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A variational model for innovation diffusion under fuzzy uncertainty

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ABSTRACT

We propose a variational approach to study the market penetration of new technologies under conditions of spatial heterogeneity in the economic factors influencing the process and imprecise knowledge about their intensities. Differently from other methodologies that describe the adoption process in terms of partial differential equations, we formulate the model as a minimization problem of an appropriate variational functional with fuzzy coefficients. This approach permits to consider in the analysis both the attractive and diffusive forces that drive the process and the subjective opinions of policy makers about the actual influence exerted by these determinants on the adoption decision. Interestingly, our results show that different degrees of uncertainty lead to significantly different predictions about the diffusion process and, therefore, our methodology could be applied to support strategic decisions concerning innovation diffusion plans. An application to the digital transition in agriculture is also provided to study the effectiveness of government policies.

1. Introduction

Forecasting the diffusion of technological innovations is of strategic importance for both market players and policy makers, such as technology-based companies, government agencies and entrepreneurs. Understanding and analysing the spread of a new technology introduced in a reference market is also an important research topic in several disciplines, such as marketing, strategy, organizational behaviour and economics [45,77,87]. Adoption rates depend on a host of factors including characteristics of the technology and characteristics of the adoption environment [43,64,26]. Cost considerations are surely important in explaining adoption patterns, although other factors may play a role as well. Relative wage rates are also important in explaining cross-country attractiveness and diffusion, as many new technologies are labour-saving and capital-using. Additionally, there may be other barriers as well regulations or tariffs that are imposed to protect older technologies [16,74].

Diffusion is a spatial as well as a temporal process [33,47,80]. New technologies spread among many different users as well as across different geographic regions and, therefore, it is crucial to study the diffusion of technological improvements among producers within a country and across international borders. Diffusion tends to be less pervasive in regions where there is a lack of adequate

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institutions and infrastructures in adopters, or markets that may not support a technology [43], while it tends to be more effective in regions closest to the regions of origin of the innovation [62,48].

Different models have been proposed in literature to capture the diffusion trend in the form of mathematical equations [51,84]. They usually refer to different stages of the adoption process classified as innovators/early adopters, early and late majority, and laggards according to the time of adoption, since the technology is introduced to the market [61,62]. However, most classical diffusion models do not consider the empirical observations regarding the spatial dimensions of technology diffusion. In particular, they do not distinguish among core, rim, and periphery regions, and do not represent important feedbacks such as knowledge or technology spillover effects where diffusion in one region facilitates adoption in other regions. Furthermore, they do not consider that the growth in the number of adoptions is also influenced by the behaviour observed in neighbouring regions [48]. In a forecast context, a diffusion agency may be interested not only in estimating "aggregate" adoptions of an innovation, but also in assessing how the innovation is penetrating in different geographical regions, in order to determine appropriate marketing strategies. While it is possible to estimate adoptions across multiple regions by developing independent diffusion models for each region, such an approach is inefficient as it ignores the richness of regional interactions. Therefore, some mathematical models have been proposed to consider spatial effects in technology diffusion [47,18].

Technological change is also characterized by a high degree of uncertainty which arises from several sources [65,62,36]. The traditional deterministic approach cannot be applied when the costs and benefits of a new technology are not clearly quantifiable [28]. Moreover, good data on diffusion are not readily available and, for many innovations, there are none at all [74,51]. In these cases information can be provided by expert opinion [64].

In this study, we focus on industrial innovation diffusion and present a variational approach to analyse the penetration process of a new technology among firms of a reference market under conditions of fuzzy uncertainty. We identify as main determinants of the adoption process the following economic factors: diffusion, innovation and resistance due to costs of substitution. Innovation and resistance factors can be viewed as forces that attract the level of adoption towards their respective targets, i.e., the full adoption of new technology and the pre-existent owned technology. Diffusion phenomenon reflects the spatial interaction effect, i.e., the influence of neighbouring firms on the adoption decision. We design the adoption process as a system that evolves in order to minimize an appropriate variational functional, with fuzzy coefficients, that takes into account both the combined action of the attractive and diffusive forces and the imprecise knowledge about their impact on the adoption decision. This approach was inspired us by the principle of minimum energy applied to the modern theory of phase transitions in fluid dynamics [32]. We interpret the market behaviour as a system that tends to reach the equilibrium by minimizing its "total energy", due to the combined action of internal (diffusion) and external (innovation, cost) forces, expressed in the form of a variational functional. Spatial heterogeneity and uncertainty are also considered by letting the coefficients of the variational functional be space-dependent and described by fuzzy numbers. To our knowledge, this approach has not previously appeared in the innovation diffusion literature.

Differently from other approaches that describe the adoption process in terms of partial differential equations, we formulate the model as an optimization problem where the objective function is the variational functional. This permits to aggregate, in a meaningful and useful manner, the various sources of information underlying the specific market scenario considered and the imprecise knowledge about the effect of the economic factors that influence the adoption process. Furthermore, the proposed flexible methodology allows policy makers to carry out scenario analyses for different degrees of uncertainty on the basis of their own subjective opinions. We will discuss this and other benefits of our methodology in Section 6.2.

We underline that we focus on the spatial aspect of the innovation diffusion under "stationary" conditions, without explicitly consider the temporal evolution of the process. The developed methodology can be used as a pre-screening tool to predict the potential market share and geographical diffusion of new technologies or even in situations where diffusion occurs in a short time or the model parameters do not vary significantly over time. The stationary solution determined in this study can also represent the starting point for a dynamic analysis in which the time variable is explicitly included in the model equations.

In agreement with the literature, our results show that spatial dimension plays a central role in understanding technology diffusion and geographic differences in productivity and/or cost can have important implications on the adoption decision. Furthermore, we find that not only the spatial dependence of the economic factors influences the adoption process, but also the ambiguity about their estimation has a significant impact on the diffusion analysis. Interestingly, the subjective opinions of the decision maker lead to significantly different predictions about the adoption process and, therefore, our methodology could be applied to support strategic planning for technological innovation.

The paper is organized as follows. Section 2 provides the basics of fuzzy numbers. In Section 3 we present our fuzzy variational approach to innovation diffusion. In Section 4 we provide a discrete solution for the addressed problem and, in Section 5, we perform and discuss numerical simulations. In Section 6 we briefly review the relevant innovation diffusion models proposed in the literature and discuss the main ideas of our proposal. In Section 7 we apply the proposed methodology to the digital transition in agriculture, in order to investigate the effectiveness of government policies to promote the adoption of smart technologies. Finally, Section 8 concludes.

2. Basic concepts on fuzzy numbers

A fuzzy subset $\tilde{a}$ of the non-empty universe set U is defined by a membership function $\mu_{\tilde{a}} : U \to [0,1]$ where, for each $z \in U$, the value $\mu_{\tilde{a}}(x)$ is the membership degree of element z to fuzzy set $\tilde{a}$. The support and the core of $\tilde{a}$ are defined, respectively, as the crisp sets $supp(\tilde{a}) = \{z \in U : \mu_{\tilde{a}}(z) > 0\}$ and $core(\tilde{a}) = \{z \in U : \mu_{\tilde{a}}(z) = 1\}$. A fuzzy subset $\tilde{a}$ is called normal if its core is nonempty, i.e. if there exists an element $z \in U$ such that $\mu_{\tilde{a}}(z) = 1$. The α-level set of $\tilde{a}$, with $0 \le \alpha \le 1$, is defined as the crisp set

$\tilde{a}(\alpha) = \{z \in U : \mu_{\tilde{a}}(z) \geq \alpha\}$ if $0 < \alpha \leq 1$ and as the closure of the support of $\tilde{a}$ if $\alpha = 0$. A fuzzy subset $\tilde{a}$ is said to be a fuzzy number if it is a fuzzy subset of the real line $\mathbb{R}$ with a normal, convex and upper-semicontinuous membership function of bounded support. Each α-level set of a fuzzy number is a closed interval $\tilde{a}(\alpha) = [a^L(\alpha), a^R(\alpha)]$, for $0 \leq \alpha \leq 1$, where $a^L(\alpha) = \inf \tilde{a}(\alpha)$ and $a^R(\alpha) = \sup \tilde{a}(\alpha)$. A triangular fuzzy number is a fuzzy number $\tilde{a} = \langle a^1, a^2, a^3 \rangle$, with $a^1 < a^2 < a^3$, defined by the α-level sets

$$\tilde{a}(\alpha) = [a^L(\alpha), a^R(\alpha)] = [a^1 + \alpha(a^2 - a^1), a^3 - \alpha(a^3 - a^2)]\,. \tag{1}$$

We consider the following ordering [29,83,9] for comparing two fuzzy numbers $\tilde{a}$ and $\tilde{b}$:

$$\tilde{a} \leq \tilde{b} \iff \int_0^1 \left(a^L(\alpha) + a^R(\alpha)\right) \alpha\, d\alpha \leq \int_0^1 \left(b^L(\alpha) + b^R(\alpha)\right) \alpha\, d\alpha\,. \tag{2}$$

In the special case when $\tilde{a} = \langle a^1, a^2, a^3 \rangle$ and $\tilde{b} = \langle b^1, b^2, b^3 \rangle$ are two fuzzy numbers, we have

$$\tilde{a} \leq \tilde{b} \iff a^2 + \frac{a^1 - 2a^2 + a^3}{6} \leq b^2 + \frac{b^1 - 2b^2 + b^3}{6}\,.$$

If $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers with α-level sets $\tilde{a}(\alpha) = [a^L(\alpha), a^R(\alpha)]$ and $\tilde{b}(\alpha) = [b^L(\alpha), b^R(\alpha)]$, respectively, then the sum $\tilde{a} + \tilde{b}$ and the difference $\tilde{a} - \tilde{b}$ are defined as the fuzzy numbers with α-level sets, respectively,

$$(\tilde{a} + \tilde{b})(\alpha) = [a^L(\alpha) + b^L(\alpha), a^R(\alpha) + b^R(\alpha)]$$

and

$$(\tilde{a} - \tilde{b})(\alpha) = [a^L(\alpha) - b^R(\alpha), a^R(\alpha) - b^L(\alpha)]\,.$$

The multiplication by a real number λ is defined as the fuzzy number $\lambda\tilde{a}$, with α-level sets given by

$$(\lambda\, \tilde{a})(\alpha) = [\lambda\, a^L(\alpha), \lambda\, a^R(\alpha)]$$

if $\lambda \geq 0$, and

$$(\lambda\, \tilde{a})(\alpha) = [\lambda\, a^R(\alpha), \lambda\, a^L(\alpha)]$$

if $\lambda < 0$. The product of two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ is the fuzzy number $\tilde{a} \odot \tilde{b}$ with α-level sets given by

$$\begin{aligned}(\tilde{a} \odot \tilde{b})(\alpha) = [\,&\min(a^L(\alpha) \cdot b^L(\alpha), a^L(\alpha) \cdot b^R(\alpha), a^R(\alpha) \cdot b^L(\alpha), a^R(\alpha) \cdot b^R(\alpha)),\\ &\max(a^L(\alpha) \cdot b^L(\alpha), a^L(\alpha) \cdot b^R(\alpha), a^R(\alpha) \cdot b^L(\alpha), a^R(\alpha) \cdot b^R(\alpha))]\,.\end{aligned}$$

In the case when $\tilde{a}$ and $\tilde{b}$ are non-negative fuzzy numbers, i.e. $a^L(\alpha), b^L(\alpha) \geq 0$ for all α, we have

$$(\tilde{a} \odot \tilde{b})(\alpha) = [a^L(\alpha) \cdot b^L(\alpha), a^R(\alpha) \cdot b^R(\alpha)]\,.$$

For further details on arithmetic operations involving fuzzy numbers we refer to [38,23,73].

