Published for SISSA by Springer
Received: *July 21, 2025*
Accepted: *August 17, 2025*
Published: *September 24, 2025*

2-loop scattering on superstring and supermembrane in flat space

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Abstract: We consider the S-matrix of transverse scalar excitations on an infinite $D = 10$ GS superstring and $D = 11$ supermembrane in flat target space. We compute the 4-particle scattering amplitude in the 2-loop approximation and demonstrate that, like in the Nambu string case, the $D = 10$ GS string S-matrix does not contain non-trivial 2d UV divergences (UV pole not accompanied by terms with logarithms of momenta is an artifact of dimensional regularization). This is consistent with underlying integrability of this model which is maintaining by adding appropriate local counterterms. In the supermembrane case there are no 1-loop divergences but we find a genuine 2-loop UV pole. This demonstrates non-finiteness of the world-volume S-matrix of the M2 brane theory.

Keywords: AdS-CFT Correspondence, M-Theory

ArXiv ePrint: 2507.09528

[1] Also at ITMP and Lebedev Institute.

https://doi.org/10.1007/JHEP09(2025)191

JHEP09(2025)191

Contents

1 Introduction

Even most-symmetric examples of AdS/CFT duality have their AdS side less well defined than the CFT side. The GS superstring [1] and BST supermembrane [2] theories are formally non-renormalizable even in the case of flat target space and thus require special definition of the corresponding quantum theory.

One may view these 2d and 3d world-volume models just as effective theories that may receive higher-derivative corrections (like what happens in the case of D-brane actions that get α' corrections controlled by open-string dynamics, cf. [3]). This point of view, however, is

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not satisfactory as it is not clear a priori which should be a more fundamental UV-consistent world-volume theory that specifies higher-order counterterms or defines an effective cutoff as these models themselves are expected to describe fundamental degrees of freedom.

Alternatively, one may hope that a large amount of supersymmetry and possible existence of hidden symmetries (in the case of special symmetric backgrounds) may provide enough constraints to define the world-volume quantum theory unambiguously. That may then allow to match its predictions (at least for some special observables) to the corresponding strong-coupling expansion on the CFT side. Indeed, in the case of the 2d GS theory in $AdS_5 \times S^5$ [4] its integrability [5] should provide such stringent constraints.[1] Indeed, the semiclassical 1-loop computations in $AdS_5 \times S^5$ GS theory gave finite consistent results (see, e.g. [10, 11]). What is much more non-trivial is that there are also examples of 2-loop GS string computations in $AdS_5 \times S^5$ [12–16] and $AdS_4 \times CP^3$ [17] which gave finite results in agreement with the integrability predictions.

In the case of the M2 brane theory in AdS backgrounds the semiclassical expansion near non-degenerate 3d world surfaces is also well defined: the 1-loop correction has no log UV divergences and matches predictions on the dual gauge theory side [18–24] (see also [25–29]).

What happens at higher loops is an important open problem. On general grounds, one may expect to find logarithmic UV divergences in M2 brane theory starting at 2-loop level making the results ambiguous. This would be problematic for checks of AdS/CFT duality as the dual gauge theory predictions for the corresponding subleading (in inverse M2 brane tension) terms in relevant observables are finite and unambiguous (cf. discussions in [19, 20]). While there is no analog of 2d integrability in the case of 3d Lorentz-invariant field theories, consistency with AdS/CFT appears to imply that 2-loop UV logs may actually cancel in special observables due to some hidden symmetry of the M2 brane theory yet to be uncovered.

With a motivation to shed light on possible existence of such hidden symmetry ref. [30] considered the M2 brane theory in flat $D = 11$ target space and computed the 1-loop on-shell $2 \to 2$ scattering amplitude of the 8 transverse massless scalars in flat infinite membrane vacuum. The resulting finite expression for the amplitude turned out to be much simpler than the corresponding one [31] in the bosonic membrane case, indicating that the M2 brane theory may have some special features.

Our aim here will be to extend the $D = 11$ M2 brane computation of [30] to the 2-loop level. We shall also find a similar 2-loop correction to the 4-point amplitude in flat $D = 10$ GS string case.

The S-matrix of massless excitations on a long string in flat target space is expected to have a special (elastic, pure phase) form reflecting the integrability of the world-sheet theory [32–35]. This implies that the direct computation of the amplitude should give the expression consistent with the pure-phase form provided one makes a specific choice of local counterterms. This means, in particular, that the amplitude cannot contain non-polynomial logarithm of momentum ($\log s$, etc.) terms associated with divergent UV behaviour.

This is indeed what happens at the 1-loop level in both bosonic string case in any dimension D [32] and in the GS string case in $D = 10$ [30]. The 2-loop amplitude in the

[1] One can draw an analogy with $T\bar{T}$ deformation [6–9] where the assumption of quantum integrability may allow to define a formally non-renormalizable 2d theory by specifying the required counterterms at each order in loop expansion.

bosonic case was computed in [35] and found also not to contain $\log s$ terms. Below we will confirm the result of [35] and extend it to the case of the $D = 10$ GS string with a similar conclusion about consistency with integrability under a specific choice of local counterterms. Like in the Nambu string case the remaining UV poles in the 2-loop GS amplitude are effectively unphysical (or artifacts of dimensional regularization, originating from an evanescent 1-loop counterterm).

In the M2 brane case we will find the non-trivial 2-loop log UV divergences that are accompanied by the associated non-analytic $\log s$ terms. This demonstrates non-renormalizability of the world-volume S-matrix in M2 brane theory in flat $D = 11$ target space. This does not a priori rule out a possibility that some other observables in the M2 brane theory (like partition function near minimal surface for M2 brane in AdS space considered in [19, 20]) may turn out to be UV finite.

Let us first review the known expressions for the 1-loop amplitudes in the (super) string and in the (super) membrane cases. We shall then summarize the 2-loop results of the present paper.

In the bosonic string ($d = 2$) and membrane ($d = 3$) cases, expanding the standard (Nambu-Goto or Dirac) d-dimensional induced volume action in the static gauge one may compute the scattering of massless scalar bosons represented by the $\widehat{D} = D - d$ transverse coordinates X^i. The amplitude for the two incoming bosons with $\mathrm{SO}(\widehat{D})$ indices i_1, i_2 and momenta p_1, p_2, and two outgoing bosons with indices i_3, i_4 and momenta p_3, p_4 has the following general structure

$$\mathcal{M}^{i_1 i_2 i_3 i_4}(s,t,u) = A(s,t,u)\,\delta^{i_1 i_2}\delta^{i_3 i_4} + B(s,t,u)\,\delta^{i_1 i_3}\delta^{i_2 i_4} + C(s,t,u)\,\delta^{i_1 i_4}\delta^{i_2 i_3}\,. \tag{1.1}$$

Here the annihilation A, transmission B and reflection C amplitudes are related by crossing[2]

$$B(s,t,u) = B(u,t,s) = A(t,s,u), \qquad C(s,t,u) = A(u,t,s), \tag{1.2}$$

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 + p_3)^2, \qquad u = -(p_1 + p_4)^2, \qquad s + t + u = 0. \tag{1.3}$$

In general, using loop or inverse tension T expansion one gets for $A(s,t,u)$

$$A = T^{-1} A^{(0)} + T^{-2} A^{(1)} + T^{-3} A^{(2)} + \dots\,, \qquad A^{(0)} = -\tfrac{1}{2} ut\,, \tag{1.4}$$

where $A^{(0)}$ is the tree-level amplitude, $A^{(1)}$ is the 1-loop one, etc. In the string case the massless $d = 2$ kinematics imposes the constraint $stu = 0$ that can be solved by choosing $t = 0$ so that $u = -s$. Then A, B, C become functions of the single variable s and crossing symmetry and the real analyticity requirements imply that

$$C(s) = A^*(-s)\,, \qquad B(s) = B^*(-s)\,, \tag{1.5}$$

where $s \to -s$ is the analytic continuation $s \to e^{i\pi} s$ through the upper half-plane. At the tree level

$$A^{(0)} = C^{(0)} = 0, \qquad B^{(0)} = \tfrac{1}{2} s^2\,. \tag{1.6}$$

[2]Note that in [30–32, 35] the directions of p_3, p_4 momenta were outward, so that t and u were defined with minus signs.

