-tuple of positive integers $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$, a $\boldsymbol{\rho}$-uniform mixed profile is a mixed profile in which each user u_i adopts a ρ_i-uniform mixed strategy and a $\boldsymbol{\rho}$-uniform mixed equilibrium is a $\boldsymbol{\rho}$-uniform mixed profile in which no user u_i can lower her cost by deviating to another ρ_i-uniform mixed strategy.

As a first step in the understanding of the properties of these equilibria, we assume that $\rho_i = \rho_j := \rho$ for every pair of users u_i and u_j, and we denote $\boldsymbol{\rho} = (\rho, \ldots, \rho)$ simply as ρ. Besides defining a simple, yet interesting case, this assumption has at least two practical applications/justifications. First, it models the case of symmetric games, in which all users share the same source-destination pair; this setting has been widely studied with respect to the analysis of efficiency of Nash equilibria [21] and to the (hardness of) computation of equilibria [17, 16]. From a theoretical point of view, it is worth noting that, in this case, the value of ρ, i.e. the maximum number of edge disjoint paths connecting the common source-destination pair, can be efficiently computed by a reduction to the max-flow problem (see [1] for further details). To illustrate the second application, observe that the desire to minimize the failure probability induces each user to add even extremely costly paths to the set of her possible alternatives. It is reasonable to assume that, in some contexts, users' priorities can be restated as follows: each user wants to select the mixed strategy of minimum expected latency among those keeping the failure probability within a certain threshold θ. For the ease of exposition, assume that the adversary can corrupt just one link and that $\theta = 1/3$. By simple calculations, it is not difficult to establish that the best strategy for each user is to play $\rho = 3$ edge disjoint paths with probability $1/3$ each.

We stress that, although the notions of 1-uniform mixed equilibria and that of pure Nash equilibria [26] coincide, there are no correlations between the set of ρ-uniform mixed equilibria and that of mixed Nash equilibria of a given game when $\rho > 1$. Moreover, for $\rho > 1$, ρ-uniform mixed strategies can be interpreted as an hybridization between the notions of pure and mixed strategies. In fact, although the cost incurred by a player needs to be evaluated in expectation (as it happens when adopting mixed strategies), the fact that probabilities are superimposed by the model limits the players' choices to deciding which strategies to play (as it happens when adopting pure strategies). To the best of our knowledge, this is the first attempt towards this direction.

1.1 Our Contribution

We study the existence and efficiency of ρ-uniform mixed equilibria in (network) congestion games by distinguish between the case in which all players need to route the same amount of traffic (*unweighted congestion games*) and the general case of different traffic rates (*weighted congestion games*). In particular, we focus on networks in which the link latency functions are affine (*affine congestion games*).

Table 1 The prices of anarchy and stability of ρ-uniform mixed equilibria in affine unweighted congestion games and the price of anarchy of ρ-uniform mixed equilibria in affine weighted congestion games, for each value of $\rho \geq 1$. Bounds labeled as $*$ holds also for parallel link networks with restricted strategies, while bounds labeled as † applies to even parallel link networks with unrestricted strategies.

ρ	unweighted games		weighted games
	price of stability	price of anarchy	price of anarchy
1	$1+1/\sqrt{3}$ [11, 13]	$5/2^*$ [3, 11, 14]	$(3+\sqrt{5})/2^\dagger$[3, 5, 11, 14]
2	$1+1/\sqrt{5}$	$5/3^*$	$2^\dagger$
3	$1+1/\sqrt{7}$	$(2\sqrt{7}-1)/3^*$	$(7+\sqrt{13})/6^\dagger$
4	$4/3^\dagger$	$4/3^\dagger$	$(9+\sqrt{17})/8^\dagger$
≥ 5	$4/3^\dagger$	$4/3^\dagger$	$4\rho^2/(3\rho^2-2\rho-1)^\dagger$

We first prove that ρ-uniform mixed equilibria do exist in affine weighted congestion games, for each $\rho \geq 1$. This is done by showing that any affine weighted congestion game in which players adopt ρ-uniform mixed strategies admits a potential function. This generalizes to every value of ρ the results by [21, 22, 27] which were proved for the classical setting in which players adopt pure strategies, i.e., for $\rho = 1$. For the case of unweighted players, existential guarantees are extended to any class of latency functions. This generalizes Rosenthal's Theorem [30] which shows existence of pure Nash equilibria, i.e., existence of ρ-uniform mixed equilibria for the basic case of $\rho = 1$.

Then, for each $\rho \geq 1$, by exploiting the primal-dual method [6], we derive tight bounds on the price of anarchy of ρ-uniform mixed equilibria in affine weighted congestion games and tight bounds on both the prices of anarchy and stability of ρ-uniform mixed equilibria in affine unweighted congestion games (see the values reported in Table 1, where many lower bounds hold even for parallel link networks). It is worth noticing that our results nicely extend the ones obtained for pure Nash equilibria, i.e., the case of $\rho = 1$. In particular, for unweighted congestion games with affine latency functions, [13] proved that the price of stability is lower bounded by $1 + 1/\sqrt{3}$, while [11] showed that this bound is tight; [3, 14] proved that the price of anarchy is $5/2$ and [11] showed that the same (lower) bound extends to the special case of parallel link networks with restricted strategies (i.e., every player can only select a link from an allowable set of alternatives). For weighted congestion games with linear latency functions, [3, 14] proved that the price of anarchy is $(3 + \sqrt{5})/2$, [11] showed that the same (lower) bound extends to the special case of parallel link networks with restricted strategies, and finally [5] proved that even parallel link networks with unrestricted strategies are enough to obtain a matching lower bound. The existential guarantee, as well as the bounds for unweighted games, are obtained by exploiting the fact that, for each $\rho \geq 1$, any unweighted congestion game in which players adopt ρ-uniform mixed strategies is isomorphic to an unweighted congestion game in which players adopt pure strategies and whose latency functions are slightly different.

Our results show that, as ρ increases, the prices of anarchy and stability of ρ-uniform mixed equilibria of affine congestion games approach the value $4/3$, that is, the price of anarchy of affine non-atomic congestion games [32]. This is in accordance with the intuition that, by arbitrarily splitting an atomic request over disjoint strategies, atomic congestion games tend to their non-atomic counterparts. The striking evidence of our findings, however, is that, for unweighted players, when such a splitting is restricted to be uniform (i.e., the same amount of traffic must be routed on each selected path), this happens for $\rho = 4$ already.