3. The variational model for new technology adoption under fuzzy uncertainty

We study the market penetration process of a new technology introduced in a reference market, focusing on the factors that drive the diffusion system and the uncertainty about their estimation. We consider as main determinants of the adoption process the following factors: diffusion, innovation and cost of substitution. The diffusion factor takes into account the network's influence on the firms' adoption strategies, i.e. the need for market firms to operate at a technology level as close as possible to that of neighbouring firms. The innovation factor refers to the strength with which the new technology imposes itself on the market; it is an aggregate combination of several dimensions, such as the advantages that the adoption of new technologies brings in terms of profits, the effectiveness of advertising policies, etc. Firms' resistance to modifying pre-existing technologies derives both from external factors, such as the costs associated with the acquisition of new technologies, and from internal factors, such as, for example, the level of firm know-how.

We assume that the market penetration of the new technology occurs in order to minimize an appropriate variational functional that takes into account both the combined action of the attractive and diffusive forces that guide the adoption process and the imprecise knowledge about their intensities. The coefficients of the functional are modelled as fuzzy variables in order to deal with the uncertain impact that the factors considered will have on the adoption process.

The main purpose of this research is to determine, at equilibrium, the spatial distribution of the new technology. Consequently, we will not focus on the time evolution of the system, but rather on the stationary solution of the diffusion problem. Moreover, following a decision-making oriented approach, we incorporate in the model the subjective opinions of decision maker by introducing suitable parameters that allow to consider different degrees of uncertainty and pessimistic/optimistic behaviours.

For convenience, in Table 1 we list symbols used for parameters or variables employed in the model.

We consider a reference market represented by a bounded open set $\Omega \subset \mathbb{R}^2$. The level of adoption of the new technology is described by a "density" function $u : \Omega \to [0, 1]$ that associates, to each spatial location $(x, y) \in \Omega$, the proportion $u(x, y) \in [0, 1]$

Table 1
List of main symbols.

Symbol	Description
Ω	Reference geographical area
$\tilde{\mathcal{F}}(u)$	Fuzzy functional defined in (3) and (4)
$u(x,y)$	Adoption level of the new technology at spatial location (x,y)
$u_0(x,y)$	Initial distribution, i.e., the adoption level observed at the initial time
$\tilde{d}(x,y)$	Fuzzy diffusion coefficient at spatial location (x,y)
$\tilde{a}(x,y)$	Fuzzy innovation investment at spatial location (x,y)
$\tilde{c}(x,y)$	Fuzzy cost (also called fuzzy resistance) coefficient at (x,y)
n	Degree of fuzzy uncertainty (the lower n, the higher the uncertainty)
λ	Pessimistic/optimistic parameter
$\varphi_n(\alpha)$	Parametric family of weighting functions $\varphi_n(\alpha)=(n+1)\,\alpha^n$
$\mathcal{F}_{\lambda,n}(u)$	Evaluation of the fuzzy functional $\tilde{\mathcal{F}}(u)$: $\mathcal{F}_{\lambda,n}(u)=E_{\lambda,n}\left[\tilde{\mathcal{F}}(u)\right]$
$u^{(p)}(i,j)$	Iterative procedure, defined in (18) and (23), for minimum problem (8)

of the firms operating at location (x,y) which has already adopted the new technology. Therefore, the function u represents the spatial distribution of the market share captured by the innovation. The level $u=0$ stands for the non adoption and $u=1$ is the total adoption level. The objective is to determine, on the basis of the observed distribution $u_0:\Omega\to[0,1]$ at a reference time t_0, the spatial distribution of the new technology at equilibrium. The initial observation u_0 can represent the deployment of the new technology after its launch, for example at the end of the first phase of an R&D program. If the new technology has not yet been introduced on the market, we will have $u_0=0$. In this setting, the market penetration process of the new technology occurs in such a way as to minimize the following three components: the variation of adoption level in the neighbour (diffusion), the distance from 1 (innovation) and the distance from u_0 (cost of substitution). This process can be modelled by using a suitable variational functional in which each of its three terms represents one of the effects considered. The imprecise intensities of these effects are described by fuzzy coefficients.

We propose to determine the equilibrium distribution $u^*:\ \Omega\to[0,1]$ as the minimum of the following variational functional with fuzzy coefficients:

$$\tilde{\mathcal{F}}(u)=\ \tilde{d}\int_{\Omega}|\nabla u(x,y)|^2\,dx\,dy+\tilde{a}\int_{\Omega}(1-u(x,y))^2\,dx\,dy \\ +\tilde{c}\int_{\Omega}\left(u(x,y)-u_0(x,y)\right)^2\,dx\,dy \tag{3}$$

where the gradient operator ∇ is defined by $\nabla u(x,y)=\left(u_x(x,y),u_y(x,y)\right)$, with $u_x(x,y)=\dfrac{\partial u}{\partial x}(x,y)$, $u_y(x,y)=\dfrac{\partial u}{\partial y}(x,y)$ and $|\nabla u(x,y)|^2=\left(u_x(x,y)\right)^2+\left(u_y(x,y)\right)^2$ is the square of the gradient modulus of $u(x,y)$. For convenience, we write $|\nabla u(x,y)|^2=u_x^2(x,y)+u_y^2(x,y)$. The fuzzy coefficients $\tilde{d},\tilde{a},\tilde{c}$ are non-negative fuzzy numbers.

The fuzzy functional $\tilde{\mathcal{F}}(u)$ defined in (3) is expressed as the sum of three terms. The first term, i.e. the diffusion term, describes the diffusion of the innovation in the space depending on the interaction with other adopters in one's neighbourhood or in one's network. The non-negative fuzzy number $\tilde{d}$ expresses the extent to which the adoption rate of innovation by firms is influenced by their dependence on the network. It can be interpreted as a measure of the mean information field or, in other words, the spatial scale of the social interaction network of firms [37]. The second term of the functional concerns the capacity of the new technology to penetrate the market; it forces the distribution to be attracted by the value $u=1$, that represents the total adoption of the new technology. The intensity of this attractive effect is described by the non-negative fuzzy number $\tilde{a}$. This coefficient takes also into account the susceptibility of firms to marketing efforts (advertising policies). We observe that $1-u(x,y)$ represents the proportion of firms, operating at location (x,y), that have not adopted the new technology. The third term of the functional reflects the factors that hinder the adoption of new technologies and imposes a penalty for deviation of u from u_0. The non-negative fuzzy number $\tilde{c}$ describes resistance to replacing old technology due to the cost of adopting the new one.

To consider heterogeneity in the spatial structure of the reference market [33,37], for example different costs in regions (countries) or different know-how of firms, we introduce spatial dependence by local factors in the fuzzy parameters $\tilde{d}$, $\tilde{a}$ and $\tilde{c}$. We therefore generalize the functional (3) by considering the fuzzy functional

$$\tilde{\mathcal{F}}(u)=\int_{\Omega}\tilde{d}(x,y)\cdot|\nabla u(x,y)|^2\,dx\,dy+\int_{\Omega}\tilde{a}(x,y)\cdot(1-u(x,y))^2\,dx\,dy \\ +\int_{\Omega}\tilde{c}(x,y)\cdot\left(u(x,y)-u_0(x,y)\right)^2\,dx\,dy, \tag{4}$$

where, for each spatial location $(x,y)\in\Omega$, the diffusion coefficient $\tilde{d}(x,y)$, the innovation coefficient $\tilde{a}(x,y)$ and the cost coefficient $\tilde{c}(x,y)$ are non-negative fuzzy numbers. We observe that functional defined in (3) can be viewed as a special case of the functional (4) when fuzzy coefficients $\tilde{d}(x,y)$, $\tilde{a}(x,y)$, $\tilde{c}(x,y)$ do not depend on spatial dimension.

For our analysis, it is convenient to express the fuzzy functional $\tilde{\mathcal{F}}(u)$ defined in (3) and (4) in terms of its α-level sets $[\tilde{\mathcal{F}}(u)](\alpha)=[(\mathcal{F}(u))^L(\alpha),(\mathcal{F}(u))^R(\alpha)]$ that, for simplicity of notation, will be denoted as

$$\tilde{\mathcal{F}}(u,\alpha)=\left[\mathcal{F}^L(u,\alpha),\mathcal{F}^R(u,\alpha)\right].$$

If we denote the α-level sets of fuzzy numbers $\tilde{d}$, $\tilde{a}$, $\tilde{c}$, by $\tilde{d}(\alpha)=[d^L(\alpha),d^R(\alpha)]$, $\tilde{a}(\alpha)=\left[a^L(\alpha),a^R(\alpha)\right]$ and $\tilde{c}(\alpha)=[c^L(\alpha),c^R(\alpha)]$, respectively, then the α-level sets of the functional $\tilde{\mathcal{F}}(u)$ defined in (3) can be expressed as

$$\begin{aligned}\mathcal{F}^L(u,\alpha)&=d^L(\alpha)\int_\Omega|\nabla u(x,y)|^2\,dx\,dy+a^L(\alpha)\int_\Omega(1-u(x,y))^2\,dx\,dy\\&\quad+c^L(\alpha)\int_\Omega\left(u(x,y)-u_0(x,y)\right)^2\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^R(u,\alpha)&=d^R(\alpha)\int_\Omega|\nabla u(x,y)|^2\,dx\,dy+a^R(\alpha)\int_\Omega(1-u(x,y))^2\,dx\,dy\\&\quad+c^R(\alpha)\int_\Omega\left(u(x,y)-u_0(x,y)\right)^2\,dx\,dy.\end{aligned}$$