The exact S-matrix of the bosonic string expanded near long string vacuum suggested in [32] corresponds to the pure transmission $A(s) = C(s) = 0$ and is simply given by a pure-phase factor, thus representing an integrable theory[3]

$$\mathbb{S}(s) = 1 + \tfrac{i}{2s}\, B(s) = e^{\frac{is}{4T}} = 1 + \tfrac{is}{4T} - \tfrac{s^2}{32T^2} - \tfrac{is^3}{2\times 192T^3} + \dots\,. \tag{1.7}$$

Here the linear in s term corresponds to the tree-level contribution $B^{(0)}$ in (1.6). At the 1-loop order one finds [32]

$$A^{(1)} = -C^{(1)} = -\tfrac{1}{192\pi}\,(D-26)\, s^3, \qquad B^{(1)} = i\tfrac{1}{16}\, s^3\,. \tag{1.8}$$

$A^{(1)}$ and $C^{(1)}$ can be cancelled [32, 36] by adding a local real *finite* PPS counterterm [37, 38][4]

$$\Delta S_{\rm PPS} = 2b \int R^{(2)} \nabla^{-2} R^{(2)} = -2b \int d^2\sigma\, (\partial^a X^i \partial_a \partial_b X^i)^2 + \dots\,, \quad b = \tfrac{1}{192\pi}\mathrm{b}\,, \quad \mathrm{b} = D - 26\,, \tag{1.9}$$

while the value of $B^{(1)}$ is indeed consistent with the s^2 term in (1.7).

A similar result for the 1-loop massless boson scattering amplitude is found in the case of the $D = 10$ GS string [30] (see also [39])

$$A^{(1)} = -C^{(1)} = \tfrac{1}{16\pi}\, s^3\,, \qquad B^{(1)} = i\tfrac{1}{16} s^3\,. \tag{1.10}$$

Here again $A^{(1)}$ and $C^{(1)}$ can be cancelled by a local counterterm as in (1.9) and the resulting 1-loop S-matrix is the same as in (1.7).

The 2-loop correction to the amplitude (1.1) in the bosonic string was found in [35] using dimensional regularization with $d = 2 - 2\varepsilon$ (including the required 1-loop evanescent counterterm $\int \sqrt{-h}\, R^{(d)}$ that is a total derivative in $d = 2$). The resulting 2-loop corrections to $A(s)$, $B(s)$, $C(s)$ that we will confirm below contain the UV pole parts as well as the finite parts (cf. (1.4), (1.5))

$$A^{(2)}_{\frac{1}{\varepsilon}} = C^{(2)}_{\frac{1}{\varepsilon}} = \tfrac{(D-12)(D-8)}{9216\pi^2\varepsilon}\, s^4, \qquad B^{(2)}_{\frac{1}{\varepsilon}} = -\tfrac{D-8}{768\pi^2\varepsilon}\, s^4\,, \tag{1.11}$$

$$A^{(2)}_{\rm fin}(s) = -i\,\tfrac{D-26}{768\pi}\, s^4 - \tfrac{6D^2-143D+448}{13824\pi^2}\, s^4\,, \quad C^{(2)}_{\rm fin}(s) = i\,\tfrac{D-26}{768\pi}\, s^4 - \tfrac{6D^2-143D+448}{13824\pi^2}\, s^4\,, \tag{1.12}$$

$$B^{(2)}_{\rm fin} = -\tfrac{1}{192} s^4 + \tfrac{11(D+4)}{4608\pi^2}\, s^4\,. \tag{1.13}$$

The imaginary parts of $A^{(2)}$ and $C^{(2)}$ in (1.12) can be cancelled for any D by including the contribution of the 1-loop counterterm (1.9) that was required for 1-loop integrability. Then the pole parts as well as the real $\frac{1}{\pi^2}s^4$ terms in the finite parts in (1.12), (1.13) can be cancelled by a local counterterm [35] of the form (here K is the extrinsic curvature)[5]

$$\int d^2\sigma \sqrt{-h}\, [c_1\, \mathrm{tr}(K^i K^j)^2 + c_2\, \mathrm{tr}(K^i K^i)^2], \qquad K^i_{ab} = \partial_a \partial_b X^i + \dots\,. \tag{1.14}$$

[3]As was shown in [33], using this S-matrix in the thermodynamic Bethe Ansatz one reproduces the expected free bosonic string spectrum.

[4]Here $R^{(2)}$ is the curvature of the induced metric in the static gauge, i.e. $h_{ab} = \eta_{ab} + \partial_a X^i \partial_b X^i$. Cancellation of conformal anomaly in any D is required for preservation of equivalence to the free string spectrum found in conformal gauge.

[5]Note that while $\mathrm{tr}(K^i K^i)^2$ is the square of the Ricci scalar, the first term $\mathrm{tr}(K^i K^j)^2$ cannot be written just in terms of the curvature of the induced metric.

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Then one ends up with

$$A^{(2)} = C^{(2)} = 0\,, \qquad\qquad B^{(2)} = -\tfrac{1}{192}s^4\,, \tag{1.15}$$

which is consistent with the pure-phase S-matrix in (1.7).

This special choice of the counterterms or "scheme" is thus required for preservation of integrability at the 2-loop level.[6] Its existence relies on cancellation of non-analytic $\log s$ terms. The presence of $\frac{1}{\varepsilon}$ poles not accompanied by $\log\frac{s}{\mu^2}$ terms suggests that these do not actually represent genuine logarithmic UV divergences and should be simply subtracted out.[7]

The 2-loop computation in the $D = 10$ GS string case that we shall carry out below gives a similar result: there is $\frac{1}{\varepsilon}$ pole but all $\log s$ terms cancel out and the expression for the amplitude is consistent with 2-particle unitarity. Explicitly, we find[8]

$$A^{(2)}_{\frac{1}{\varepsilon}} = C^{(2)}_{\frac{1}{\varepsilon}} = -\tfrac{1}{1536\pi^2\,\varepsilon}\,s^4, \qquad\qquad B^{(2)}_{\frac{1}{\varepsilon}} = -\tfrac{1}{128\pi^2\,\varepsilon}\,s^4\,, \tag{1.16}$$

$$A^{(2)}_{\rm fin}(s) = i\,\tfrac{1}{64\pi}\,s^4 + \tfrac{145}{9216\pi^2}\,s^4\,, \qquad\qquad C^{(2)}_{\rm fin}(s) = -i\,\tfrac{1}{64\pi}\,s^4 + \tfrac{145}{9216\pi^2}\,s^4\,, \tag{1.17}$$

$$B^{(2)}_{\rm fin} = -\tfrac{1}{192}s^4 + \tfrac{25}{768\pi^2}\,s^4\,. \tag{1.18}$$

Like in the bosonic string case, the imaginary parts in $A^{(2)}$ and $C^{(2)}$ are again cancelled by the contribution of the finite 1-loop PPS counterterm (1.9) that was needed to remove the non-vanishing $A^{(1)}$ and $C^{(1)}$ in (1.10). Then the remaining real s^4 terms can be cancelled by a local counterterm like (1.14) so that at the end we get the same integrability-consistent expression (1.15) as in the bosonic string case.