1.2 Related Work

Penn, Polukarov and Tennenholtz [28, 29] introduced congestion games with failures. In their model, each player has a task that can be executed on any resource, i.e. players only adopt singleton strategies, and each resource may fail with a certain probability, hence, for reliability reasons, a player may choose to simultaneously use multiple resources. The cost of a player is given by the minimum of the costs payable on all the selected resources that do not fail. In this setting, the existence, properties and efficiency of pure Nash equilibria are investigated.

The setting of adversarial behavior in congestion games was introduced by Karakostas and Viglas [23] for network congestion games. Babaioff, Kleinberg and Papadimitriou [4] studied the impact of malicious players on the quality of Nash equilibria for non-atomic games. In particular, [4, 23] considered two classes of players, i.e., rational players and malicious players; while rational players act aiming at minimizing their own cost, malicious ones aim at maximizing the average delay experienced by the rational players. Roth [31] applied this adversarial setting to the class of linear congestion games. Also Moscibroda et al. [25] analyzed an adversarial behavior in a different game.

1.3 Paper Organization

The paper is organized as follows. In the next section we provide the notation and definitions, together with some basic results. Section 3 is devoted to the study of affine weighted congestion games, while Section 4 to the analysis of the unweighted case. Finally, Section 5 gives some conclusive remarks and lists some interesting open problems. Due to space limitations, some proofs are omitted (see the full version of the paper).

2 Definitions and Notation

Given two integers $0 \leq k_1 \leq k_2$, let $[k_2]_{k_1}$ denote the set $\{k_1, k_1+1, \ldots, k_2-1, k_2\}$ and let $[k_1]$ denote the set $[k_1]_1$. Moreover, let $\vec{1}^n$ denote the vector $(1, \ldots, 1) \in \mathbb{R}^n_{\geq 0}$.

A *weighted congestion model* is defined by a tuple $\mathsf{CM} = (\mathsf{N}, E, (\ell_e)_{e \in E}, (w_i)_{i \in \mathsf{N}}, (\Sigma_i)_{i \in \mathsf{N}})$, where N is a set of $n \geq 2$ players, E is a set of resources, $\ell_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the latency function of resource $e \in E$, and, for each $i \in \mathsf{N}$, $w_i \geq 0$ is the weight of player i and $\Sigma_i \subseteq 2^E \setminus \emptyset$ is her set of strategies. A weighted congestion model is *symmetric* if $\Sigma_i = \Sigma$ for each $i \in \mathsf{N}$, i.e., if all players share the same strategy space. A *weighted load balancing model* is a weighted congestion model in which for each $i \in \mathsf{N}$ and $S \in \Sigma_i$, $|S| = 1$, that is, all players' strategies are singleton sets. Observe that a weighted load balancing model corresponds to a parallel link network. A weighted congestion model is *affine* if its latency functions are of the form $\ell_e(x) := \alpha_e x + \beta_e$, with $\alpha_e, \beta_e \geq 0$. An *unweighted congestion model* is a weighted congestion model such that $w_i = 1$ for each $i \in \mathsf{N}$.

Depending on the types of strategies adopted by the players, a congestion model CM may induce different classes of congestion games.

A *strategy profile* is an n-tuple of strategies $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$, that is a state in which each player $i \in \mathsf{N}$ adopts pure strategy $s_i \in \Sigma_i$. When players adopt pure strategies, CM induces a congestion game $\mathsf{CG}(\mathsf{CM})$ (usually, when CM is clear from the context, we shall drop it from the notation). For a strategy profile $\boldsymbol{s}$, the *congestion* of resource $e \in E$ in $\boldsymbol{s}$, denoted as $k_e(\boldsymbol{s}) := \sum_{i \in \mathsf{N}: e \in s_i} w_i$, is the total weight of the players using resource e in $\boldsymbol{s}$, (observe that, for unweighted games, $k_e(\boldsymbol{s})$ coincides with the number of users selecting resource e in $\boldsymbol{s}$). The cost of player i in $\boldsymbol{s}$ is defined as $cost_i^{\mathsf{CG}}(\boldsymbol{s}) = \sum_{e\in s_i} \ell_e(k_e(\boldsymbol{s}))$ (usually, when CG is clear from the context, we shall drop it from the notation). The quality of a strategy profile in $\mathsf{CG(CM)}$ is measured by using the *social function* $\mathsf{SUM}(\boldsymbol{s}) = \sum_{i\in\mathsf{N}} w_i cost_i(\boldsymbol{s}) = \sum_{e\in E} k_e(\boldsymbol{s})\ell_e(k_e(\boldsymbol{s}))$, that is, the weighted sum of the players' costs. A *pure Nash equilibrium* for $\mathsf{CG(CM)}$ is a strategy profile $\boldsymbol{s}$ such that, for any player $i \in \mathsf{N}$ and strategy $s_i' \in \Sigma_i$, $cost_i(\boldsymbol{s}) \leq cost_i(\boldsymbol{s}_{-i}, s_i')$. We denote by $\mathsf{Eq(CG(CM))}$ the set of pure Nash equilibria of a weighted congestion game $\mathsf{CG(CM)}$. The *price of anarchy* (resp. *price of stability*) of a weighted congestion game $\mathsf{CG(CM)}$ is defined as $\mathsf{PoA(CG(CM))} = \max_{\boldsymbol{s}\in\mathsf{Eq(CG(CM))}} \left\{\frac{\mathsf{SUM}(\boldsymbol{s})}{\mathsf{SUM}(\boldsymbol{s}^*)}\right\}$ (resp. $\mathsf{PoS(CG(CM))} = \min_{\boldsymbol{s}\in\mathsf{Eq(CG(CM))}} \left\{\frac{\mathsf{SUM}(\boldsymbol{s})}{\mathsf{SUM}(\boldsymbol{s}^*)}\right\}$), where $\boldsymbol{s}^*$ is a *social optimum* for $\mathsf{CG(CM)}$, that is a strategy profile minimizing the social function.