Furthermore, by denoting the α-level sets of $\tilde{d}(x,y)$, $\tilde{a}(x,y)$, $\tilde{c}(x,y)$ as $\tilde{d}(x,y,\alpha)=[d^L(x,y,\alpha),d^R(x,y,\alpha)]$, $\tilde{a}(x,y,\alpha)=\left[a^L(x,y,\alpha),a^R(x,y,\alpha)\right]$ and $\tilde{c}(x,y,\alpha)=[c^L(x,y,\alpha),c^R(x,y,\alpha)]$, respectively, the α-level sets of the functional $\tilde{\mathcal{F}}(u)$ defined in (4) are given by

$$\begin{aligned}\mathcal{F}^L(u,\alpha)&=\int_\Omega d^L(x,y,\alpha)\cdot|\nabla u(x,y)|^2\,dx\,dy+\int_\Omega a^L(x,y,\alpha)\cdot(1-u(x,y))^2\,dx\,dy\\&\quad+\int_\Omega c^L(x,y,\alpha)\cdot\left(u(x,y)-u_0(x,y)\right)^2\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^R(u,\alpha)&=\int_\Omega d^R(x,y,\alpha)\cdot|\nabla u(x,y)|^2\,dx\,dy+\int_\Omega a^R(x,y,\alpha)\cdot(1-u(x,y))^2\,dx\,dy\\&\quad+\int_\Omega c^R(x,y,\alpha)\cdot\left(u(x,y)-u_0(x,y)\right)^2\,dx\,dy.\end{aligned}$$

In order to include in the analysis the pessimistic/optimistic behaviour of the policy maker and her/his subjective perceptions about the degree of fuzzy uncertainty, we suggest to evaluate the fuzzy functional $\tilde{\mathcal{F}}(u)$ by applying the operator $E_{\lambda,n}[\cdot]$ introduced in [83,9,44]. Hence, for each fixed pair of parameters $(\lambda,n)\in[0,1]\times]0,+\infty[$, where λ represents a pessimistic/optimistic parameter and n reflects the subjective judgement about the degree of fuzzy uncertainty, we define

$$\mathcal{F}_{\lambda,n}(u)=E_{\lambda,n}\left[\tilde{\mathcal{F}}(u)\right]=\int_0^1\mathcal{F}_\lambda(u,\alpha)\,\varphi_n(\alpha)\,d\alpha \tag{5}$$

where $\mathcal{F}_\lambda(u,\alpha)$ is the point of the α-level set $\tilde{\mathcal{F}}(u,\alpha)=\left[\mathcal{F}^L(u,\alpha),\mathcal{F}^R(u,\alpha)\right]$ computed as

$$\mathcal{F}_\lambda(u,\alpha)=E_\lambda\left[\tilde{\mathcal{F}}(u,\alpha)\right]=(1-\lambda)\,\mathcal{F}^L(u,\alpha)+\lambda\,\mathcal{F}^R(u,\alpha)$$

and the weighting function $\varphi_n\geq 0$, with $\int_0^1\varphi_n(\alpha)\,d\alpha=1$, is defined by the parametric family $\varphi_n(\alpha)=(n+1)\,\alpha^n$. As an immediate application of Fubini theorem, we obtain for the functional (5) the following expression

$$\begin{aligned}\mathcal{F}_{\lambda,n}(u)=\int_\Omega&\left[d_{\lambda,n}(x,y)\cdot|\nabla u(x,y)|^2+a_{\lambda,n}(x,y)\cdot(1-u(x,y))^2\right.\\&\left.+c_{\lambda,n}(x,y)\cdot\left(u(x,y)-u_0(x,y)\right)^2\right]dx\,dy\end{aligned} \tag{6}$$

where

$$d_{\lambda,n}(x,y) = E_{\lambda,n}\left[\tilde{d}(x,y)\right] = \int_0^1 [(1-\lambda)\, d^L(x,y,\alpha) + \lambda\, d^R(x,y,\alpha)]\, \varphi_n(\alpha)\, d\alpha$$

$$a_{\lambda,n}(x,y) = E_{\lambda,n}\left[\tilde{a}(x,y)\right] = \int_0^1 [(1-\lambda)\, a^L(x,y,\alpha) + \lambda\, a^R(x,y,\alpha)]\, \varphi_n(\alpha)\, d\alpha \tag{7}$$

$$c_{\lambda,n}(x,y) = E_{\lambda,n}\left[\tilde{c}(x,y)\right] = \int_0^1 [(1-\lambda)\, c^L(x,y,\alpha) + \lambda\, c^R(x,y,\alpha)]\, \varphi_n(\alpha)\, d\alpha\,.$$

For each pair (λ, n) fixed, we say that $u^*_{\lambda,n} : \Omega \to [0,1]$ is an equilibrium distribution if it solves the minimization problem

$$\min_u \mathcal{F}_{\lambda,n}(u) \quad \longrightarrow \quad u^*_{\lambda,n}\,, \tag{8}$$

where the functional $\mathcal{F}_{\lambda,n}(u)$ is given by (6). For each value $\lambda \in [0,1]$ and for each degree of uncertainty $n > 0$, the solution $u^*_{\lambda,n}$ of the minimization problem (8) represents the spatial distribution of the adoption rate of the new technology at equilibrium, that is the stationary solution of the innovation diffusion process, determined according to the subjective opinions of the policy maker.

For theoretical studies about the existence of minimizers for the problem (8), we refer the reader to [17,67].

4. Discrete model and numerical procedure

In this section we develop a methodology for finding the equilibrium distribution as solution of the minimization problem (8) for the functional $\mathcal{F}_{\lambda,n}$ defined in (6). We analyse both the case in which the fuzzy coefficients do not depend on the spatial dimension and the case in which they are dependent.

We approach the minimization problem using the Euler-Lagrange equations associated to functional $\mathcal{F}_{\lambda,n}$. We recall (see, e.g., [17] for details) that if u is a minimizer of the functional

$$\mathcal{G}(u) = \int_\Omega f\left(x, y, u, u_x, u_y\right)\, dx\, dy\,, \tag{9}$$

where we have denoted $u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$, then a necessary condition is that the derivative of $\mathcal{G}$, applied to u, is equal to zero. Consequently, the minimizer u has to satisfy the Euler-Lagrange equations

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} = 0\,. \tag{10}$$

4.1. The case when fuzzy coefficients do not depend on spatial dimension

First, we study the minimization problem (8) assuming that the fuzzy coefficients $\tilde{d}$, $\tilde{a}$ and $\tilde{c}$ do not depend on spatial dimension. In this case the functional (6) can be expressed as

$$\mathcal{F}_{\lambda,n}(u) = \int_\Omega \left[d_{\lambda,n} \cdot |\nabla u(x,y)|^2 + a_{\lambda,n} \cdot (1 - u(x,y))^2 + c_{\lambda,n} \cdot \left(u(x,y) - u_0(x,y)\right)^2\right] dx\, dy \tag{11}$$

where

$$d_{\lambda,n} = \int_0^1 [(1-\lambda)\, d^L(\alpha) + \lambda\, d^R(\alpha)]\, \varphi_n(\alpha)\, d\alpha$$

$$a_{\lambda,n} = \int_0^1 [(1-\lambda)\, a^L(\alpha) + \lambda\, a^R(\alpha)]\, \varphi_n(\alpha)\, d\alpha \tag{12}$$

$$c_{\lambda,n} = \int_0^1 [(1-\lambda)\, c^L(\alpha) + \lambda\, c^R(\alpha)]\, \varphi_n(\alpha)\, d\alpha\,.$$

Observing that functional (11) is of the form (9) with

$$f\left(x, y, u, u_x, u_y, \alpha\right) = d_{\lambda,n}\left(u_x^2 + u_y^2\right) + a_{\lambda,n}\,(1-u)^2 + c_{\lambda,n}\left(u - u_0\right)^2\,,$$

from (10) we obtain, by computation, that the Euler–Lagrange equation associated to the functional $\mathcal{F}_{\lambda,n}$ can be expressed as the partial differential equation (PDE)

$$-2\, d_{\lambda,n}\, \Delta u(x,y) - 2\, a_{\lambda,n}\,(1 - u(x,y)) + 2\, c_{\lambda,n}\,(u(x,y) - u_0(x,y)) = 0$$

that is

$$d_{\lambda,n}\,\Delta u(x,y) + a_{\lambda,n}\,(1-u(x,y)) - c_{\lambda,n}\,(u(x,y)-u_0(x,y)) = 0 \tag{13}$$

where $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$ is the two-dimensional Laplacian operator.

We can determine a numerical solution of PDE (13) by using the finite difference approximation methods [2,79]. Hence, we discretize the spatial domain by placing a uniform grid over the domain, with grid spacing h. In particular, we represent the spatial domain Ω, i.e. the reference market, as the open rectangle $]0,n+1[\,\times\,]0,m+1[$ and denote the points of the discretized domain as $(x_i,y_j)=(ih,jh)$, with $i=1,\dots,n$ and $j=1,\dots,m$. Moreover, for simplicity of notation, we denote

$$u(i,j) = u(x_i,y_j) = u(ih,jh)\,, \qquad i=1,\dots,n, \quad j=1,\dots,m\,.$$

We adopt the following approximation scheme

$$\begin{aligned} u_x(i,j) &\approx \frac{u(i+1,j)-u(i-1,j)}{2h} \\ u_y(i,j) &\approx \frac{u(i,j+1)-u(i,j-1)}{2h} \end{aligned} \tag{14}$$

and

$$\begin{aligned} u_{xx}(i,j) &\approx \frac{u(i+1,j)-2u(i,j)+u(i-1,j)}{h^2} \\ u_{yy}(i,j) &\approx \frac{u(i,j+1)-2u(i,j)+u(i,j-1)}{h^2} \end{aligned} \tag{15}$$

where $u(i+1,j) = u(x_{i+1},y_j) = u((i+1)h,jh)$. From (15) we get the approximation

$$\Delta u(i,j) \approx \frac{u(i+1,j)+u(i-1,j)+u(i,j+1)+u(i,j-1)-4u(i,j)}{h^2}\,. \tag{16}$$

By substituting (16) in (13) we obtain, by computation,

$$u(i,j) \approx \frac{\frac{4}{h^2}\,d_{\lambda,n}\cdot \bar{u}(i,j) + a_{\lambda,n}\cdot 1 + c_{\lambda,n}\cdot u_0(i,j)}{\frac{4}{h^2}\,d_{\lambda,n} + a_{\lambda,n} + c_{\lambda,n}} \tag{17}$$

where the amount

$$\bar{u}(i,j) = \frac{u(i+1,j)+u(i-1,j)+u(i,j+1)+u(i,j-1)}{4}$$

approximates the mean value of u in the neighbour of (i,j).