Let us now turn to the membrane case. Here the momentum invariants are subject only to $s+t+u=0$ but otherwise generic and the amplitudes B and C are related to A by (1.2) so it is enough to present just $A(s,t,u)$. For the bosonic membrane in general dimension D, one finds that the 1-loop correction to the massless scalar tree-level amplitude A in (1.4) is given by the following finite expression [31]

$$A^{(1)} = \tfrac{1}{256}\Big[(-s)^{3/2}\Big(\tfrac{3D-41}{32}s^2 - \tfrac{D-3}{4}tu\Big) + (-t)^{5/2}(3t+2s) + (-u)^{5/2}(3u+2s)\Big]\,. \tag{1.19}$$

[6] The need to introduce special counterterms may be related to dimensional regularization not preserving higher conserved charges manifestly (indeed, integrability is expected to be present only in $d = 2$ theory). One may wonder if there may exist an alternative approach (e.g. based on interpreting the Nambu action in the static gauge as a $T\bar{T}$ deformation of a free scalar theory, cf. [8, 9] and using conformal perturbation theory) in which preservation of integrability may be more manifest. We thank R. Tateo for a related comment.

[7] In particular, the cancellation of $\log\mu$ terms means the absence of ambiguous scheme-dependent coefficients in the finite part which allows one to maintain the underlying symmetry. The remaining poles may be viewed as an artifact of dimensional regularization that requires the introduction of an evanescent counterterm which produces extra $\frac{1}{\varepsilon}$ poles not accompanied by $\log s$ terms. Examples when the coefficients of $\frac{1}{\varepsilon}$ poles in dimensional regularization are not correlated with those of $\log\frac{s}{\mu^2}$ terms were discussed in 4d case in [40, 41]. The model considered in [41] was the $\mathcal{N} = 1$ supergravity with one matter multiplet where the 2-loop UV poles are present but do not have any physical consequences since all momentum logarithms cancel out. This mismatch is again due to the effect of evanescent operator (here the Gauss-Bonnet term). As was shown in [41], the absence of $\log\mu^2$ terms related to $\log s$ terms can be seen directly using 4d unitarity cuts ($\log\mu^2$ term itself is not associated to a cut). This approach avoids the need for an ultraviolet regularization and thus for the introduction of evanescent operators.

[8] Considering GS model in general dimension D one finds that $\log s$ terms cancel only for $D = 10$.

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As was found in [30], the M2 brane analog of (1.19) takes particularly simple form for $D = 11$

$$A^{(1)} = \tfrac{1}{32}\big[(-s)^{3/2} + (-t)^{3/2} + (-u)^{3/2}\big]\, A^{(0)}\,, \qquad\qquad A^{(0)} = -\tfrac{1}{2}tu\,. \tag{1.20}$$

As we will show below the 2-loop correction to the $D = 11$ M2 brane amplitude computed in dimensional regularization with $d = 3 - 2\varepsilon$ is given by

$$\begin{aligned} A^{(2)} = \Big[\Big(-\tfrac{1}{256\pi^2\varepsilon} + \tfrac{3}{2048}\Big)\, stu + \tfrac{1}{384\pi^2}\big[s^3\log(-\tfrac{s}{\mu^2}) + t^3\log(-\tfrac{t}{\mu^2}) + u^3\log(-\tfrac{u}{\mu^2})\big]\Big]\; tu \\ + \tfrac{1}{80640\pi^2}\, s(6s^4 + 124s^3t - 1833s^2t^2 - 3914st^3 - 1957t^4)\,. \end{aligned} \tag{1.21}$$

This expression is simpler than the corresponding one in the bosonic membrane case (see (2.29), (2.30)) but still contains the UV pole.[9] Here it is accompanied by the corresponding $\log s$-like terms and thus represents a genuine logarithmic UV divergence. Some coefficients in the second line in (1.21) are scheme-dependent.[10] Note that like in (1.20) the scheme-independent part of (1.21) is again proportional to the tree-level amplitude $A^{(0)} = -\frac{1}{2}tu$.

The presence of the pole indicates non-renormalizability of the M2 brane theory.[11] At the moment there is no indication of some hidden symmetry that would guide the definition of this theory. One may speculate (cf. [30]) that after adding some specific counterterms to remove the pole and part of rational finite part of (1.21) to make the whole amplitude (1.4) look as $A(s,t,u) = F(s,t,u)A^{(0)}$ where $F = 1 + T^{-1}F^{(1)} + T^{-2}F^{(2)} + \ldots$ is a totally symmetric function, the latter may "exponentiate" by analogy with the 2d case in (1.7).

In section 2 we shall discuss the 2-loop S-matrix for the bosonic string and the membrane. We shall reproduce the string 2-loop amplitude in [35] clarifying the origin of the remaining UV pole term and find the analogous result for the bosonic membrane.

In section 3 we shall present the expansion of the GS string and the M2 brane actions in static gauge to quartic order in bosons and fermions fixing a particular κ-symmetry gauge in which there is no cubic interaction vertices (generalizing the expressions in [30] to quartic fermionic terms). This will be the starting point for computing 4-scalar 2-loop amplitudes.

In section 4 and 5 we shall compute the 2-loop GS string and M2 brane amplitudes treating both cases in parallel. We shall find extra fermionic loop contributions that should be added to the bosonic loop expressions in section 2. In the GS case there will be extra 1-loop counterterm contributions related to two fermion propagators in the loop or on the external lines.

In appendix A we will present details of the expansion of the M2 brane and GS string actions. Appendix B will contain some useful fermionic traces and 1-loop integrals. In

[9]Note that the double-pole terms cancel out which is a consequence of the 1-loop finiteness in $d = 3$.

[10]This scheme dependence is due to the well-known ambiguity in treating products of 3d Levi-Civita tensor in dimensional regularization. An alternative approach using dimensional reduction regularization [42] is beyond the scope of the present work.

[11]One could try to argue that supersymmetry and special structure of the GS and M2 actions may imply that despite being power-counting non-renormalizable they may represent UV finite theories. Indeed, the WZ terms in the actions should not be renormalized but then κ-symmetry may relate it to the volume part of the action. This argument implies only non-renormalization of the tension but does not a priori exclude the presence of higher supersymmetric and κ-symmetric invariants (κ-symmetry may also be deformed at the quantum level). An attempt to argue for finiteness of M2 brane theory was in [43]. Note also that possible higher derivative corrections to M2 brane action were discussed in [44, 45].

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appendix C we will review few relations for the extrinsic curvature and the scalar curvature for the induced world-volume metric. Appendix D will contain details of the computation of 2-loop diagrams in the bosonic case.

2 Massless scattering on bosonic string and membrane

We shall start with the bosonic case treating the string and membrane case in parallel. We will first review the results for the 1-loop string [32] and membrane [31] amplitudes. We will then reproduce the 2-loop contribution [35] to the string amplitude and generalize it to the membrane case.

2.1 Classical action

A brane moving in a D-dimensional target space has the induced volume action

$$S = -T \int d^d\sigma \, \sqrt{-\det h_{ab}}, \qquad h_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{2.1}$$

where $a, b = 0, 1, \ldots, d-1$ and $\mu, \nu = 0, 1, \ldots, D-1$. We shall consider its expansion near an infinite flat brane vacuum and fix the static gauge $X^a = \sigma^a$ so that the induced metric is

$$h_{ab} = \eta_{ab} + \partial_a X^i \partial_b X^i, \qquad\qquad i = 1, \ldots, \widehat{D}, \qquad \widehat{D} = D - d. \tag{2.2}$$

Expanding in powers of derivatives gives (dropping constant term)

$$S = -T \int d^d\sigma \Big[\, \tfrac{1}{2}\partial^a X^i \partial_a X^i + \tfrac{1}{8}(\partial^a X^j \partial_a X^j)^2 - \tfrac{1}{8}(\partial_a X^i \partial_b X^i)(\partial^a X^j \partial^b X^j) + \mathcal{O}((\partial X)^6)\Big]. \tag{2.3}$$

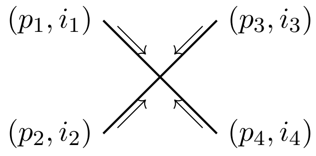
We shall not need $(\partial X)^6$ terms as they will not contribute non-trivially to 2-loop 4-point scattering amplitude (such terms may be relevant only for tadpole contributions that vanish in dimensional regularization that we will use).