A mixed strategy for player i is a probability distribution σ_i defined over Σ_i, so that $\sigma_i(s)$ is the probability that player i plays strategy $s \in \Sigma_i$. We denote by $\mathsf{supp}(\sigma_i) = \{s \in \Sigma_i : \sigma_i(s) > 0\}$ the set of strategies played with positive probability in σ_i. A *mixed profile* $\boldsymbol{\sigma}$ is an n-tuple of mixed strategies, i.e., $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Informally, $\boldsymbol{\sigma}$ is a state in which each player $i \in \mathsf{N}$ picks her strategy according to probability distribution σ_i, independently from the choices of other players. If σ_i is such that a pure strategy s_i is picked with probability one by player i, we write s_i instead of σ_i.

Given an integer $\rho \geq 1$ and a weighted congestion model CM in which for each player $i \in N$ there exist at least ρ pairwise disjoint strategies in Σ_i, a *ρ-uniform mixed strategy* for player i is a mixed strategy σ_i such that $|\mathsf{supp}(\sigma_i)| = \rho$, $s_1 \cap s_2 = \emptyset$ for any $s_1, s_2 \in \mathsf{supp}(\sigma_i)$ with $s_1 \neq s_2$, and $\sigma_i(s) = 1/\rho$ for each $s \in \mathsf{supp}(\sigma_i)$, i.e., a mixed strategy in which player i plays exactly ρ pairwise disjoint strategies with uniform probability. Denote by $\Delta_i^\rho(\mathsf{CM})$ the set of ρ-uniform mixed strategies for player i. A *ρ-uniform mixed profile* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an n-tuple of ρ-uniform mixed strategies, one for each player. When players adopt *ρ-uniform mixed strategies*, CM induces a ρ-uniform congestion game ρ-$\mathsf{CG(CM)}$ (again, when CM is clear from the context, we shall drop it from the notation). For a ρ-uniform mixed profile $\boldsymbol{\sigma}$, the *expected congestion* of resource $e \in E$ in $\boldsymbol{\sigma}$, denoted as $k_e(\boldsymbol{\sigma}) := \mathbb{E}_{\boldsymbol{s}\sim\boldsymbol{\sigma}}\left(\sum_{i\in\mathsf{N}:e\in s_i} w_i\right)$, is the expected total weight of the players using resource e in $\boldsymbol{\sigma}$. The cost of player i in $\boldsymbol{\sigma}$ is defined as $cost_i^{\rho\text{-}\mathsf{CG}}(\boldsymbol{\sigma}) = \mathbb{E}_{\boldsymbol{s}\sim\boldsymbol{\sigma}}\left(\sum_{e\in s_i} \ell_e(k_e(\boldsymbol{s}))\right)$ (again, when ρ-CG is clear from the context, we shall drop it from the notation). The quality of a ρ-uniform mixed profile in ρ-$\mathsf{CG(CM)}$ becomes $\mathsf{SUM}(\boldsymbol{\sigma}) = \mathbb{E}_{\boldsymbol{s}\sim\boldsymbol{\sigma}}\left(\sum_{i\in\mathsf{N}} w_i cost_i(\boldsymbol{s})\right) = \sum_{e\in E} \mathbb{E}_{\boldsymbol{s}\sim\boldsymbol{\sigma}}\left(k_e(\boldsymbol{s})\ell_e(k_e(\boldsymbol{s}))\right)$, that is, the expected weighted sum of the players' costs. A *ρ-uniform mixed equilibrium* for ρ-$\mathsf{CG(CM)}$ is a ρ-uniform mixed profile $\boldsymbol{\sigma}$ such that, for any player $i \in \mathsf{N}$ and ρ-uniform mixed strategy $\sigma_i' \in \Delta_i^\rho(\mathsf{CM})$, $cost_i(\boldsymbol{\sigma}) \leq cost_i(\boldsymbol{\sigma}_{-i}, \sigma_i')$. We denote by $\mathsf{Eq}(\rho\text{-}\mathsf{CG(CM)})$ the set of ρ-uniform mixed equilibria of a weighted congestion game ρ-$\mathsf{CG(CM)}$. The price of anarchy (resp. price of stability) of a ρ-uniform weighted congestion game ρ-$\mathsf{CG(CM)}$ is defined as $\mathsf{PoA}_\rho(\rho\text{-}\mathsf{CG(CM)}) = \max_{\boldsymbol{\sigma}\in\mathsf{Eq}(\rho\text{-}\mathsf{CG(CM)})} \left\{\frac{\mathsf{SUM}(\boldsymbol{\sigma})}{\mathsf{SUM}(\boldsymbol{\sigma}^*)}\right\}$ (resp. $\mathsf{PoS}_\rho(\rho\text{-}\mathsf{CG(CM)}) = \min_{\boldsymbol{\sigma}\in\mathsf{Eq}(\rho\text{-}\mathsf{CG(CM)}} \left\{\frac{\mathsf{SUM}(\boldsymbol{\sigma})}{\mathsf{SUM}(\boldsymbol{\sigma}^*)}\right\}$), where $\boldsymbol{\sigma}^*$ is a *ρ-uniform social optimum* for ρ-$\mathsf{CG(CM)}$, that is a ρ-uniform mixed profile minimizing the social function.

Given a ρ-uniform mixed strategy σ_i, let $E(\sigma_i) = \bigcup_{s\in\mathsf{supp}(\sigma_i)} s$ denote the set of resources contained by all strategies belonging to $\mathsf{supp}(\sigma_i)$[2]. For a ρ-uniform mixed profile $\boldsymbol{\sigma}$, the *ρ-maximum congestion* of resource e in $\boldsymbol{\sigma}$, denoted as $k_{\rho,e}(\boldsymbol{\sigma}) = \sum_{i:e\in E(\sigma_i)} w_i$, is the congestion of e obtained if all players assigning non-null probability to a strategy s containing e pick s.

[2] Given $e \in E(\sigma_i)$, there exists a unique strategy of σ_i containing e, since strategies selected with non-null probability by each player are pairwise disjoint.