From (17), and using the above equations, we can easily derive the following iterative procedure, for $p=1,2,\dots,$

$$u^{(p)}(i,j) = w_1\cdot \bar{u}^{(p-1)}(i,j) + w_2\cdot 1 + w_3\cdot u_0(i,j) \tag{18}$$

where

- $u^{(p)}(i,j)$ is the value at (i,j) of the discrete version of u at iteration p;
- $\bar{u}^{(p-1)}(i,j)$ is the average value of $u^{(p-1)}$ in the neighbour of (i,j); it is computed as

$$\bar{u}^{(p-1)}(i,j) = \frac{u^{(p-1)}(i+1,j)+u^{(p-1)}(i-1,j)+u^{(p-1)}(i,j+1)+u^{(p-1)}(i,j-1)}{4}\,; \tag{19}$$

- the weights $w_1,w_2,w_3\ge 0$, such that $w_1+w_2+w_3=1$, are defined by

$$\begin{aligned} w_1 &= \frac{\frac{4}{h^2}\,d_{\lambda,n}}{\frac{4}{h^2}\,d_{\lambda,n}+a_{\lambda,n}+c_{\lambda,n}} \\ w_2 &= \frac{a_{\lambda,n}}{\frac{4}{h^2}\,d_{\lambda,n}+a_{\lambda,n}+c_{\lambda,n}} \\ w_3 &= \frac{c_{\lambda,n}}{\frac{4}{h^2}\,d_{\lambda,n}+a_{\lambda,n}+c_{\lambda,n}}\,. \end{aligned} \tag{20}$$

In the expressions (20), the coefficients $d_{\lambda,n}$, $a_{\lambda,n}$ and $c_{\lambda,n}$, as defined in (12), can be computed using the following approximation

$$d_{\lambda,n} \approx \frac{\displaystyle\sum_{q=1}^{N}[(1-\lambda)\, d^L(q/N) + \lambda\, d^R(q/N)]\left(\frac{q}{N}\right)^n}{\displaystyle\sum_{q=1}^{N}\left(\frac{q}{N}\right)^n}$$

for a suitable fixed N. Similar approximations hold for coefficients $a_{\lambda,n}$ and $c_{\lambda,n}$.

Starting from $u^{(0)} = u_0$, at each iterative step $p = 1, 2, \dots$, the update adoption function $u^{(p)}(i,j)$ is obtained, according to (18), as the weighted average of the values $\bar{u}^{(p-1)}(i,j)$, 1 and $u_0(i,j)$ with weights w_1, w_2 and w_3, respectively. The value $\bar{u}^{(p-1)}(i,j)$, that approximates the mean value of $u^{(p-1)}$ in the neighbour of (i,j), is related to diffusion effect, the attracting value $u = 1$, i.e. the total adoption level, is related to innovation effect, and the value $u_0(i,j)$, related to the friction effect, represents the technology distribution observed ad initial time. We observe that $0 \le u^{(p)}(i,j) \le 1$, as it can be easily verified taking into account that $0 \le u_0(i,j) \le 1$ and using (18) and (19).

The above iterative procedure allows to determine a numerical solution of PDE (13), since when $p \to +\infty$ then $u^{(p)}(i,j) \to u^*(i,j)$ where $u^*(i,j) = u^*_{\lambda,n}(x,y)$ is a discrete solution of the PDE (13).

4.2. The cases when fuzzy coefficients are dependent on spatial dimension

The Euler–Lagrange equation (10) associated to the functional $\mathcal{F}_\lambda$ defined in (6), can be expressed as the PDE

$$\begin{aligned} &\operatorname{div}\left(d_{\lambda,n}(x,y)\cdot \nabla u(x,y)\right) + a_{\lambda,n}(x,y)\,(1-u(x,y)) \\ &\qquad - c_{\lambda,n}(x,y)\,(u(x,y)-u_0(x,y)) = 0 \end{aligned} \tag{21}$$

where the coefficients are defined in (7) and the divergence operator of a vector function $g = (g_1, g_2)$ is defined by $\operatorname{div} g = (g_1)_x + (g_2)_y$. By using the identity

$$\operatorname{div}\left(d_{\lambda,n}(x,y)\,\nabla u(x,y)\right) = d_{\lambda,n}(x,y)\,\Delta u(x,y) + \nabla d_{\lambda,n}(x,y)\cdot\nabla u(x,y)\,, \tag{22}$$

where $\nabla d_{\lambda,n}\cdot\nabla u = (d_{\lambda,n})_x\, u_x + (d_{\lambda,n})_y\, u_y$ denotes the inner product between the gradients, we can rewrite the equation (21) as

$$\begin{aligned} &d_{\lambda,n}(x,y)\,\Delta u(x,y) + \nabla d_{\lambda,n}(x,y)\cdot\nabla u(x,y) + a_{\lambda,n}(x,y)\,(1-u(x,y)) \\ &\qquad - c_{\lambda,n}(x,y)\,(u(x,y)-u_0(x,y)) = 0\,. \end{aligned}$$

By employing the finite difference schemes (14) and (16) we can easily obtain the following iterative procedure

$$u^{(p)}(i,j) = w_1(i,j)\cdot\bar{u}^{(p-1)}(i,j) + w_2(i,j)\cdot 1 + w_3(i,j)\cdot u_0(i,j) + \frac{1}{w(i,j)}V^{(p-1)}(i,j) \tag{23}$$

where

- the average value $\bar{u}^{(p-1)}(i,j)$ is the same as defined in (19);
- the value $V^{(p-1)}(i,j) = \nabla d_{\lambda,n}(i,j)\cdot\nabla u^{(p-1)}(i,j)$ can be computed, using (14), as

$$\begin{aligned} V^{(p-1)}(i,j) = \frac{1}{4h^2}\Big[&(d_{\lambda,n}(i+1,j) - d_{\lambda,n}(i-1,j))\,(u^{(p-1)}(i+1,j) - u^{(p-1)}(i-1,j)) \\ &+(d_{\lambda,n}(i,j+1) - d_{\lambda,n}(i,j-1))\,(u^{(p-1)}(i,j+1) - u^{(p-1)}(i,j-1))\Big]\ ; \end{aligned}$$

- we have denoted

$$w(i,j) = \frac{4}{h^2}\, d_{\lambda,n}(i,j) + a_{\lambda,n}(i,j) + c_{\lambda,n}(i,j)$$

  and

$$w_1(i,j) = \frac{\frac{4}{h^2}\, d_{\lambda,n}(i,j)}{w(i,j)}\,, \qquad w_2(i,j) = \frac{a_{\lambda,n}(i,j)}{w(i,j)}\,, \qquad w_3(i,j) = \frac{c_{\lambda,n}(i,j)}{w(i,j)}\,.$$

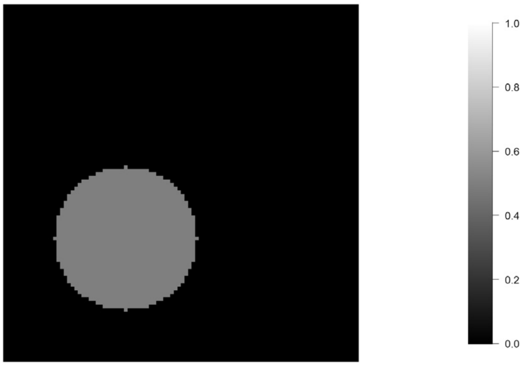


Fig. 1. The technology adoption distribution u_0 observed at initial time.

The coefficients $d_{\lambda,n}$, $a_{\lambda,n}$ and $c_{\lambda,n}$ defined in (7) can be computed using the approximation, by setting a suitable N,

$$d_{\lambda,n}(i,j) = \int_0^1 [(1-\lambda)\, d^L(i,j,\alpha) + \lambda\, d^R(i,j,\alpha)] \cdot \varphi_n(\alpha)\, d\alpha$$

$$\approx \frac{\sum_{q=1}^{N} [(1-\lambda)\, d^L(i,j,q/N) + \lambda\, d^R(i,j,q/N)] \left(\frac{q}{N}\right)^n}{\sum_{q=1}^{N} \left(\frac{q}{N}\right)^n} .$$

Similar approximations hold for coefficients $a_{\lambda,n}$ and $c_{\lambda,n}$.

It is worth noting that when the fuzzy coefficients do not depend on spatial dimension, the iterative procedure defined by formula (23) agrees with the procedure described by formula (18) since, being $d_{\lambda,n}(i,j) = d_{\lambda,n}$ a constant, we have $V^{(p-1)}(i,j) = 0$.

Remark 4.1. Decomposition (22) suggests that the diffusion effect can be described by the sum of two terms. The first is a smoothing term and drives the process towards the average adoption level observed in the neighbourhood. The second term, that estimates the spatial variation of the diffusion coefficient using a similarity measure between the two gradients, adjusts the previous smoothing effect making it stronger in the directions where the coefficient is greater.

We observe that the second term is not present in equation (18). It appears when fuzzy coefficients are dependent on the spatial dimension, as indicated by equation (23).

5. Numerical simulations

In this section we will apply the proposed methodology to the analysis of the diffusion process of a new technology considering two different market scenarios. First, we will investigate the spatial diffusion of a new technology that has previously been introduced in a reference market region. We will perform the analysis under conditions of spatial homogeneity in the market structure and, therefore, we will assume that the fuzzy parameters of the model do not depend on the spatial dimension. Second, we will analyse a situation where the new technology has not yet been introduced into the market. In this case we will consider a market characterized by the presence of two groups of firms, located in two different spatial positions, capable of influencing the decision of the other firms to adopt the new technology. We also assume spatial heterogeneity regarding the profitability due to the use of the new technology. In accordance with these market conditions, we will consider spatial dependence in fuzzy parameters.

It is interesting to note that, in the light of our results, different degrees of uncertainty lead us to obtain different evolutions of the adoption process. Furthermore, also the subjective opinions of the decision-maker lead to significantly different predictions regarding the diffusion process.