After rescaling $X^i \to \frac{1}{\sqrt{T}} X^i$, the factors of inverse tension T^{-1} will appear in the interaction vertices and thus in the corresponding scattering amplitudes.

2.2 Tree amplitude

Starting with (2.3) the expression for the scattering amplitude $X^{i_1}(p_1)X^{i_2}(p_2) \to X^{i_3}(p_3)X^{i_4}(p_4)$ follows from the quartic vertex

$V^{i_1 i_2 i_3 i_4}(p_1, p_2, p_3, p_4) =$



$$\begin{aligned} &= -i\Big[(p_1 \cdot p_2\, p_3 \cdot p_4 - p_1 \cdot p_4\, p_2 \cdot p_3 - p_1 \cdot p_3\, p_2 \cdot p_4)\, \delta^{i_1 i_2}\delta^{i_3 i_4} \\ &\quad + (p_1 \cdot p_3\, p_2 \cdot p_4 - p_1 \cdot p_4\, p_2 \cdot p_3 - p_1 \cdot p_2\, p_3 \cdot p_4)\, \delta^{i_1 i_3}\delta^{i_2 i_4} \\ &\quad + (p_1 \cdot p_4\, p_2 \cdot p_3 - p_1 \cdot p_3\, p_2 \cdot p_4 - p_1 \cdot p_2\, p_3 \cdot p_4)\, \delta^{i_1 i_4}\delta^{i_2 i_3}\Big]. \end{aligned} \tag{2.4}$$

The resulting tree amplitude is (cf. (1.1), (1.4))

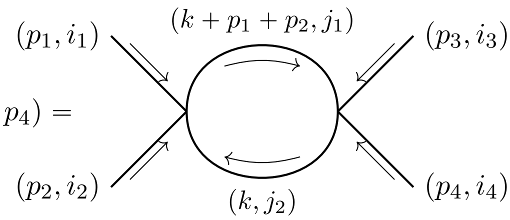
$$\mathcal{M}^{(0)\,i_1i_2i_3i_4} = \tfrac{1}{i}V^{i_1i_2i_3i_4} = -\tfrac{1}{2}tu\,\delta^{i_1i_2}\delta^{i_3i_4} - \tfrac{1}{2}su\,\delta^{i_1i_3}\delta^{i_2i_4} - \tfrac{1}{2}st\delta^{i_1i_4}\delta^{i_2i_3}. \tag{2.5}$$

It has the same universal form for any d, simplifying in the string case (under the $t = 0$ choice) to (1.6).

2.3 1-loop amplitude

The non-trivial 1-loop diagram is

$$D^{i_1i_2i_3i_4}(p_1, p_2, p_3, p_4) = \tag{2.6}$$



The corresponding amplitude is given by the sum over crossing[12]

$$\mathcal{M}^{(1)\,i_1i_2i_3i_4}(s,t,u) = D^{i_1i_2i_3i_4}(s,t,u) + D^{i_1i_3i_4i_2}(t,u,s) + D^{i_1i_4i_2i_3}(u,s,t). \tag{2.7}$$

In dimensional regularization

$$D^{i_1i_2i_3i_4}(s,t,u) = -\tfrac{1}{2i}\int \frac{d^dk}{(2\pi)^d}\, \frac{V^{i_1i_2j_1j_2}(p_1,p_2,-k-p_1-p_2,k)\,V^{j_1j_2i_3i_4}(k+p_1+p_2,-k,p_3,p_4)}{k^2\ (k+p_1+p_2)^2}, \tag{2.8}$$

where V is given in (2.4). This integral may be evaluated in the standard way by introducing Feynman parameters and doing the loop integration (see appendix B) or by reducing it to simpler integrals using the Integration By Parts (IBP) and tensor reduction [46–49] implemented in the automated Mathematica software FIRE [50–54] that we will use also below. The result is ($s = -2p_1 \cdot p_2$)

$$D^{i_1i_2i_3i_4}(s,t,u) = -\frac{i\,s^2}{16(d-1)}G_{1,1}(s)\left[\frac{(d-2)[d(\widehat{D}-8)-8]\,s^2 - 8\widehat{D}\,tu}{8(d+1)}\,\delta^{i_1i_2}\delta^{i_3i_4} \right.$$
$$\left. + s(ds+2t)\delta^{i_1i_3}\delta^{i_2i_4} + s(ds+2u)\delta^{i_1i_4}\delta^{i_2i_4}\right], \tag{2.9}$$

$$G_{1,1}(s) \equiv \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k+p_1+p_2)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)\left[\Gamma\left(\frac{d}{2}-1\right)\right]^2}{\Gamma(d-2)}(-s)^{\frac{d}{2}-2}. \tag{2.10}$$

In the 2d string case we are to set $d = 2 - 2\varepsilon$ and $\widehat{D} = D - 2$.[13] Then

$$G_{1,1}(s) = \frac{i}{2\pi\, s}\,\frac{1}{\varepsilon} + \dots\,, \tag{2.11}$$

[12] We set $D^{i_1i_2i_3i_4}(s,t,u) \equiv D^{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4)$.

[13] Note that while we introduced in general $\widehat{D}$ as $D - d$ we are not continuing d there away from 2 or 3 as that would break the target-space supersymmetry (the number of bosons will change, but the number of fermionic components is always the same). Thus in the general expressions we set $\widehat{D} = D - 2$ in the string case and $\widehat{D} = D - 3$ in the membrane case.

and we get the following UV pole contribution to (2.7)

$$\mathcal{M}^{(1)\,i_1i_2i_3i_4}_{\frac{1}{\varepsilon}}(s,t,u) = -\tfrac{1}{96\pi\,\varepsilon}\,(D-8)\,stu\,(\delta^{i_1i_2}\delta^{i_3i_4}+\delta^{i_1i_3}\delta^{i_2i_4}+\delta^{i_1i_4}\delta^{i_2i_3})\,. \tag{2.12}$$

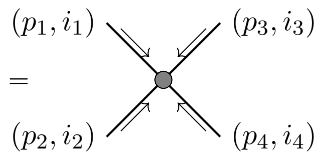
This pole represents the standard UV divergences so that the finite part contains also the corresponding $stu\log s$ term [31, 32]. These contributions vanish in $d=2$ due to the kinematic $stu=0$ relation and then the finite part is given by (1.8).