▶ Remark. According to the first application described in Section 1, given a symmetric congestion model CM such that the maximum number of disjoint strategies is $\rho* > \rho$, we can consider a congestion model CM′ such that CG(CM′) is equivalent to CG(CM), and such that the maximum number of disjoint strategies of CG(CM′) is ρ. To this aim, it suffices considering a congestion model CM′ in which $E' := E \cup \{e'_1, e'_2, \dots, e'_\rho\}$, where e'_j is a dummy resource with $\ell'(e'_j) = 0$ for any $j \in [\rho]$, $\Sigma_i{}' := \{s \cup \{e'_j\} : s \in \Sigma_i, j \in [\rho]\}$ for any $i \in \mathsf{N}$, and all the other quantities are defined as in CM. Observe that, given ρ-disjoint strategies $s_1, s_2, \dots, s_\rho$ in Σ, we have that $s_1 \cup \{e'_1\}, s_2 \cup \{e'_2\}, \dots, s_\rho \cup \{e'_\rho\}$ are disjoint strategies of $\Sigma_i{}'$. Furthermore, there are no $\rho+1$ disjoint strategies in $\Sigma_i{}'$, since, given $\rho+1$ strategies of $\Sigma_i{}'$, there are necessarily at least two strategies $s'_1, s'_2 \in \Sigma_i{}'$ such that $e'_j \in s'_1 \cap s'_2$ for some $j \in [\rho]$. Thus, ρ is the maximum number of disjoint strategies in CG(CM′). Finally, since each strategy of $\Sigma_i{}'$ is defined as union of some strategy of Σ_i and some dummy resource having null cost, games CG(CM′) and CG(CM) are completely equivalent.

We conclude the section by providing useful equations to express the players' costs in ρ-uniform congestion games as a function of the ρ-maximum congestions only, thus getting rid of expected values. Towards this end, as shown in [6], we can assume without loss of generality that the latency functions of the games we consider are restricted to be linear, that is, of the form $\ell(x) = \alpha_e x$ for some $\alpha_e \geq 0$.

▶ **Lemma 1.** *Given an affine weighted congestion model* CM *and a ρ-uniform strategy profile $\boldsymbol{\sigma}$ for ρ-*CG(CM)*, we have*

$$cost_i(\boldsymbol{\sigma}) = \sum_{e \in E(\sigma_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2} + \left(\frac{\rho - 1}{\rho^2} \right) w_i \right) \tag{1}$$

and, given $\sigma'_i \in \Delta_i^\rho(\mathsf{CM})$, we have

$$\begin{aligned} cost_i(\boldsymbol{\sigma}_{-i}, \sigma'_i) &= \sum_{e \in E(\sigma'_i) \setminus E(\sigma_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2} + \frac{w_i}{\rho} \right) + \sum_{e \in E(\sigma'_i) \cap E(\sigma_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2} + \left(\frac{\rho-1}{\rho^2} \right) w_i \right) \\ &\leq \sum_{e \in E(\sigma'_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2} + \frac{w_i}{\rho} \right). \end{aligned} \tag{2}$$

3 Weighted Games

In this section, we consider the general case of ρ-uniform congestion games induced by affine weighted congestion models. We start by showing that ρ-uniform mixed equilibria are always guaranteed to exist, for each $\rho \geq 1$. In particular, by resorting to a potential function argument, we prove that, for each affine weighted congestion model CM, any better-response dynamics in ρ-CG(CM) converges to a ρ-uniform mixed equilibrium after a finite number of steps.

▶ **Theorem 2.** *For each affine weighted congestion model* CM *and $\rho \geq 1$, ρ-*CG(CM) *admits a potential function.*

Proof. Given an affine weighted congestion model CM and an integer $\rho \geq 1$, consider the function Φ_ρ defined on the set of ρ-uniform mixed profiles for ρ-CG(CM):

$$\Phi_\rho(\boldsymbol{\sigma}) := \sum_{e \in E} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2}{2\rho^2} + \frac{2\rho - 1}{2\rho^2} \sum_{i: e \in E(\sigma_i)} w_i^2 \right). \tag{3}$$

We prove that Φ_ρ is a weighted potential function for ρ-CG. Fix a ρ-uniform mixed profile $\boldsymbol{\sigma}$, a player $i \in \mathsf{N}$, and a ρ-uniform mixed strategy $\sigma'_i \in \Delta_i^\rho(\mathsf{CM})$. Let $\mathbb{I}_e = 1$ if $e \in \sigma'_i \setminus \sigma_i$, $\mathbb{I}_e = -1$ if $e \in \sigma_i \setminus \sigma'_i$, $\mathbb{I}_e = 0$ if $e \in \sigma'_i \cap \sigma_i$. We have

$$
\begin{aligned}
&\Phi_\rho(\boldsymbol{\sigma}_{-i}, \sigma'_i) - \Phi_\rho(\boldsymbol{\sigma}) \\
=&\sum_{e \in E} \alpha_e \left(\frac{(k_{\rho,e}(\boldsymbol{\sigma}) + \mathbb{I}_e w_i)^2}{2\rho^2} + \frac{2\rho - 1}{2\rho^2} \left(\sum_{j: e \in E(\sigma_j)} w_j^2 + \mathbb{I}_e w_i^2 \right) \right) \\
&- \sum_{e \in E} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2}{2\rho^2} + \frac{2\rho - 1}{2\rho^2} \sum_{j: e \in E(\sigma_j)} w_j^2 \right) \\
=& \sum_{e \in E(\sigma'_i) \setminus E(\sigma_i)} \alpha_e \left(\frac{(k_{\rho,e}(\boldsymbol{\sigma}) + w_i)^2 - k_{\rho,e}(\boldsymbol{\sigma})^2}{2\rho^2} + \frac{2\rho - 1}{2\rho^2} w_i^2 \right) \\
&- \sum_{e \in E(\sigma_i) \setminus E(\sigma'_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2 - (k_{\rho,e}(\boldsymbol{\sigma}) - w_i)^2}{2\rho^2} + \frac{2\rho - 1}{2\rho^2} w_i^2 \right) \\
=& \sum_{e \in E(\sigma'_i) \setminus E(\sigma_i)} \alpha_e w_i \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2} + \frac{1}{\rho} w_i \right) - \sum_{e \in E(\sigma_i) \setminus E(\sigma'_i)} \alpha_e w_i \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2} + \frac{\rho - 1}{\rho^2} w_i \right) \\
=& w_i (cost_i(\boldsymbol{\sigma}_{-i}, \sigma'_i) - cost_i(\boldsymbol{\sigma})), \qquad (4)
\end{aligned}
$$

where (4) comes from Lemma 1. Thus, Φ_ρ is a weighted potential function for ρ-CG(CM). ◀

3.1 Price of Anarchy

In this subsection, we derive exact bounds on the price of anarchy of ρ-uniform congestion games induced by affine weighted congestion models.