5.1. Innovation diffusion in markets with homogeneous spatial structure

We consider the problem to forecast the diffusion process of a new technology that has been already introduced in the circular region shown in Fig. 1. Assuming homogeneity in the economic spatial structure of the market, we model the fuzzy coefficients as independent of the spatial dimension. Furthermore, we describe the diffusion coefficient $\tilde{d} =< d^1, d^2, d^3 >=< 2, 30, 90 >$, the attraction coefficient $\tilde{a} =< a^1, a^2, a^3 >=< 1, 10, 70 >$ and the cost coefficient $\tilde{c} =< c^1, c^2, c^3 >=< 4, 5, 10 >$ as triangular fuzzy numbers, whose α-level sets are given by (1). In order to apply the procedure iterative established in (18), we consider as domain Ω the square obtained by setting the grid spacing to $h = 1$. The numerical results, for different values of parameters λ and n, are displayed in Fig. 2.

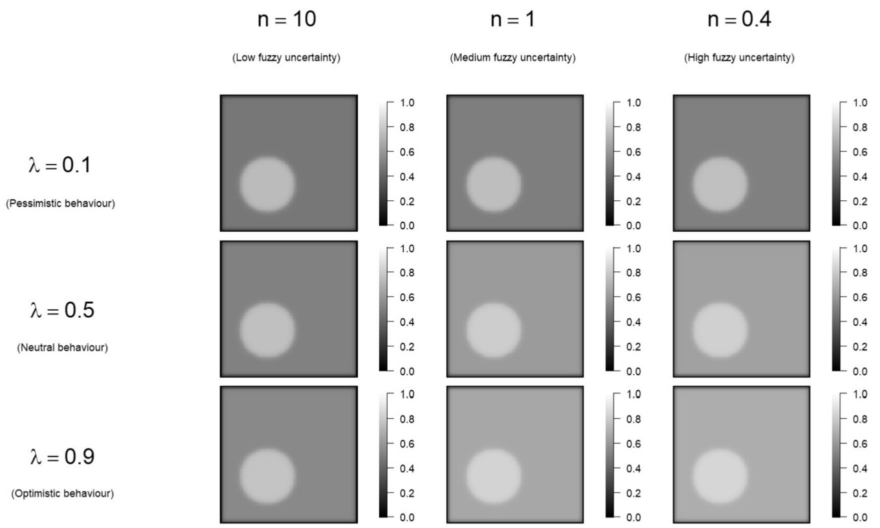


Fig. 2. The adoption distribution $u^*_{\lambda,n}$ for different values of parameters λ and n.

(a) (b)

Fig. 3. (a) The reference market Ω. (b) The distribution u_0 observed at initial time.

We observe that the adoption distribution changes as n varies, i.e. when different levels of uncertainty are considered. It is interesting to note that even the pessimistic/optimistic point of view of the decision-maker leads to the identification of different evolutions of the innovation process.

5.2. Innovation diffusion in markets with heterogeneous spatial structure

Let us now consider the case in which the market Ω is mainly influenced by two groups of firms operating, respectively, at spatial locations $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, as shown in Fig. 3 (a). Accordingly, we suppose that the diffusion effect has greater intensity near P_1 and P_2 and, moreover, that the intensity of this effect decreases as one moves away from these two spatial locations. Consequently, we model the fuzzy diffusion coefficient as

$$\tilde{d}(x,y) = \tilde{d}(P) = \tilde{D}_1\, e^{-\tilde{k}_D\,|P-P_1|^2} + \tilde{D}_2\, e^{-\tilde{k}_D\,|P-P_2|^2}$$

for all $P(x,y) \in \Omega$, where $|P-P_1|^2 = (x-x_1)^2 + (y-y_1)^2$, $|P-P_2|^2 = (x-x_2)^2 + (y-y_2)^2$ and $\tilde{D}_1, \tilde{D}_2, \tilde{k}_D$ are non-negative fuzzy numbers. The attractive coefficient, related to the strength of the new technology to penetrate the market, is described by

$$\tilde{a}(x,y) = \tilde{a}(P) = \tilde{\pi}_1\, e^{-\tilde{k}_\pi\,|P-P_1|^2} + \tilde{\pi}_2\, e^{-\tilde{k}_\pi\,|P-P_2|^2}$$

for all $P(x,y) \in \Omega$, where $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{k}_\pi$ are non-negative fuzzy numbers. We assume that the basic cost of equipment of new technology is non fuzzy (i.e., a crisp value) and constant, that is

$$\tilde{c}(x,y) = c$$

for all $P(x,y) \in \Omega$.

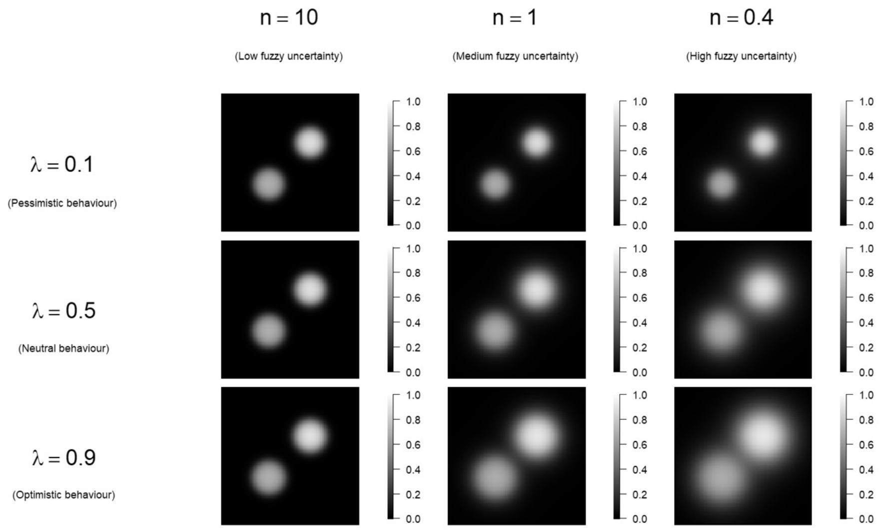


Fig. 4. The technology adoption function $u^*_{\lambda,n}$ for different values of parameters λ and n.

We suppose that the neighbour influence exerted by firms located at P_1 is greater than that exerted by firms at P_2, while the profitability due to the adoption of the new technology is greater near P_2 than P_1. So we assume $\tilde{D}_1 > \tilde{D}_2$ and $\tilde{\pi}_2 > \tilde{\pi}_1$ (where the ordering $>$ has been defined in (2)).

The α-level sets $\tilde{d}(x,y,\alpha) = [d^L(x,y,\alpha), d^R(x,y,\alpha)]$ of $\tilde{d}(x,y)$ and $\tilde{a}(x,y,\alpha) = [a^L(x,y,\alpha), a^R(x,y,\alpha)]$ of $\tilde{a}(x,y)$, can be expressed, respectively, as

$$\begin{aligned}\tilde{d}(x,y,\alpha) = [&D_1^L(\alpha)\, e^{-k_D^R(\alpha)((x-x_1)^2+(y-y_1)^2)} + D_2^L(\alpha)\, e^{-k_D^R(\alpha)((x-x_2)^2+(y-y_2)^2)},\\ &D_1^R(\alpha)\, e^{-k_D^L(\alpha)((x-x_1)^2+(y-y_1)^2)} + D_2^R(\alpha)\, e^{-k_D^L(\alpha)((x-x_2)^2+(y-y_2)^2)}]\end{aligned}$$

and

$$\begin{aligned}\tilde{a}(x,y,\alpha) = [&\pi_1^L(\alpha)\, e^{-k_\pi^R(\alpha)((x-x_1)^2+(y-y_1)^2)} + \pi_2^L(\alpha)\, e^{-k_\pi^R(\alpha)((x-x_2)^2+(y-y_2)^2)},\\ &\pi_1^R(\alpha)\, e^{-k_\pi^L(\alpha)((x-x_1)^2+(y-y_1)^2)} + \pi_2^R(\alpha)\, e^{-k_\pi^L(\alpha)((x-x_2)^2+(y-y_2)^2)}],\end{aligned}$$

where $\tilde{k}_D(\alpha) = [k_D^L(\alpha), k_D^R(\alpha)]$ and $\tilde{k}_\pi(\alpha) = [k_\pi^L(\alpha), k_\pi^R(\alpha)]$ denote the α-level sets of $\tilde{k}_D$ and $\tilde{k}_\pi$, respectively. For our simulation, we assume that $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are the triangular fuzzy numbers given by $\tilde{D}_1 =< 6, 10, 18 >$, $\tilde{D}_2 =< 1, 3, 5 >$, $\tilde{k}_D =< 0.001, 0.020, 0.050 >$, $\tilde{\pi}_1 =< 3, 6, 9 >$, $\tilde{\pi}_2 =< 4, 7, 15 >$, $\tilde{k}_\pi =< 0.001, 0.020, 0.050 >$. Moreover, we set the cost coefficient $c = 1$. We analyse a scenario in which the new technology is not present in the market at the observation time, that is $u_0(x,y) = 0$ for all $(x,y) \in \Omega$, as shown in Fig. 3 (b). The numerical results, for different values of parameters λ and n, are displayed in Fig. 4.

Our results show that the proposed fuzzy approach produces different results from those obtained with the traditional crisp method and, moreover, leads to interesting interpretations. In particular, observing Fig. 4, we deduce that, differently from the traditional crisp model (which can be identified with the case $n = 10$), the fuzzy approach (cases $n = 1$ and $n = 0.4$) indicates that the adoption of new technologies occurs in a wider region. This difference in results stems from the fact that the uncertainty of the impact of economic variables on the diffusion process, such as, for example, the interaction between networked firms, was also incorporated into the model.

6. Conceptual framework and main ideas outline

In this section we explain the main ideas that inspired us to approach the innovation diffusion problem using a fuzzy variational framework. First, we briefly review the relevant literature on this topic.

6.1. Innovation adoption models

Many drivers influence an innovation's market process, including innovation (types and characteristics) [64], the adopter (characteristics and attributes) [52], communication (information channels and media) [82], contextual factors (social system and communication behaviours) [53], marketing activities [10] and profitability [52]. Technical factors and economic parameters as well as government policy (for example, financial incentives for end users) play key roles in the adoption of new technologies [20].

Several market penetration models have been proposed in the literature with the main aim of forecasting the time period for the market penetration of a new technology and the achievable market share in a period of time [5,31,39,57]. For example, subjective estimation methods [42], cost models [77], market surveys, diffusion models [62], historical analogy methods and econometric models [49]. The choice of one over the other depends on several factors such as the specific nature of the technology, its maturity level (early stage of development, market introduction, market acceptance, maturity) and availability of historical data [61].