Keeping d general in the numerator of (2.12) the pole term (2.12) can be cancelled by adding an "evanescent" scalar curvature counterterm which is trivial only in $d=2$

$$\delta S = \kappa\int d^d\sigma\,\sqrt{-h}\,R^{(d)} = \kappa\int d^d\sigma\,\Big[(\partial_a\partial_b X^i\,\partial_c X^i)\,(\partial^a\partial^c X^j\,\partial^b X^j)+\dots\Big]\,, \qquad \kappa=-\frac{1}{48\pi\,\varepsilon}\mathrm{c}\,. \tag{2.13}$$

Here we ignored the contribution $\sim\partial^2 X$ that vanishes on the free equation of motion (see appendix C.2).[14] The corresponding vertex is

$$E^{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4) =$$



$$\begin{aligned}
&= -2i\kappa(p_1\cdot p_3\,p_1\cdot p_4\,p_2\cdot p_3 + p_1\cdot p_3\,p_1\cdot p_4\,p_2\cdot p_4 + p_1\cdot p_3\,p_2\cdot p_3\,p_2\cdot p_4 + p_1\cdot p_4\,p_2\cdot p_3\,p_2\cdot p_4)\delta^{i_1i_2}\,\delta^{i_3i_4}\\
&\quad - 2i\kappa(p_1\cdot p_2\,p_1\cdot p_4\,p_2\cdot p_3 + p_1\cdot p_2\,p_1\cdot p_4\,p_3\cdot p_4 + p_1\cdot p_2\,p_2\cdot p_3\,p_3\cdot p_4 + p_1\cdot p_4\,p_2\cdot p_3\,p_3\cdot p_4)\delta^{i_1i_3}\,\delta^{i_2i_4}\\
&\quad - 2i\kappa(p_1\cdot p_2\,p_1\cdot p_3\,p_2\cdot p_4 + p_1\cdot p_2\,p_1\cdot p_3\,p_3\cdot p_4 + p_1\cdot p_2\,p_2\cdot p_4\,p_3\cdot p_4 + p_1\cdot p_3\,p_2\cdot p_4\,p_3\cdot p_4)\delta^{i_1i_4}\delta^{i_2i_3}.
\end{aligned} \tag{2.14}$$

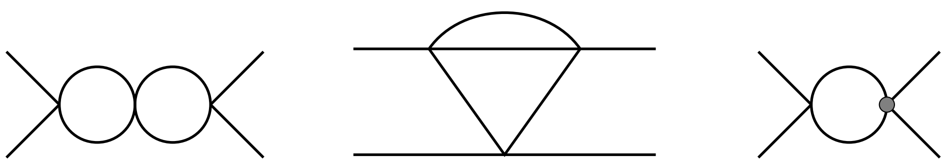
Its contribution cancels the pole in (2.12) if we choose (cf. [35] and also footnote 5 in [31])

$$\mathrm{c} = D-8\,. \tag{2.15}$$

In the membrane case, setting $d=3-2\varepsilon$ and $\widehat{D}=D-3$ in (2.9), we automatically get a finite expression [30] given in (1.19), so that no "evanescent" counterterm is needed here.

2.4 2-loop amplitude: string

Since the expansion in (2.3) starts with quartic vertex, non-trivial contributions to the 2-loop amplitude may come from the following diagrams[15]



"Double-bubble" "Wine-glass" "Counterterm" (2.16)

[14]The term in (2.13) proportional to $\partial^2 X$ may be ignored also when inserting it into 2-loop counterterm diagram (third in (2.16)) as it gives extra tadpole contribution that vanishes in dimensional regularization. In general, there may be contributions to higher point functions that may be accounted for by replacing $\partial^2 X$ by X^2 terms using equations of motion (or doing field redefinitions).

[15]As was already mentioned, we are ignoring massless tadpole diagrams as they vanish in dimensional regularization.

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Detailed evaluation of these 2-loop diagrams is presented in appendix D. The last diagram in (2.16) represent the insertion of the 1-loop counterterm (2.13), i.e. its contribution is proportional to c.

The $\frac{1}{\varepsilon^2}$ double pole terms that may come only from the first two diagrams in (2.16) cancel out, reflecting the fact that there are no $\frac{1}{\varepsilon}$ divergences at 1-loop level in $d=2$ (this is true even off shell as the candidate $\int R^{(2)}$ counterterm (2.13) is trivial directly in $d=2$ so that there are no $\frac{1}{\varepsilon}$ sub-divergences).[16]

The simple pole contributions to the amplitude are found to be as in [35] (assuming as usual $t=0$ choice of 2d kinematics)

$$A^{(2)}_{\frac{1}{\varepsilon}} = C^{(2)}_{\frac{1}{\varepsilon}} = -\frac{1}{9216\pi^2\,\varepsilon}\,(D-12)(D-8-2\mathrm{c})\,s^4, \qquad B^{(2)}_{\frac{1}{\varepsilon}} = \frac{1}{768\pi^2\,\varepsilon}\,(D-8-2\mathrm{c})\,s^4. \tag{2.17}$$

For the finite parts we find (we absorb $4\pi e^{-\gamma_E}$ into the normalization factor μ^2)

$$\begin{aligned} A^{(2)}_{\rm fin}(s) =& -\frac{1}{13824\pi^2}\left[1024-255D+11D^2+(72-5D)\mathrm{c}\right]\, s^4 + i\frac{1}{768\pi}(34-2D+\mathrm{c})\,s^4 \\ &+\frac{1}{4608\pi^2}(D-12)(D-8-\mathrm{c})\; s^4\log(-\tfrac{s}{\mu^2}), \end{aligned} \tag{2.18}$$

$$\begin{aligned} C^{(2)}_{\rm fin}(s) =& -\frac{1}{13824\pi^2}\left[1024-255D+11D^2+(72-5D)\mathrm{c}\right] s^4 - i\frac{1}{4608\pi}\left[108+8D-D^2+(D-6)\mathrm{c}\right] s^4 \\ &+\frac{s^4}{4608\pi^2}(D-12)(D-8-\mathrm{c})\; s^4\log(-\tfrac{s}{\mu^2}), \end{aligned} \tag{2.19}$$

$$\begin{aligned} B^{(2)}_{\rm fin}(s) =& -\frac{1}{192}\,s^4 - \frac{1}{4608\pi^2}(148-35D+24\mathrm{c}) - i\frac{1}{768\pi}(D-8-\mathrm{c})\,s^4 \\ &-\frac{1}{384\pi^2}(D-8-\mathrm{c})\; s^4\log(-\tfrac{s}{\mu^2}). \end{aligned} \tag{2.20}$$

For $\mathrm{c}=D-8$ all $\log s$ terms cancel out but the pole terms survive and we get the expressions in (1.11)(1.12), (1.13) [35], i.e.

$$A^{(2)}_{\frac{1}{\varepsilon}} = C^{(2)}_{\frac{1}{\varepsilon}} = \frac{1}{9216\pi^2\,\varepsilon}\,(D-12)(D-8)\,s^4, \qquad B^{(2)}_{\frac{1}{\varepsilon}} = -\frac{1}{768\pi^2\,\varepsilon}\,(D-8)\,s^4, \tag{2.21}$$

$$A^{(2)}_{\rm fin}(s) = [C^{(2)}_{\rm fin}(-s)]^* = -i\,\frac{1}{768\pi}\,(D-26)\,s^4 - \frac{1}{13824\pi^2}\,(6D^2-143D+448)\,s^4, \tag{2.22}$$

$$B^{(2)}_{\rm fin} = -\frac{1}{192}s^4 + \frac{11}{4608\pi^2}\,(D+4)\,s^4\,. \tag{2.23}$$

Note that for $\mathrm{c}=0$ the coefficients of the pole terms in (2.17) are the same as of the $\log s$ terms in (2.18)–(2.20). That means that the pole contributions of the genuine 2-loop diagrams (the first two in (2.16)) represent the standard log UV divergences, like that was the case for the 1-loop pole terms in (2.12). At the same time, half of the pole terms among those proportional to c which come from the evanescent counterterm diagram and are not accompanied by the $\log s$ terms should have a different nature and thus their presence may be viewed as an artifact of dimensional regularization (cf. [40, 41]). They should be just subtracted as they have no physical consequences.[17]

[16] In general, the coefficient of $\frac{1}{\varepsilon^2}$ is proportional to stu (like the 1-loop pole term in (2.12)) and thus vanishes due to 2d kinematics [35].