▶ **Theorem 3.** *Fix an affine weighted congestion model* CM*. For any $\rho \geq 1$, we have*

$$
\mathsf{PoA}_\rho(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \leq \begin{cases} \frac{\sqrt{4\rho+1}+2\rho+1}{2\rho} & \text{if } \rho < 5, \\ \frac{4\rho^2}{3\rho^2 - 2\rho - 1} & \text{if } \rho \geq 5. \end{cases}
$$

Proof. Given an integer $\rho \geq 1$, let CM be an arbitrary affine weighted congestion model. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ be a ρ-uniform mixed equilibrium and a ρ-uniform social optimum for ρ-CG(CM), respectively. By exploiting the primal-dual technique we get the following linear program:

$$
\begin{aligned}
\max \quad & \mathsf{SUM}(\boldsymbol{\sigma}) = \sum_{i \in \mathsf{N}} w_i cost_i(\boldsymbol{\sigma}) \\
\text{s.t.} \quad & \sum_{i \in \mathsf{N}} w_i cost_i(\boldsymbol{\sigma}) \leq \sum_{i \in \mathsf{N}} w_i cost_i(\boldsymbol{\sigma}_{-i}, \sigma_i^*) && (5) \\
& \mathsf{SUM}(\boldsymbol{\sigma}^*) = \sum_{i \in \mathsf{N}} w_i cost_i(\boldsymbol{\sigma}^*) = 1 && (6) \\
& \alpha_e \geq 0, \quad \forall e \in E
\end{aligned}
$$

where:

- (5) has been obtained by multiplying each inequality $cost_i(\boldsymbol{\sigma}) \leq cost_i(\boldsymbol{\sigma}_{-i}, \sigma_i^*)$ by w_i, and then summing them up for each $i \in \mathsf{N}$;
- the linear coefficients α_e's are the variables of the linear program, and the other quantities are fixed parameters;

- (6) normalizes the optimal social function, so that the maximum value of the objective function (i.e. the social function of the ρ-uniform mixed equilibrium) is an upper bound on the price of anarchy.

By using Lemma 1 in the previous linear program, we get the following relaxation (relaxation comes from inequality (2), that may not be tight):

$$\text{LP:}\quad \max \quad \sum_{e\in E}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)\sum_{i:e\in E(\sigma_i)}w_i^2\right) \tag{7}$$

$$\text{s.t.}\quad \sum_{e\in E}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)\sum_{i:e\in E(\sigma_i)}w_i^2\right)\leq$$

$$\leq\sum_{e\in E}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma}^*)k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2}+\sum_{i:e\in E(\sigma_i^*)}\frac{w_i^2}{\rho}\right) \tag{8}$$

$$\sum_{e\in E}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma}^*)^2}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)\sum_{i:e\in E(\sigma_i^*)}w_i^2\right)=1 \tag{9}$$

$$\alpha_e\geq 0,\quad \forall e\in E$$

where (7) comes from (1), as $\sum_{i\in \mathsf{N}} w_i cost_i(\boldsymbol{\sigma}) = \sum_{i\in\mathsf{N}} w_i \sum_{e\in E(\sigma_i)}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)w_i\right) = \sum_{e\in E}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)\sum_{i:e\in E(\sigma_i)}w_i^2\right)$, and (8) comes from (2), as $\sum_{i\in\mathsf{N}} w_i cost_i(\boldsymbol{\sigma}_{-i},\sigma_i^*) \leq \sum_{i\in\mathsf{N}} w_i\sum_{e\in E(\sigma_i^*)}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2}+\frac{w_i}{\rho}\right) = \sum_{e\in E}\alpha_e\left(\frac{k_{\rho,e}(\boldsymbol{\sigma}^*)k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2}+\sum_{i:e\in E(\sigma_i^*)}\frac{w_i^2}{\rho}\right)$. By taking the dual of LP, where we associate the dual variable x to the primal constraint (8) and the dual variable γ to the primal constraint (9), we get

$$\text{DLP}:\quad \min\quad \gamma$$

$$\text{s.t.}\quad \gamma\left(\frac{k_{\rho,e}(\boldsymbol{\sigma}^*)^2}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)\sum_{i:e\in E(\sigma_i^*)}w_i^2\right)\geq$$

$$\geq -(x-1)\left(\frac{k_{\rho,e}(\boldsymbol{\sigma})^2}{\rho^2}+\left(\frac{\rho-1}{\rho^2}\right)\sum_{i:e\in E(\sigma_i)}w_i^2\right)+$$

$$+x\left(\frac{k_{\rho,e}(\boldsymbol{\sigma}^*)k_{\rho,e}(\boldsymbol{\sigma})}{\rho^2}+\sum_{i:e\in E(\sigma_i^*)}\frac{w_i^2}{\rho}\right),\quad \forall e\in E \tag{10}$$

$$x\geq 0$$

By choosing $x > 1$, we have that, if $k_{\rho,e}(\boldsymbol{\sigma}^*) = 0$, (10) is always satisfied. Thus, assume that $k_{\rho,e}(\boldsymbol{\sigma}^*) > 0$. Let us now manipulate (10) as follows: we divide both sides by $k_{\rho,e}(\boldsymbol{\sigma}^*)^2$, so that we can rewrite it as a function of a variable $t := k_{\rho,e}(\boldsymbol{\sigma})/k_{\rho,e}(\boldsymbol{\sigma}^*)$ and of some new player's weights $u_i = w_i/k_{\rho,e}(\boldsymbol{\sigma}^*)$. Furthermore, by setting $\sum_{i:e\in E(\sigma_i)} w_i^2 = 0$ we have stronger constraints. By observing that $\sum_{i:e\in E(\sigma_i^*)} w_i = k_{\rho,e}(\boldsymbol{\sigma}^*)$, and then $\sum_{i:e\in E(\sigma_i^*)} u_i = 1$, we obtain that the following value $\gamma(x)$ is a feasible solution of DLP for any $x > 1$:[3]

$$\gamma(x) = \sup_{\{n\in\mathbb{N},t\geq 0,u_i\geq 0,\sum_{i=1}^n u_i=1\}} \frac{\frac{t^2}{\rho^2} + x\left(-\frac{t^2}{\rho^2} + \frac{t}{\rho^2} + \frac{1}{\rho}\sum_{i=1}^n u_i^2\right)}{\frac{1}{\rho^2} + \frac{\rho-1}{\rho^2}\sum_{i=1}^n u_i^2}.$$