Spatial diffusion models have been proposed in order to explain differences in adoption patterns across firms (e.g., producers) and across regions or countries [33]. For example, in [74] the Author discusses two models that focus on different sources of profitability differentials across producers. The first, which features heterogeneity in producer size and productivity, is directed toward explaining diffusion across producers in a single country. The second, which highlights the fact that newer technologies are generally labour-saving and capital-using, features heterogeneity in wage rates and wage growth, and is directed toward explaining cross-country diffusion. Both models draw on model of tractors [50,74] and on model of agricultural technologies in developing countries [13,74]. In [30] a diffusion model for hybrid corn is discussed in order to explain differences in adoption patterns across regions: adoption was more complete in regions where yields were higher and acreage in corn was larger. The industrial technologies diffusion model suggested in [49] aims to explain differences in the speed of adoption across innovations.

Mathematical formulations of innovation diffusion models have been advanced in the form of systems of differential, differential–integral or integral equations describing boundary-initial value problems [51]. They mainly refer to different stages of consumers' adoption during market development classified as innovators/early adopters, early and late majority, and laggards according to the time of adoption, since the new technology is introduced in the market [61,62]. Non linear models have also been introduced to consider the spatial aspects in technology diffusion [47,18,37,8], under the general assumption that the growth of the number of adoptions in a give region is influenced by the behaviour observed in neighbouring regions [48]. The involved equations are often non-linear and, consequently, no analytical solutions can be obtained in non-trivial cases. So, different numerical procedures have been implemented [19].

However, classical innovation diffusion models suffer from some limitations since they assume that historical data are precisely known and that the model parameters are precisely estimable. But in real life scenario, data are uncertain and imprecise in nature, making the model's parameters unstable and fluctuate. The high degree of uncertainty present in the market penetration process [65,81] has lead many Authors to study market penetration patterns [85], life cycle curves of products [41] and market penetration curves [59] under different sources of uncertainty [62] and by considering many variables, such as price [35], market potential [46], advertising and promotion [22], government support [3] and research and development (R&D) activities [66]. In this context, subjective estimation methods have been used since they imply decisions on the introduction of a new technology on the market based on the perspectives and available data. Subjective estimation can be carried out by involving an expert panel [25] or, alternatively, by acquiring information by distributing questionnaires to experts [63].

Fuzzy logic has been applied in [68,11,66,12,14,71,58] to describe the imprecise knowledge involved in innovation diffusion models. In [40] the Authors employ fuzzy set theory in order to qualitatively model the characteristics of an innovation.

6.2. The main ideas of our proposal

In this study we have focused on the spatial aspects of the market penetration of a new technology and we have determined the stationary solutions, i.e. the stable configurations, of the adoption process as minima of a suitable variational functional. The variational functional (3) proposed in this paper has been inspired us by the functional

$$J(u) = \gamma \int_{\Omega} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} + \int_{\Omega} W(u(\mathbf{x})) \, d\mathbf{x} \,,$$

with $\Omega \subset \mathbb{R}^N$, applied in [32,55,4] to describe the total energy of a dynamical system made of a single fluid in the framework of Van der Waals-Cahn-Hilliard theory of *phase transitions*. The functional $J(u)$ consists of two components: the "potential energy" W, which is a non-negative function of the density distribution u, and the "density gradient" component, which is the diffusion term. The stable configurations of the system are obtained by minimizing the total energy of the fluid expressed by $J(u)$, that forces the system to be attracted by the wells (i.e., the zeros) of potential W, taking also into account the energy due to the variation of the density u in the neighbour. A modified version of functional $J(u)$, that is

$$J(u) = \gamma_1 \int_{\Omega} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} + \gamma_2 \int_{\Omega} |u(\mathbf{x}) - g(\mathbf{x})|^2 \, d\mathbf{x} + \gamma_3 \int_{\Omega} W(u(\mathbf{x})) \, d\mathbf{x} \,,$$

where g is the observed data, has been applied in [67] to image classification problems.

Starting from the previous framework, we have transposed the principle of minimum energy to the market penetration system for determining the stable configurations of the innovation adoption process. In this vision, the variational functional $\tilde{\mathcal{F}}(u)$ defined in (3) can be interpreted as the "total energy" $J(u)$ of the system, where u represents the "density" of the new technology, the diffusion term describes the energy associated to the network influence, the data term reflects the resistance due to cost of substitution and the "potential energy" refers to the attracting force towards the total adoption level $u = 1$ that represents the well of the potential W. Moreover, in order to consider spatial heterogeneity and uncertainty, we have extended the initial idea by introducing the functional (4) in which coefficients are spatial dependent and described by fuzzy numbers.

An advantage offered by our methodology is that it starts from the formulation of the variational functional that describes the process and this permits to aggregate, in a meaningful and useful manner, the various sources of information underlying the specific scenario considered. Differently from the spatio-temporal models based on reaction-diffusion partial differential equations [37], we obtain the differential equations that drive the innovation adoption system as a secondary step, by considering the Euler–Lagrange

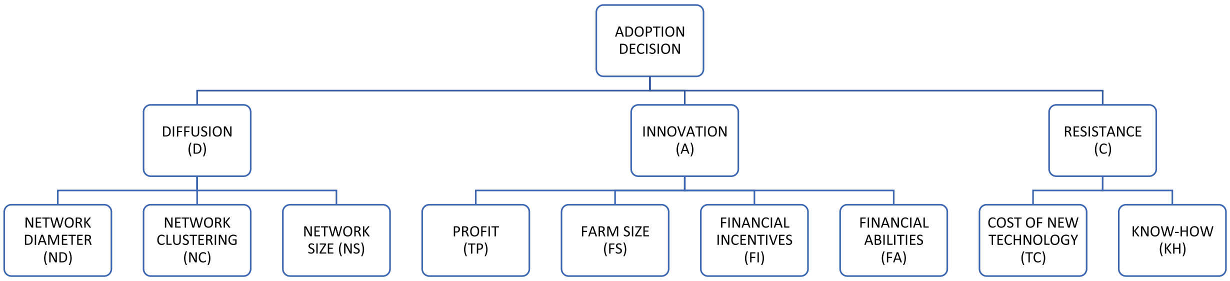


Fig. 5. Conceptual model of the digital farming technologies adoption process.

equations associated to the variational functional. This approach permits to achieve interesting results. For example, as already discussed in Remark 4.1, comparing the Euler-Lagrange equations obtained in cases of spatial independence and spatial dependence of fuzzy coefficients, we can observe that a new term appears. This new term cannot be obtained through a straightforward extension of the equation, by introducing the spatial dependence of the coefficients. Furthermore, another advantage of our approach is that it provides a flexible framework that allows to forecast the spatial diffusion of a new technology for different degrees of uncertainty also taking into account the subjective opinions of policy makers.

7. An application to the adoption process of digital agricultural technologies

We apply the fuzzy logic-based framework developed in the previous sections to analyse the effectiveness of government policies to promote digital transformation in the agricultural sector. The high degree of heterogeneity and complexity of the problem results in great uncertainty in estimating the variables involved [15,60,70,72]. In addition, the digital agriculture transition poses different issues and challenges from those faced in past transformations of the agricultural sector, which makes the available data insufficient or inadequate to conduct an appropriate analysis of the problem. Knowledge must therefore be derived from other sources, such as information provided by experts. For this reason, the use of fuzzy logic to handle this kind of information can be particularly useful.

7.1. Factors influencing digital technologies adoption in agriculture

Digital farming, also called smart agriculture or agriculture 4.0, has the potential to transform current agricultural systems to make them more sustainable (for example, by reducing the use of agrochemicals). Examples of technologies for digital agriculture are: mobile applications for decision support, field sensors and remote sensing technologies for data collection, drones and robots for process automation. The impact of disruptive events, such as climate change and the global pandemic, has highlighted the role played by innovation capabilities in building resilience and flexibility into the food supply chain [7]. For these reasons, it is important to study the mechanisms of adoption and diffusion of digital agricultural technologies [56].

As highlighted in [78] the adoption of digital farming is a multi-dimensional and spatially differentiated process. Heterogeneity of natural resources influences performance and subsequent adoptive decision making [21]. Adoption behaviour not only depends on farm and operator characteristics but is also influenced by structural, political and economic conditions of the agricultural system. Farms' interactions and organized networks can also play a role in the adoption and diffusion of digital agriculture technologies [1,27,69]. In addition, government interventions, the costs of digital transformation and the relative difficulty of technology substitution are identified as key factors influencing the adoption process.

7.2. Conceptual framework underlying the digital farming adoption decision

Based on the innovation diffusion model introduced in the previous sections, we conceptualize that the adoption decision is influenced by the following three dimensions: diffusion, innovation, and resistance due to replacement cost. Furthermore, stemming from the literature [7,27], we decompose each dimension into several aspects (sub-criteria) specific to the problem of digital agricultural technology adoption, as shown in Fig. 5. Specifically, we model the main coefficients as follows:

DIFFUSION: The intensity of the diffusion effect, that is, the influence exerted by neighbouring firms on the adoption decision, depends on the structure of the social network. As outlined in [6], the main characteristics that influence the diffusion process are: *network diameter* (ND), also known as the "small-world" property, which reflects the size of the network; *network clustering* (NC), which refers to the strength of connections; and *network structure* (NS), that is, the extent to which hubs are interconnected to facilitate interaction within the network.

INNOVATION: Determinant factors for innovation are the *profit* (TP) due to the adoption of new technology [54] and the *farm size* (FS), since large farms can take advantage of economies of scale and are more likely to be able to afford the high initial investment of new technologies [76]. Also, institutional policies of *financial incentives* (FI), such as the accessibility of subsidy/credit, can have a positive effect on adoption of smart technologies [69]. In addition, the *financial ability* (FA) of the farm can facilitate access to credit and the use of subsidies, and thus promote the adoption of innovations [69].

Table 2
Description of the employed symbols.

Symbol	Description
Ω	Target rural area
$\mathcal{R}$	Set of identified regions
R	Generic region $R \in \mathcal{R}$
$\tilde{D}, \tilde{D}_R$	Fuzzy diffusion coefficient, also denoted (for simplicity of notation) D, D_R
$\tilde{A}, \tilde{A}_R$	Fuzzy innovation coefficient, also denoted as A, A_R
$\tilde{C}, \tilde{C}_R$	Fuzzy resistance coefficient, also denoted as C, C_R
$\tilde{d}(x, y)$	Fuzzy diffusion coefficient at spatial location (x, y)
$\tilde{a}(x, y)$	Fuzzy innovation investment at spatial location (x, y)
$\tilde{c}(x, y)$	Fuzzy resistance coefficient at spatial location (x, y)
$u(x, y)$	Adoption level of digital farming technology at spatial location (x, y)
U	Rate of adoption of digital farming technology in the area Ω

RESISTANCE: Farms' resistance to modify pre-existing technologies derives both from external factors, such as the costs of acquisition of new technologies (TC), and from internal factors, such as the farm's level of know-how (KH) development. Know-how (i.e., knowledge and capacity) refers to the farmer's ability to implement digital technologies. Lack of knowledge in new technologies is a barrier to innovation [75].