[17] In more detail, the counterterm contribution is given by the $\frac{1}{\varepsilon}$ factor in (2.13) (which has 1-loop UV origin) times the combination $J=(d-2)(sG_{1,1}+G_{1,0}+G_{0,1})$ where the first integral $G_{1,1}$ is the contribution of the bubble diagram (2.10) and the other two are the tadpoles (we assume they are IR-regularized by

The pole terms in (2.17) or (2.21) may be canceled by adding a counterterm (1.14) whose contribution to the amplitudes are given by (as usual, here $t = 0$)

$$\Delta A^{(2)} = \Delta C^{(2)} = \tfrac{1}{4}(c_1 + 2c_2)\, s^4, \qquad \Delta B^{(2)} = \tfrac{1}{2}c_1\, s^4\,. \tag{2.24}$$

Thus to cancel pole terms in (2.21) we need to choose the constants in (1.14) as

$$c_1 = \tfrac{1}{384\pi^2\varepsilon}(D-8)\,, \qquad c_2 = -\tfrac{1}{4608\pi^2\varepsilon}(D-6)(D-8)\,. \tag{2.25}$$

The cancellation of $\log s$ terms that is possible due to the inclusion of the contribution of the evanescent counterterm (2.13) is crucial for preservation of integrability. Indeed, for $D = 26$ the imaginary part in (2.22) vanishes and then all the real s^4 terms in (2.21)–(2.23) apart from $-\frac{1}{192}s^4$ in $B^{(2)}$ required for consistency with (1.7), (1.15) may be removed by adding a specific real local counterterm [35] (see below).

To preserve integrability for generic D we need also to take into account the *finite* counterterm (1.9) that was required for integrability for any D at the 1-loop order. Its contribution to 2-loop diagram is again given by the third diagram in (2.16) with the vertex coming from (1.9) given by

$$\begin{aligned} \mathrm{V}^{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4) = 4ib\Big[&p_1\cdot p_2\, p_3\cdot p_4 (p_1+p_2)\cdot(p_3+p_4)\delta^{i_1i_2}\delta^{i_3i_4} \\ &+ p_1\cdot p_3\, p_2\cdot p_4 (p_1+p_3)\cdot(p_2+p_4)\delta^{i_1i_3}\delta^{i_2i_4} \\ &+ p_1\cdot p_4\, p_2\cdot p_3 (p_1+p_4)\cdot(p_2+p_3)\delta^{i_1i_4}\delta^{i_2i_3}\Big]\,. \end{aligned} \tag{2.26}$$

Its tree-level contribution changes the 1-loop amplitudes by

$$\Delta A^{(1)} = -\Delta C^{(1)} = b\, s^3, \qquad \Delta B^{(1)} = 0\,. \tag{2.27}$$

Choosing $b = \frac{D-26}{192\pi}$ we cancel the $D-26$ terms in (1.8). Its contribution to the 1-loop counterterm diagram in (2.16) is finite and changes the finite part of the 2-loop amplitudes in (2.22), (2.23) as (see appendix D.4)

$$\begin{aligned} &\Delta A^{(2)} = \tfrac{1}{4}ib\, s^4 + \tfrac{1}{48\pi}(6D-37)\, b\, s^4\,, \qquad \Delta C^{(2)} = -\tfrac{1}{4}ib\, s^4 + \tfrac{1}{48\pi}(6D-37)\, b\, s^4\,, \\ &\Delta B^{(2)} = \tfrac{1}{6\pi}b\, s^4\,. \end{aligned} \tag{2.28}$$

introducing a mass m). The $\log m$ IR divergence of the bubble diagram cancels against the $\log m$ terms in the tadpoles, and the remainder is a $\frac{1}{d-2}$ UV divergence of the tadpoles. $J \sim \varepsilon \times \frac{1}{\varepsilon}$ is then finite, and we get just a simple UV pole factor remaining. The reason why part of the $\frac{1}{\varepsilon}$ poles in the counterterm contribution are not accompanied by $\log s$ terms can be understood also by tracing the dependence on the dimensional regularization scale parameter μ. The factor of $\mu^{-2\varepsilon}$ is introduced to compensate for the dimension of each $\int d^d\sigma = \int d^{2-2\varepsilon}\sigma$ or each factor of T and thus its inverse appears in the vertices that are multiplied by T^{-1} factors. In general, at L-loop order we then get from a simple pole factor $\frac{1}{\varepsilon}\mu^{2L\varepsilon} = \frac{1}{\varepsilon} + L\log\mu^2 + \ldots$. Thus for $L = 2$ there is a factor of 2 difference between the coefficient of $\frac{1}{\varepsilon}$ and of $\log\mu^2$, in agreement with what one has for the c-independent terms in (2.17)–(2.20). At the same time, the contribution of the 1-loop counterterm (2.13) contains no $\mu^{2\varepsilon}$ factor, i.e. there is just one factor of $\mu^{-2\varepsilon}$ coming from the left 4-vertex (2.4) in the last diagram in (2.16) (the one from the Nambu action (2.3)). The dependence on μ is correlated with the dependence on s in (2.18)–(2.20) and that explains why $\frac{1}{\varepsilon}$ factor still survives after all $\log s$ terms are cancelled out due to choosing $\mathrm{c} = D-8$.

These contributions cancel the imaginary parts in $A^{(2)}$ and $C^{(2)}$ and the remaining real s^4 terms in them can be eliminated by adding the local counterterm (1.14) with finite c_1 and c_2 (cf. (2.24)).[18]

2.5 2-loop amplitude: membrane

Starting with the general expressions for the two 2-loop diagrams in (2.16) we may set $d = 3 - 2\varepsilon$ and $\widehat{D} = D - 3$ (for $d = 3$ there is no 1-loop counterterm contribution). We find that the first double-bubble diagram is finite, while the wine-glass diagram gives the following single UV pole contribution (double-pole terms cancel)

$$A^{(2)}_{\frac{1}{\varepsilon}}(s,t,u) = -\frac{1}{322560\pi^2\varepsilon}\Big[(27D+455)\,st^3(2s+t) + 3(D-3)\,s^3(4s^2+st+10t^2) \\ + 8s^3(-8s^2+4st+71t^2)\Big], \tag{2.29}$$

with $B^{(2)}$ and $C^{(2)}$ related to $A^{(2)}$ as in (1.2). The finite part contains log momentum terms that accompany the pole terms in (2.29), i.e. the latter represent genuine log UV divergences. We find explicitly

$$A^{(2)}_{\rm fin}(s,t,u) = \\ \frac{1}{161280\pi^2}\Big\{s^3\big[3(7D-25)s^2 - 2(9D-17)(t^2+u^2)\big]\log(-\tfrac{s}{\mu^2}) \\ -t^4\big[s(250-6D)+t(191+3D)\big]\log(-\tfrac{t}{\mu^2}) + u^4\big[t(191+3D)+s(-59+9D)\big]\log(-\tfrac{u}{\mu^2})\Big\} \\ -\frac{1}{45158400\pi^2}\,s\Big\{\big[2(8869D-97807)tu + 8(1027D-8745)s^2\big](t^2+u^2) + (57777D+74797)t^2u^2\Big\} \\ +\frac{1}{4194304}\,s\Big\{\big[2(-289-138D+7D^2)tu + 3(617-38D+D^2)s^2\big](t^2+u^2) \\ +4(1055-138D+7D^2)t^2u^2\Big\}. \tag{2.30}$$

One can check that the total sum of coefficients of $\log\mu^2$ here is twice that of the $\frac{1}{\varepsilon}$ term in (2.29) (cf. footnote 17).

The presence of the UV divergence in the 2-loop S-matrix (2.29) demonstrates the non-renormalizability of the bosonic membrane theory. While this divergence may be eliminated by a local $\partial^{10}X^4$ counterterm there is no known underlying principle (like integrability in the $d = 2$ string case) that would fix the coefficients of the remaining finite terms in the amplitude order by order in loop expansion.