Since $\sum_{i=1}^n u_i = 1$ and $u_i \geq 0$ for each $i \in [n]$, one can replace $\sum_{i=1}^n u_i^2$ with a variable $u \in [0,1]$, so that we can set $\gamma(x) = \sup_{t\geq 0, u\in[0,1]} \gamma(x,t,u)$, where

$$\gamma(x,t,u) = \frac{\frac{t^2}{\rho^2} + x\left(-\frac{t^2}{\rho^2} + \frac{t}{\rho^2} + \frac{u}{\rho}\right)}{\frac{1}{\rho^2} + \frac{\rho-1}{\rho^2}u}.$$

We have that the monotonicity of $\gamma(x,t,u)$ with respect to u does not depend on u, thus $\gamma(x,t,u)$ is maximized either by $u = 0$ or by $u = 1$. So, we get $\gamma(x) = \sup_{t\geq 0, u\in\{0,1\}} \gamma(x,t,u)$. Note that $t = \frac{x}{2x-2}$ is the unique maximum point of $\gamma(x,t,u)$ for $u \in \{0,1\}$. Thus, we can conclude

$$\gamma(x) = \max\left\{\gamma\left(x, \frac{x}{2x-2}, 0\right), \gamma\left(x, \frac{x}{2x-2}, 1\right)\right\}.$$

Now, if $\rho < 5$, set $x = 1 + \frac{1}{\sqrt{4\rho+1}}$, otherwise set $x = \frac{4\rho}{3\rho+1}$. If $\rho < 5$, we get

$$\gamma\left(1 + \frac{1}{\sqrt{4\rho+1}}\right) = \frac{\sqrt{4\rho+1} + 2\rho + 1}{2\rho} \geq \mathsf{PoA}_\rho(\rho\text{-}\mathsf{CG}),$$

otherwise, for $\rho \geq 5$, we get

$$\gamma\left(\frac{4\rho}{3\rho+1}\right) = \frac{4\rho^2}{3\rho^2 - 2\rho - 1} \geq \mathsf{PoA}_\rho(\rho\text{-}\mathsf{CG}),$$

thus showing the claim. ◀

We show that the derived upper bounds are tight, even when restricting to games induced by symmetric load balancing models.

▶ **Theorem 4.** *For any $\rho \geq 1$ and $\epsilon > 0$, there exists an affine weighted symmetric load balancing model* CM := CM(ρ, ϵ) *such that*

$$\mathsf{PoA}_\rho(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \geq \begin{cases} \frac{\sqrt{4\rho+1}+2\rho+1}{2\rho} - \epsilon & \text{if } \rho < 5, \\ \frac{4\rho^2}{3\rho^2-2\rho-1} - \epsilon & \text{if } \rho \geq 5. \end{cases}$$

4 Unweighted Games

In this section, we consider the case of ρ-uniform congestion games induced by unweighted congestion models. First, we show that uniform mixed equilibria are always guaranteed to exist for any class of latency functions.

Given an unweighted congestion model CM and an integer $\rho \geq 1$, let f be a function mapping CM and ρ to another congestion model $f(\mathsf{CM}, \rho)$, according to the following definition.

[3] To simplify the notation, we have written $\sum_{i=1}^n u_i$, instead of $\sum_{i:e\in E(\sigma_i^*)} u_i$.

▶ **Definition 5.** Given an unweighted congestion model

$$\mathsf{CM} = \left(\mathsf{N}, E, (\ell_e)_{e\in E}, \vec{1}^n, (\Sigma_i)_{i\in\mathsf{N}}\right),$$

define $\mathsf{CM}' = f(\mathsf{CM}, \rho) = \left(\mathsf{N}', E', (\ell'_e)_{e\in E}, \vec{1}^n, (\Sigma'_i)_{i\in\mathsf{N}}\right)$ as the unweighted congestion model such that $\mathsf{N}' = \mathsf{N}$, $E' = E$, $\Sigma'_i = \{E(\sigma_i) : \sigma_i$ is a ρ-uniform mixed strategy for player i in ρ-$\mathsf{CG}(\mathsf{CM})\}$ for each $i \in \mathsf{N}$, and

$$\ell'_e(x) := \frac{1}{\rho}\left(\sum_{j=0}^{x-1}\binom{x-1}{j}\left(\frac{1}{\rho}\right)^j\left(\frac{\rho-1}{\rho}\right)^{x-1-j}\ell_e(j+1)\right) \tag{11}$$

for each $e \in E'$. Moreover, given a latency function ℓ_e, let $\ell_e^{f(\rho)}$ denote the latency function defined in (11), and let $\ell_e^{-f(\rho)}$ denote the function $\overline{\ell}_e$ such that $\overline{\ell}_e^{f(\rho)} = \ell_e$.

For instance, if $\mathsf{CG}(\mathsf{CM})$ is a symmetric load balancing game, then $\mathsf{CG}(f(\mathsf{CM}, \rho))$ is a ρ-uniform matroid congestion game [15], i.e. the strategies of each player are arbitrary subsets of ρ resources.