7.3. Modelling the digital farming transition process

We investigate the adoption process in a given area Ω of a country, by supposing that, from the perspective of the agricultural sector, a number of geographical regions, characterized by different features, can be identified in the considered area. We denote by $\mathcal{R}$ the set of regions considered for the analysis.

We approach the problem from a variational point of view, considering the fuzzy functional $\tilde{\mathcal{F}}(u)$ defined in (4), where the diffusion, innovation and resistance coefficients are described by space-dependent fuzzy parameters. According to the conceptual scheme illustrated in Fig. 5, we determine the diffusion, innovation and resistance dimensions as an aggregate combination of the specific variables that influence their impact on the adoption process. In order to handle with the imprecise estimation of these variables, we use appropriate fuzzy inference systems, with input variables the specific regional factors influencing the coefficients, that are obtained as output of the systems. This approach handles both the uncertainty due to the imprecise estimation of coefficients and the ambiguity of assigning a geographical location (x, y) to a single region R. The computed fuzzy coefficients are then used to determine the spatial distribution of adopters $u = u(x, y)$ by solving the minimization problem (8).

The main steps of the proposed procedure can be summarized as follows:

1. For each region R, we determine the corresponding diffusion, innovation and resistance fuzzy coefficients $\tilde{D}_R$, $\tilde{A}_R$ and $\tilde{C}_R$, as outputs of appropriate rule-based inference systems, having as input variables the specific factors influencing the coefficients, according to the conceptual scheme illustrated in Fig. 5.
2. For each spatial location (x, y) and for each region R, we determine the degree of belonging of (x, y) to R, denoted by $\mu_R(x, y)$.
3. For each spatial location (x, y), we determine the overall fuzzy coefficients $\tilde{d}(x, y)$, $\tilde{a}(x, y)$ and $\tilde{c}(x, y)$ as the fuzzy weighted average of the regional fuzzy coefficients, computed in step 1, with weights membership degrees $\mu_R(x, y)$, computed in step 2.
4. We determine the adoption distribution of digital farming technologies by implementing the iterative procedure established in (23) using fuzzy coefficients computed in step 3.

In the following, we will detail the steps of the proposed methodology. We refer to Table 2 for a description of the notations employed in the analysis.

As first step, we determine the fuzzy coefficients $\tilde{D}$, $\tilde{A}$ and $\tilde{C}$ as the outputs of three inference systems. In all three systems considered, the input variables are described by three fuzzy linguistic terms classified as Low (L), Medium (M) and High (H), while the output variables, namely $\tilde{D}$, $\tilde{A}$ and $\tilde{C}$, are described by five fuzzy terms classified as Very Low (VL), Low (L), Medium (M), High (H) and Very High (VH). For each considered fuzzy system the inference is preformed through a set of IF-THEN statements, also called IF-THEN rules. Using the inference mechanism, we can obtain, for each region R, the associated fuzzy coefficients $\tilde{D}_R$, $\tilde{A}_R$ and $\tilde{C}_R$.

The rules of the systems can be established using information provided by one or more experts [24]. For our analysis, we have deduced the inference rules using information gathered from the relevant literature.

In the following we illustrate the three fuzzy systems used for determining the diffusion, innovation and resistance coefficients of each region R.

Evaluation of diffusion coefficient. As previously discussed, diffusion coefficient (D) can be inferred from the following variables: *network diameter* (ND), *network clustering* (NC) and *network structure* (NS). As established in [6], the network clustering variable has a greater impact on the diffusion effect than the others. Therefore, we construct the fuzzy system by assigning a higher weight to the variable NC. We omit to list all the rules of the fuzzy system, but present only some of them:

IF ND is L and NC is L and NS is M THEN D is VL

IF ND is L and NC is M and NS is L THEN D is L

IF ND is L and NC is H and NS is L THEN D is M

IF ND is L and NC is H and NS is M THEN D is H

Evaluation of innovation coefficient. The innovation coefficient (A) can be obtained as the output of a fuzzy system with input variables: *profitability* (TP), *farm size* (FS), *financial incentives* (FI) and *financial ability* (FA). As pointed out in [7,27,69,86], the effectiveness of financial incentives, such as the opportunity to access credit or subsidies, also depends on the financial ability of the farm. Therefore, we establish inference rules by considering the interaction (synergy) between the variables FI and FA. Some of the rules are as follows:

IF TP is L and FS is L and FI is L and FA is M THEN A is VL

IF TP is M and FS is H and FI is H and FA is H THEN A is VH

Evaluation of resistance coefficient. The resistance (C) of farms to replace pre-existing technologies depends on the cost of acquiring new technologies (TC) and the farm's level of know-how (KH). We model the inference rules taking into account that the variable KH represents a decreasing criterion, i.e., the higher KH the lower C. Examples of the adopted system rules are:

IF TC is M and HK is L THEN C is H

IF TC is M and HK is M THEN C is M

As next step, we compute, for each spatial location (x, y), the corresponding overall fuzzy coefficients $\tilde{d}(x, y)$, $\tilde{a}(x, y)$ and $\tilde{c}(x, y)$. To deal with the imprecise boundaries between regions, we introduce, for each region R, a fuzzy membership function $\mu_R(x, y)$ representing the belonging degree of location (x, y) to region R. Fuzzy coefficients $\tilde{d}(x, y)$, $\tilde{a}(x, y)$ and $\tilde{c}(x, y)$ are then determined as the fuzzy weighted average of the regional fuzzy coefficients $\tilde{D}_R, \tilde{A}_R, \tilde{C}_R$, with weights expressed by $\mu_R(x, y)$. Specifically, the fuzzy diffusion coefficient at spatial location (x, y) is given

$$\tilde{d}(x, y) = \frac{\sum_{R \in \mathcal{R}} \mu_R(x, y) \cdot \tilde{D}_R}{\sum_{R \in \mathcal{R}} \mu_R(x, y)}, \tag{24}$$

the fuzzy innovation coefficient by

$$\tilde{a}(x, y) = \frac{\sum_{R \in \mathcal{R}} \mu_R(x, y) \cdot \tilde{A}_R}{\sum_{R \in \mathcal{R}} \mu_R(x, y)}, \tag{25}$$

and the fuzzy resistance coefficient by

$$\tilde{c}(x, y) = \frac{\sum_{R \in \mathcal{R}} \mu_R(x, y) \cdot \tilde{C}_R}{\sum_{R \in \mathcal{R}} \mu_R(x, y)}, \tag{26}$$

where $\mathcal{R}$ denotes the set of the regions considered for the analysis.

As final step of the procedure, we study the adoption process of digital farming technologies by implementing the methodology developed in the previous sections, using the fuzzy coefficients computed in (24), (25) and (26).

7.4. Simulations and results

We analyse the process of digital technology adoption in a given rural area Ω. We assume that, from the perspective of the agricultural sector, five main geographic regions, shown in Fig. 6, can be identified in the target area: a central region R_C and four peripheral regions, namely the northwestern R_{NW}, northeastern R_{NE}, southeastern R_{SE} and southwestern R_{SW} regions. We suppose that farms operating in central region R_C are characterized by large size and well-organized network structures. Because of their size and developed organizational structure, these farms can realize high profit by adopting the new digital technology. Northern regions R_{NW} and R_{NE} are mostly characterized by medium-sized farms that have high financial ability but, unlike those in the central regions, do not have well-developed networks. Differently, in the southern regions R_{SE} and R_{SW} there are mainly small

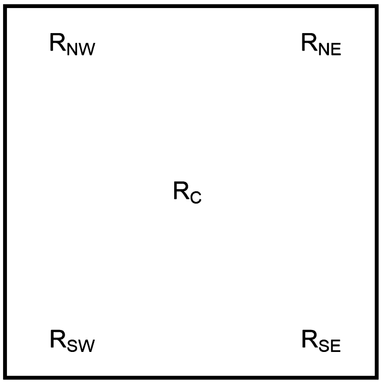


Fig. 6. Map of the identified agricultural regions in the area Ω.

Table 3
Values of factors associated to individual regions.

Region	Variables							
	Diffusion			Innovation			Resistance	
	ND	NC	NS	TP	FS	FA	TC	KH
R_C	M	M	L	H	H	H	H	M
R_{NW}	L	L	L	M	M	H	M	M
R_{NE}	M	L	L	M	M	H	M	M
R_{SE}	H	H	H	M	L	M	L	M
R_{SW}	H	H	M	M	L	M	L	M

Table 4
Fuzzy terms for Diffusion, Innovation and Resistance factors.

Fuzzy terms	Factors		
	Diffusion (D)	Innovation (A)	Resistance (C)
VL	$< 2, 2, 4 >$	$< 20, 20, 32.5 >$	$< 20, 20, 25 >$
L	$< 2, 4, 6 >$	$< 20, 32.5, 45 >$	$< 20, 25, 30 >$
M	$< 4, 6, 8 >$	$< 32.5, 45, 57.5 >$	$< 25, 30, 35 >$
H	$< 6, 8, 10 >$	$< 45, 57.5, 70 >$	$< 30, 35, 40 >$
VH	$< 8, 10, 10 >$	$< 57.5, 70, 70 >$	$< 35, 40, 40 >$

farms with low financial ability, but organized in well-connected and highly structured networks. All the farms in the considered area have a medium level of skills and knowledge. It is assumed that the acquisition cost of the new technology is related to the size of the farm. We denote by $\mathcal{R} = \{R_C, R_{NW}, R_{NE}, R_{SE}, R_{SW}\}$ the set of the considered regions.

We employ the methodology proposed in Section 7.3 to study the adoption process of digital technology in the area Ω, on the basis of the regional farms' features. First, for every region R, we assign an evaluation (Low (L), Medium(M) or High (H)) to each of the variables that influence the adoption decision. These evaluations, shown in Table 3, will be used as input values of the rule-based inference systems. Then, we describe as triangular fuzzy numbers the linguistic values (Very Low, Low, Medium, High and Very High) associated with the output variables of the systems, that is diffusion, innovation and resistance factors. The corresponding triangular fuzzy terms as shown in Table 4. Furthermore, as an illustration, in Fig. 7 we have plotted the membership functions of the triangular fuzzy terms assigned to the diffusion coefficient.