In general, starting with the brane action (2.1) that has the coupling given by (inverse) tension T that has mass dimension d one may classify possible counterterms expected at L-loop order. They should be covariant, i.e. should be built out the extrinsic curvature and its covariant derivatives (see, e.g., [35, 36, 55] and appendix C.1)[19] Assuming X^i have the same dimension (of length) as σ^a so that $\partial_a X^i$ and h_{ab} are dimensionless, a local logarithmically

[18] Note that the signs in (2.27) and (2.28) are consistent with (1.5). Also, the non-trivial part of B that should be consistent with (1.7) has imaginary 1-loop s^3 term but has real 2-loop s^4 term.

[19] Using the Gauss-Codazzi relation in the case of flat target space (C.5), one can express the Riemann tensor of the induced metric h_{ab} in terms of the extrinsic curvature, see (C.5).

divergent term at the L loop order contribution to the quantum effective action of the theory (2.1) should have the following structure

$$T^{1-L}\sum_n \int d^d\sigma\ \partial^{d\,L}(\partial X)^{2n}\,. \tag{2.31}$$

Here $d\,L$ derivatives should be distributed between $2n$ factors of X^i. The 4-point scattering amplitude probes the terms with $n=2$, i.e. with the integrand $\partial^{d\,L+4}X^4$ corresponding to $s^{\frac{1}{2}d\,L+2}$, etc., in (1.1). Such terms may originate from two covariant structures built out of the extrinsic curvature K^i_{ab} and covariant derivative (cf. (1.14))

$$I_1=\sqrt{-h}\,\nabla^{dL-2}KK, \qquad\qquad I_2=\nabla^{dL-4}KKKK\,. \tag{2.32}$$

In I_1 we should pick up the leading $\partial^2 X$ term from the expansion of one factor of K and the subleading ∂^4X^3 term from the second factor (cf. appendix C.1). In I_2 we should pick up just the leading ∂^2X term in each of the 4 factors of K.

In the string case ($d=2$) at the 1-loop order the only possibility is $\sqrt{-h}\,KK$, or, equivalently $\sqrt{-h}R$ (cf. on-shell relation (C.6)) which is trivial being a total derivative. At the 2-loop order we may have $\sqrt{-h}\,\nabla^2KK$ and $\sqrt{-h}\,KKKK$. The first invariant reduces to the second one at the X^4 order[20] so that we end up with the same counterterm (1.14) as in [35].

In the membrane case ($d=3$) there is no candidate counterterm (2.31) at $L=1$ order (in general, for odd d there are no relevant invariants at odd number of loops).[21] At $L=2$ we have the two candidate invariants in (2.32), i.e. $\sqrt{-h}\,\nabla^4KK$ or $\sqrt{-h}\,\nabla^2KKKK$. Again, we can reduce the first one to the second (restricting to on-shell X^4 vertex).[22] The divergent part in (2.29) is reproduced by the following counterterm (here $\partial_{ab\ldots c}\equiv\partial_a\partial_b\ldots\partial_c$ and repeated 3d indices are contracted by η_{ab})

$$-\tfrac{1}{80640\pi^2\varepsilon}(27D+455)\partial_{ab}X^i\partial_{ac}X^j\partial_{de}X^j\partial_{debc}X^i-\tfrac{1}{10080\pi^2\varepsilon}(3D-25)\partial_{ab}X^i\partial_{be}X^j\partial_{acd}X^i\partial_{ecd}X^j$$
$$+\tfrac{1}{161280\pi^2\varepsilon}(3D-191)\partial_{ab}X^i\partial_{ac}X^i\partial_{de}X^j\partial_{debc}X^j\,, \tag{2.33}$$

where we ignore terms with ∂^2X that vanish on-shell. A terms like $\partial_{ab}X^i$ come from the leading term in the expansion of K^i_{ab}, while terms like $\partial_{debc}X^i$ may come from both $\nabla_d\nabla_eK^i_{bc}$ and $\nabla_b\nabla_cK^i_{de}$, etc. Note that the corresponding covariant $\nabla K\nabla KKK$ counterterm in the effective action will cancel also log divergences present in higher-point 2-loop scattering amplitudes.

3 Superstring and supermembrane: classical action

Let us now consider similar scattering of massless scalars for $D=10$ GS string and $D=11$ M2 brane. The tree-level amplitude is the same as in the bosonic case (2.5) while the 1-loop correction was computed in [30].

[20] From (C.10) we see that we only need to consider $\nabla^aK^i_{bc}\nabla_a(K^i)^{bc}$. Using (C.9) this is same as $\nabla^cK^i_{ab}\nabla^a(K^i)_c{}^b+$ $KKKK$ terms where the first term vanishes upon integration by parts.

[21] We are assuming that parity invariance is preserved, i.e. that one cannot use odd number of ϵ^{abc} to contract the indices.

[22] One may commute derivatives until they hit K getting $\nabla K\nabla KKK$ term (commutator also gives such term). Equivalently, we may first assume that all derivatives are flat ones and use that $\partial^a\partial_aX\to 0$ on-shell. Then replacing derivatives by the covariant ones makes them non-commuting but difference is just extra KK or curvature terms.

Our aim will be to find the corresponding 2-loop amplitude. The total amplitude will be given by the bosonic contribution from the previous section plus the contribution from the loops involving fermions. The latter will be our main focus in what follows.

Like in [30] it will be convenient to treat the GS string and the M2 brane cases in parallel given that they are directly related by the double dimensional reduction [56]. The supermembrane action in flat target space is given by [2, 57][23]

$$S = S_1 + S_2, \qquad S_1 = -T\int d^3\sigma\,\sqrt{-\det g}\,, \tag{3.1}$$

$$S_2 = -T\int d^3\sigma\,\tfrac{i}{2}\varepsilon^{abc}\,\bar\theta\Gamma_{\mu\nu}\,\partial_a\theta\,\big(\Pi^\mu_b\Pi^\nu_c + i\,\Pi^\mu_b\,\bar\theta\Gamma^\nu\partial_c\theta - \tfrac{1}{3}\bar\theta\Gamma^\mu\partial_b\theta\,\bar\theta\Gamma^\nu\partial_c\theta\big)\,, \tag{3.2}$$

$$g_{ab} = \eta_{\mu\nu}\Pi^\mu_a\Pi^\nu_b, \qquad \Pi^\mu_a = \partial_a X^\mu - i\bar\theta\Gamma^\mu\partial_a\theta, \qquad \bar\theta = \theta^\dagger\Gamma^0, \qquad \{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3.3}$$

The first term in the action S_1 is manifestly invariant under the global supersymmetry $\delta X^\mu = i\bar\varepsilon\Gamma^\mu\theta,\ \ \delta\theta = \varepsilon$. The WZ part S_2 can be made supersymmetric in $D = 4,5,7,11$ with an appropriate choice of spinors [58]. In $D = 11$ spinors are Majorana $\bar\theta = \theta^t C$, where C satisfies $C\Gamma^\mu C = (\Gamma^\mu)^t,\ C^t = -C,\ C^2 = -1$, so that $(\bar\theta\Gamma^{\mu_1\cdots\mu_n}\partial_\mu\theta)^t = (-1)^n\partial_\mu\bar\theta\Gamma^{\mu_1\cdots\mu_n}\theta$. In $D = 4,5,7,11$ the action has local fermionic κ-symmetry

$$\delta X^\mu = i\bar\theta\Gamma^\mu(1+\mathbf{\Gamma})\kappa, \qquad \delta\theta = (1+\mathbf{\Gamma})\kappa, \qquad \mathbf{\Gamma} \equiv \tfrac{1}{6\sqrt{-g}}\varepsilon^{abc}\Pi^\mu_a\Pi^\nu_b\Pi^\rho_c\Gamma_\mu\Gamma_\nu\Gamma_\rho, \qquad \mathbf{\Gamma}^2 = 1\,. \tag{3.4}$$