We show that ρ-$\mathsf{CG}(\mathsf{CM})$ is equivalent to $\mathsf{CG}(f(\mathsf{CM}, \rho))$ for each $\rho \geq 1$. For a ρ-uniform mixed profile $\boldsymbol{\sigma}$ for ρ-$\mathsf{CG}(\mathsf{CM})$, define $\mathbf{s}'(\boldsymbol{\sigma})$ as the strategy profile for $\mathsf{CG}(f(\mathsf{CM}, \rho))$ such that $\mathbf{s}'(\boldsymbol{\sigma}) := (E(\sigma_1), E(\sigma_2), \ldots, E(\sigma_n))$.

▶ **Theorem 6.** *Given $\rho \geq 1$ and an unweighted congestion model CM, we have that, for each ρ-uniform mixed profile $\boldsymbol{\sigma}$ for ρ-$\mathsf{CG}(\mathsf{CM})$ and $i \in \mathsf{N}$, $cost_i^{\rho\text{-}\mathsf{CG}(\mathsf{CM})}(\boldsymbol{\sigma}) = cost_i^{\mathsf{CG}(f(\mathsf{CM},\rho))}(\mathbf{s}'(\boldsymbol{\sigma}))$.*

As a corollary, we obtain existence of uniform mixed equilibria for each uniform congestion games induced by unweighted congestion models, regardless of which are their latency functions. In particular, we extend Rosenthal's Theorem [30], by showing that, for each $\rho \geq 1$, any ρ-uniform unweighted congestion game admits an exact potential.

▶ **Corollary 7.** *For each $\rho \geq 1$ and unweighted congestion model CM, ρ-$\mathsf{CG}(\mathsf{CM})$ admits an exact potential.*

Proof. By Rosenthal's Theorem [30], $\mathsf{CG}(f(\mathsf{CM}, \rho))$ admits an exact potential function Φ. Because of Theorem 6, we have that $\Phi \circ \boldsymbol{s}'$ is an exact potential function for ρ-$\mathsf{CG}(\mathsf{CM})$. Indeed, given $i \in \mathsf{N}$, a strategy profile $\boldsymbol{\sigma}$ of ρ-$\mathsf{CG}(\mathsf{CM})$, and $\sigma'_i \in \Delta_i^{\rho}(\mathsf{CM})$, we get $cost_i^{\rho\text{-}\mathsf{CG}(\mathsf{CM})}(\boldsymbol{\sigma}) - cost_i^{\rho\text{-}\mathsf{CG}(\mathsf{CM})}(\boldsymbol{\sigma}_{-i}, \sigma'_i) = cost_i^{\mathsf{CG}(f(\mathsf{CM},\rho))}(\boldsymbol{s}'(\boldsymbol{\sigma})) - cost_i^{\mathsf{CG}(f(\mathsf{CM},\rho))}(\boldsymbol{s}'(\boldsymbol{\sigma}_{-i}, \sigma'_i)) = \Phi(\boldsymbol{s}'(\boldsymbol{\sigma})) - \Phi(\boldsymbol{s}'(\boldsymbol{\sigma}_{-i}, \sigma'_i)) = (\Phi \circ \boldsymbol{s}')(\boldsymbol{\sigma}) - (\Phi \circ \boldsymbol{s}')(\boldsymbol{\sigma}_{-i}, \sigma'_i)$. ◀

4.1 Price of Anarchy

In this subsection, we derive exact bounds on the price of anarchy of ρ-uniform congestion games induced by affine unweighted congestion models.

▶ **Theorem 8.** *Fix an affine unweighted congestion model CM. For any $\rho \geq 1$, we have*

$$\mathsf{PoA}_{\rho}(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \leq \begin{cases} \frac{5}{\rho+1} & \text{if } \rho \leq 2, \\ \frac{2\sqrt{7}-1}{3} & \text{if } \rho = 3, \\ \frac{4}{3} & \text{if } \rho \geq 4. \end{cases}$$

Proof. Given an integer $\rho \geq 1$, let CM be an arbitrary affine unweighted congestion model. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ be a ρ-uniform mixed equilibrium and a ρ-uniform social optimum for

ρ-CG(CM), respectively. By exploiting (1) and (2), we have that, for each $i \in \mathsf{N}$, the inequality $cost_i(\boldsymbol{\sigma}) \leq cost_i(\boldsymbol{\sigma}_{-i}, \sigma_i^*)$ becomes

$$\sum_{e \in E(\sigma_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma}) + \rho - 1}{\rho^2} \right) - \sum_{e \in E(\sigma_i^*)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma}) + \rho}{\rho^2} \right) \leq 0.$$

By also using (1) within $\mathsf{SUM}(\boldsymbol{\sigma})$ and $\mathsf{SUM}(\boldsymbol{\sigma}^*)$, we get the following linear program:

$$\begin{aligned} \mathsf{LP}: \quad \max \quad & \sum_{e \in E} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma})(k_{\rho,e}(\boldsymbol{\sigma}) + \rho - 1)}{\rho^2} \right) \\ \text{s.t.} \quad & \sum_{e \in E(\sigma_i)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma}) + \rho - 1}{\rho^2} \right) - \sum_{e \in E(\sigma_i^*)} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma}) + \rho}{\rho^2} \right) \leq 0, \quad \forall i \in \mathsf{N} && (12) \\ & \sum_{e \in E} \alpha_e \left(\frac{k_{\rho,e}(\boldsymbol{\sigma}^*)(k_{\rho,e}(\boldsymbol{\sigma}^*) + \rho - 1)}{\rho^2} \right) = 1 && (13) \\ & \alpha_e \geq 0, \quad \forall e \in E \end{aligned}$$

By taking the dual of LP, where we associate the dual variable x_i to the ith primal constraint in (12) and the dual variable γ to the primal constraint (13), we get:

$$\begin{aligned} \mathsf{DLP}: \quad \min \quad & \gamma \\ \text{s.t.} \quad & \sum_{i: e \in E(\sigma_i)} \left(x_i \frac{k_{\rho,e}(\boldsymbol{\sigma}) + \rho - 1}{\rho^2} \right) - \sum_{i: e \in E(\sigma_i^*)} \left(x_i \frac{k_{\rho,e}(\boldsymbol{\sigma}) + \rho}{\rho^2} \right) \\ & + \gamma \frac{k_{\rho,e}(\boldsymbol{\sigma}^*)(k_{\rho,e}(\boldsymbol{\sigma}^*) + \rho - 1)}{\rho^2} \\ & \geq \frac{k_{\rho,e}(\boldsymbol{\sigma})(k_{\rho,e}(\boldsymbol{\sigma}) + \rho - 1)}{\rho^2}, \quad \forall e \in E && (14) \\ & x_i \geq 0, \quad \forall i \in \mathsf{N} \end{aligned}$$