We adopt a fuzzy threshold approach to handle the uncertainty associated with imprecise boundaries between the regions under consideration. Therefore, we describe the degree to which a location $P(x, y) \in \Omega$ belongs to a given region R as a function of the distance of P from the "pilot" location P_R in the region, i.e.

$$\mu_R(x, y) = \mu_R(P) = f(\text{dist}(P, P_R)).$$

The function f, modelling the imprecise thresholds, is defined by

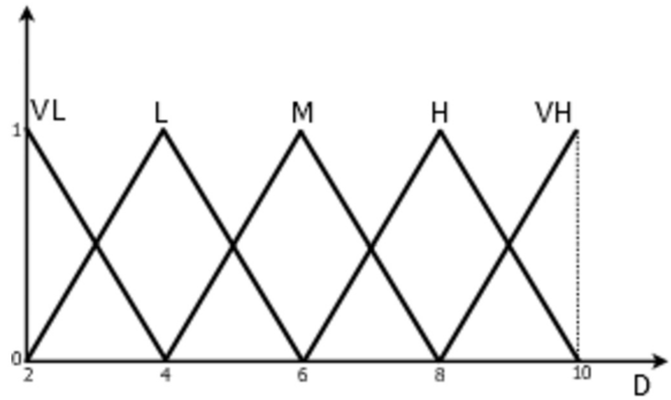


Fig. 7. Fuzzy terms associated to the diffusion coefficient.

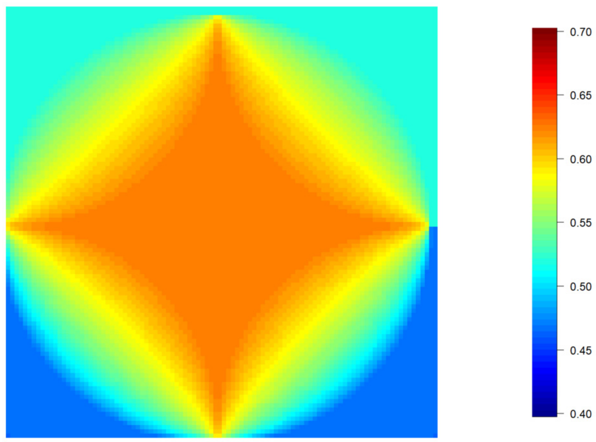


Fig. 8. Adoption map: $FI = L$ and $KH = M$.

$$f(x) = \begin{cases} 1 & x < t_1 \\ (t_2 - x)/(t_2 - t_1) & t_1 \leq x < t_2 \\ 0 & x \geq t_2 \end{cases}$$

where the values t_1, t_2, with $t_1 < t_2$, have to be appropriately chosen. Function f allows both to avoid setting sharp thresholds and to achieve smooth transitions between regions. We assume, with reference to Fig. 6, that the P_R pilot sites of the regions $R_C, R_{NW}, R_{NE}, R_{SE}, R_{SW}$ are located, respectively, at the centre and corresponding vertices of the considered area.

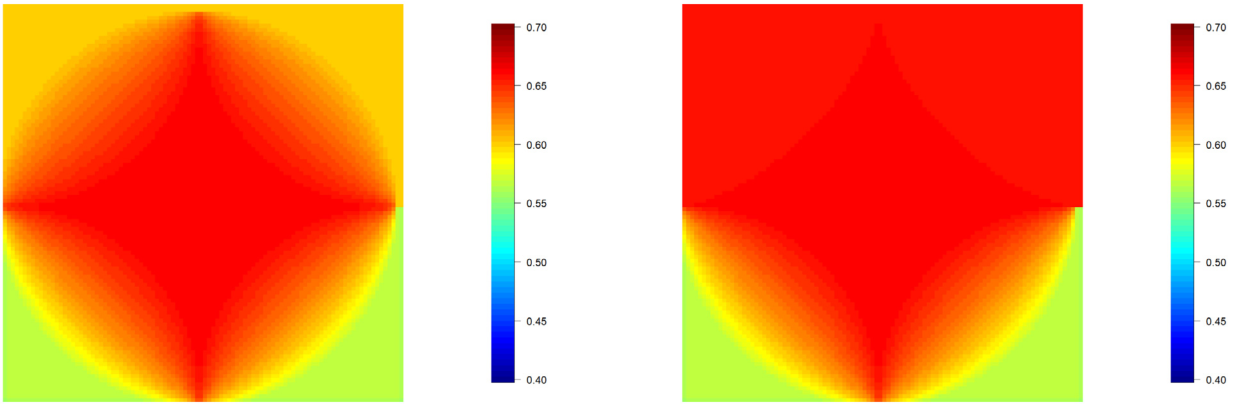
In order to investigate the effectiveness of government financial incentives to promote digital transformation in the agricultural sector, we determine the adoption distribution for different values of the Financial Incentives (FI) variable. In Fig. 8, Fig. 9(a) and Fig. 9(b) we have shown the adoption distributions obtained, respectively, for $FI = LOW$, $FI = MEDIUM$ and $FI = HIGH$. These simulations were performed with reference to farms having a medium level of know-how, that is $KH = MEDIUM$, according to evaluations assigned in Table 3. In addition, we set the uncertainty level and the pessimistic/optimistic parameters, as defined in Section 3, at $n = 1$ and $\lambda = 0.5$, respectively.

The results of the analysis indicate that policies to promote digital technology adoption based on financial incentives, such as subsidies and access to credit, are most effective in areas where farms have greater financial ability. However, incentive-based policies can still be effective, due to the diffusion factor, in regions where farms have low financial ability but are organized in well-structured networks.

As an additional indicator, we define the adoption rate $U \in [0, 1]$ related to considered area Ω by

$$U = \frac{1}{|\Omega|} \int_{\Omega} u(x, y)\, dx\, dy \tag{27}$$

where $|\Omega|$ denotes the measure of Ω. The adoption rates for scenarios described in Fig. 8, Fig. 9(a) and Fig. 9(b) are, respectively, given by $U = 56.4\%$ for $FI = LOW$, $U = 62.4\%$ for $FI = MEDIUM$ and $U = 63.9\%$ for $FI = HIGH$. Therefore, the effectiveness of financial incentive policy is greater when medium incentives are offered compared to high incentives. This effect can be understood by observing that the high increase in financial incentives promotes the adoption of new technologies especially among medium and



(a) (b)

Fig. 9. Adoption map: (a) $FI = M$ and $KH = M$; (b) $FI = H$ and $KH = M$. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

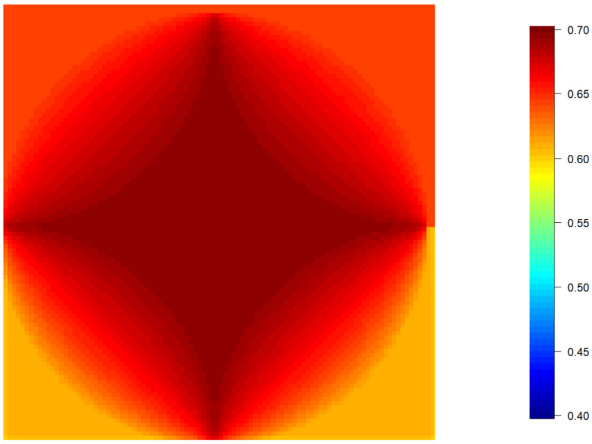


Fig. 10. Adoption map: $FI = M$ and $KH = H$.

large farms (which, for example, can buy more drones to monitor agricultural crops). In contrast, the effect on small farms is less significant.

In order to explore the effectiveness of other measures to support the digital transition in the agricultural sector, we study the effect of a policy that not only provides financial incentives but also aims to develop the skills and abilities of farms. In Fig. 10 we have plotted the adoption distribution referred to $FI = MEDIUM$ and $HK = HIGH$. The corresponding adoption rate, as defined in (27), is $U = 66.3\%$. Therefore, the adoption rate obtained for $FI = MEDIUM$ and $KH = HIGH$ is greater than that obtained for $FI = HIGH$ and $KH = MEDIUM$. This result suggests that to encourage the adoption of new digital technologies, it may be more effective to implement policies that, in addition to offering financial incentives, also aim to increase the level of knowledge and skills of farms. In fact, a high level of know-how reduces barriers (i.e., resistance) to adopting the new technology. Moreover, by comparing Fig. 9(b) and Fig. 10, the adoption distribution is more uniform for the case $FI = MEDIUM$ and $HK = HIGH$. This result indicates that the implementation of policies aimed at raising the skill level of farms also has the effect of wider adoption in different regions of the considered area. In addition, improving skills also meets the more general goal of helping to raise the level of culture in the relevant social environment [86].

7.5. Comments

We applied our methodology to study the effectiveness of government policies, in terms of financial incentives and skill enhancement, on the digital transition in agriculture, under conditions of uncertainty regarding the impact of economic variables and the characteristics of rural geographic regions in the reference area. Our study offers interesting insights and useful suggestions on how to set up an appropriate farm digitization strategy, focusing on the main determinants of adoption and diffusion and considering the effects that interactions in social networks have on the adoption process [34]. More generally, the proposed methodology can provide useful information for policy makers to understand how different components can influence the process of technological innovation and, therefore, can be used as a tool to support strategic decisions.

8. Conclusions

In this study, we have introduced a variational formulation of the innovation diffusion problem under conditions of spatial heterogeneity in the economic factors influencing the process and fuzzy uncertainty about their intensities. Different market scenarios have been considered and analysed. Our results have highlighted that both the different degrees of uncertainty and the different subjective opinions of the decision maker lead to significantly different predictions about the diffusion process. Therefore, our methodology could be applied to support strategic decisions concerning innovation diffusion plans.

As concluding remarks, we observe that our approach, based on variational functionals, requires less regularity for the distribution function than models based on differential equations. This can be useful for analysing scenarios with a high degree of spatial heterogeneity, where diffusion of new technologies is not expected to be smooth. We will explore this aspect in our future research.

We note that a disadvantage of our methodology, which needs to be improved, is that it does not explicitly consider the temporal evolution of the adoption process. Therefore, it can mainly be used as a pre-screening tool to forecast the potential market share and geographic penetration of new technologies, or even when adoption occurs over a short period of time or the model parameters do not vary significantly over time.

As future work, we intend to extend the model by explicitly considering the time variable and, furthermore, by including in the functional other factors that may influence the adoption process, not necessarily related to firms but also to consumers, such as the imitation effect related to the so-called word-of-mouth phenomenon [47].

CRediT authorship contribution statement

Luca Anzilli: Conceptualization, Formal analysis, Investigation, Methodology, Software, Visualization, Writing – original draft, Writing – review & editing. **Antonio Farina:** Formal analysis, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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