By using $x_i = x$ for each $i \in \mathsf{N}$, $k := k_{\rho,e}(\boldsymbol{\sigma})$ and $o := k_{\rho,e}(\boldsymbol{\sigma}^*)$ in (14), and multiplying both sides by ρ^2, we obtain the following relaxed dual constraint:

$$xk(k + \rho - 1) - xo(k + \rho) + \gamma o(o + \rho - 1) \geq k(k + \rho - 1). \tag{15}$$

To complete the proof, we are left to provide, for each $\rho \geq 1$, a suitable value $x \geq 0$ satisfying inequality (15) where γ is set to be equal to the claimed upper bound on the ρ-uniform price of anarchy. We now proceed by case analysis.

For $\rho \leq 2$, for which we have $\gamma = \frac{5}{\rho+1}$, set $x = \frac{\rho+2}{\rho+1}$. By substituting these values in (15), we get the inequality $k^2 - k\left(o(\rho + 2) + \rho + 1\right) + o(5o - \rho^2 + 3\rho - 5) \geq 0$ which can be easily shown to be satisfied for any pair of non-negative integers k, o when $\rho = 1, 2$. In fact, the discriminant of the associated equality is negative for each integer $o \geq 2$, while the cases of $o \in \{0, 1\}$ can be checked by inspection.

For $\rho = 3$, for which we have $\gamma = \frac{2\sqrt{7}-1}{3}$, set $x = 2\sqrt{7} - 4$. By substituting these values in (15), we get the inequality

$$(6\sqrt{7} - 15)k^2 - 6k\left(o(\sqrt{7} - 2) - 2\sqrt{7} + 5\right) + o\left(o(2\sqrt{7} - 1) - 14\sqrt{7} + 34\right) \geq 0,$$

which can be easily shown to be satisfied for any pair of non-negative integers k, o. In fact, the discriminant of the associated equality is negative for each integer $o \geq 2$, while the cases of $o \in \{0, 1\}$ can be checked by inspection.

For $\rho \geq 4$, for which we have $\gamma = \frac{4}{3}$, set $x = \frac{4}{3}$. By substituting these values in (15), we get the inequality $k^2 - k(4o - \rho + 1) + 4o(o - 1) \geq 0$ whose left-hand side is increasing in ρ. Hence, we only need to prove that it gets satisfied for the case of $\rho = 4$, by which we get the inequality $k^2 - k(4o - 3) + 4o(o - 1) \geq 0$ which can be easily shown to be satisfied for any pair of non-negative integers k, o. Again, the discriminant of the associated equality is negative for each integer $o \geq 2$, while the cases of $o \in \{0, 1\}$ can be checked by inspection. ◀

We show matching lower bounds for each $\rho \leq 3$. For $\rho \geq 4$, we show in the next subsection a matching lower bound holding even for the price of stability.

▶ **Theorem 9.** *For any $\rho \leq 3$ and $\epsilon > 0$, there exists an affine unweighted load balancing model* $\mathsf{CM} := \mathsf{CM}(\rho, \epsilon)$ *such that*

$$\mathsf{PoA}_\rho(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \geq \begin{cases} \frac{5}{\rho+1} - \epsilon & \text{if } \rho \leq 2, \\ \frac{2\sqrt{7}-1}{3} - \epsilon & \text{if } \rho = 3. \end{cases}$$

4.2 Price of Stability

In this subsection, we exhibit exact bounds on the price of stability of ρ-uniform congestion games induced by affine unweighted congestion models.

▶ **Theorem 10.** *Fix an affine unweighted congestion model* CM*. For any $\rho \geq 1$, we have*

$$\mathsf{PoS}_\rho(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \leq \begin{cases} 1 + \frac{1}{\sqrt{2\rho+1}} & \text{if } \rho \leq 3, \\ \frac{4}{3} & \text{if } \rho \geq 4. \end{cases}$$

We also have matching lower bounds. We first consider the case of $\rho \leq 3$.

▶ **Theorem 11.** *For each $\rho \leq 3$ and $\epsilon > 0$, there exists an affine unweighted congestion model* $\mathsf{CM} := \mathsf{CM}(\rho, \epsilon)$ *such that* $\mathsf{PoS}_\rho(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \geq 1 + \frac{1}{\sqrt{2\rho+1}} - \epsilon$.

For $\rho \geq 4$, the upper bounds are tight even when restricting to games induced by symmetric load balancing models.

▶ **Theorem 12.** *For each $\rho \geq 1$ and $\epsilon > 0$, there exists an affine unweighted symmetric load balancing model* $\mathsf{CM} := \mathsf{CM}(\rho, \epsilon)$ *such that* $\mathsf{PoS}_\rho(\rho\text{-}\mathsf{CG}(\mathsf{CM})) \geq \frac{4}{3} - \epsilon$.

5 Open Problems

In this paper, motivated by possible applications in fault-tolerant routing, we have introduced the notion of uniform mixed equilibria, and we have applied it to the well-studied class of (network) congestion games with affine latency functions, by providing existential results of these equilibria and by deriving tight bounds to the prices of anarchy and stability.

The main left open problem is to consider the more general definition of $\boldsymbol{\rho}$-uniform mixed equilibria, where players can use uniform mixed strategies of different support size. Another important question is the determination of lower bounds for the price of stability of ρ-unform mixed equilibria, in the setting of weighted congestion games. However, this question is open even for the price of stability of pure Nash equilibria (i.e., $\rho = 1$), for which only an upper bound equal to 2 is known, as a direct consequence of the potential function given in [20]. Following the approach used in [12, 10, 19, 9] for $\rho = 1$, it could be interesting investigating resource taxation or other strategies to improve the performance of ρ-unform mixed equilibria. Another interesting research direction is that of extending the results to other latency functions, e.g., polynomial functions, or decreasing functions as the ones inducing the Shapley cost sharing game [2, 24, 8, 7, 18].

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