Fixing the static gauge as in the bosonic case (so that $h_{ab} = \eta_{ab} + \partial_a X^i\partial_b X^i$) and the κ-symmetry gauge as in [30, 59, 60]

$$P_+\theta = 0, \qquad P_-\theta = \theta, \qquad P_\pm \equiv \tfrac{1}{2}(1\pm\Gamma^\star)\,, \tag{3.5}$$

$$\Gamma^\star \equiv \Gamma^{012}, \qquad (\Gamma^\star)^2 = 1, \qquad \varepsilon^{abc}\Gamma_{bc} = 2\Gamma^\star\Gamma^a, \qquad \varepsilon^{abc}\Gamma_{ic} = \Gamma^\star\Gamma_i\Gamma^{ab}\,, \tag{3.6}$$

and expanding the Lagrangian in (3.1), (3.2) to quartic order in both bosons and fermions we find that the cubic $X\theta\theta$ term vanishes. The remaining terms to quartic order in fermions may be written as (see appendix A)

$$L = L_B + L_{\theta^2} + L_{X^2\theta^2} + L_{\theta^4} + \ldots\,, \qquad L_B = -\sqrt{-h}\,, \qquad L_{\theta^2} = i\bar\theta\partial\!\!\!/\theta\,, \tag{3.7}$$

$$L_{X^2\theta^2} = \tfrac{i}{4}\partial_a X^i\partial^a X^i\,\bar\theta\partial\!\!\!/\theta - \tfrac{i}{2}\partial_a X^i\partial_b X^i\bar\theta\Gamma^a\partial^b\theta - \tfrac{i}{4}\varepsilon^{abc}\,\partial_a X^i\,\partial_b X^j\,\bar\theta\Gamma_{ij}\,\partial_c\theta\,, \tag{3.8}$$

$$L_{\theta^4} = -\tfrac{1}{4}\bar\theta\Gamma_a\partial_b\theta\,\bar\theta\Gamma^b\partial^a\theta + \tfrac{1}{4}\bar\theta\partial\!\!\!/\theta\,\bar\theta\partial\!\!\!/\theta\,. \tag{3.9}$$

Here $\partial\!\!\!/ = \Gamma^a\partial_a$ and we rescaled the fermions $\theta \to \frac{1}{\sqrt2}\theta$ to get the canonical normalization of their kinetic term.[24]

---

[23] We assume that $\Gamma^0(\Gamma^\mu)^\dagger\Gamma^0 = \Gamma^\mu$ and also that $\varepsilon^{012} = -1$ and $\Gamma^{\mu_1\cdots\mu_n} = \Gamma^{[\mu_1\cdots\mu_n]} = \frac{1}{n!}(\Gamma^{\mu_1}\ldots\Gamma^{\mu_n}+\ldots)$.

[24] The interaction terms with $\bar\theta\partial\!\!\!/\theta$ factors in (3.8) and (3.9) can be, in principle, eliminated by a field redefinition at the expense of introducing higher-point vertices that should not contribute to the 4-scalar amplitude at 2-loop level consider here. Explicitly, we may redefine $\theta \to \big(1 + a_1\partial_a X^i\partial^a X^i + a_2\bar\theta\partial\!\!\!/\theta\big)\theta + \ldots$. Then instead of $\bar\theta\partial\!\!\!/\theta$ terms in (3.8), (3.9) we will get 6-point terms like $(\partial X)^4(\theta\partial\theta) + (\partial X)^2(\theta\partial\theta)^2 + (\theta\partial\theta)^3$. These will not give non-trivial contributions to the 2-loop 4-scalar amplitude. Thus the $\bar\theta\partial\!\!\!/\theta$ terms in (3.8), (3.9) can be simply omitted.

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The double dimensional reduction of the classical $D = 11$ supermembrane action gives the type IIA GS superstring action [56]. Setting $X^2 = \sigma^2$ we get from (3.1), (3.2)

$$S = S_1 + S_2, \qquad\qquad S_1 = -T \int d^2\sigma \sqrt{-\det g}, \tag{3.10}$$

$$S_2 = -T \int d^2\sigma \, i\, \varepsilon^{ab}\, \bar\theta \Gamma_\mu \Gamma_2\, \partial_a \theta\, (\Pi^\mu_b + \tfrac{i}{2}\, \bar\theta \Gamma^\mu \partial_b \theta)\,. \tag{3.11}$$

Here g_{ab} is defined as in (3.3), $\varepsilon^{ab} = \varepsilon^{ab2}$ and θ is the same as in supermembrane action. Here $a, b = 0, 1,\ \ \mu = 0, 1, 3, \ldots, \mathrm{D},\ \ \mathrm{D} \equiv D - 1$, so that we will have the same number $\mathrm{D} - 2 = D - 3$ of physical "transverse" coordinates. This action is supersymmetric and κ-symmetric in dimension $\mathrm{D} = 3, 4, 6, 10$.[25]

Fixing the static gauge and the κ-symmetry gauge

$$X^\mu = (\sigma^a, X^i),\ i = 3, \ldots, \mathrm{D}, \qquad P_+\theta = 0, \qquad P_\pm = \tfrac{1}{2}(1 \pm \Gamma^\star), \quad \Gamma^\star = \Gamma^{01}\Gamma^2\,, \tag{3.12}$$

we find that the expansion of the GS string Lagrangian to quartic order in fermions is the same as in the M2 brane case in (3.7)–(3.9) with the ε^{abc} term dropped (see appendix A)

$$L = L_B + L_{\theta^2} + L_{X^2\theta^2} + L_{\theta^4} + \ldots, \qquad L_B = -\sqrt{-h}, \qquad L_{\theta^2} = i\bar\theta \partial\!\!\!/ \theta\,, \tag{3.13}$$

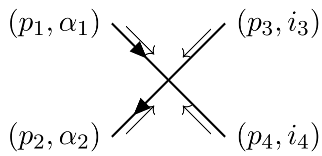
$$L_{X^2\theta^2} = \tfrac{i}{4}\partial_a X^i \partial^a X^i \bar\theta \partial\!\!\!/ \theta - \tfrac{i}{2}\partial_a X^i \partial_b X^i \bar\theta \Gamma^a \partial^b \theta, \qquad L_{\theta^4} = -\tfrac{1}{4}\bar\theta \Gamma_a \partial_b \theta\ \bar\theta \Gamma^b \partial^a \theta + \tfrac{1}{4}\bar\theta \partial\!\!\!/ \theta\ \bar\theta \partial\!\!\!/ \theta\,. \tag{3.14}$$

Note that in contrast to (3.8) in the membrane case this Lagrangian contains only Γ_a matrices with 2d indices which allows to relate it to the corresponding expansion of the NSR action [39]. Like in the membrane case, here the terms with $\bar\theta \partial\!\!\!/ \theta$ can be redefined away.

4 2-loop amplitude on superstring

In addition to the bosonic contributions we need to include also the fermionic loop ones. The fermion propagator following from (3.13) in the gauge (3.12) is given by $\frac{iP_- p\!\!\!/}{p^2}$.,[26] The $X^2\theta^2$ vertex following from (3.8) or (3.14) may be represented as

$$(V^{i_3 i_4})^{\alpha_1}{}_{\alpha_2}(p_1, p_2, p_3, p_4) = \tag{4.1}$$



Explicitly, in the spinor matrix notation it is given by

$$V^{i_3 i_4}(p_1, p_2, p_3, p_4) = \tfrac{i}{2}\, P_- \big[\delta^{i_3 i_4}(p_3 \cdot p_4\, p\!\!\!/_1 - p_3 \cdot p_1\, p\!\!\!/_4 - p_4 \cdot p_1\, p\!\!\!/_3) - \varepsilon^{abc} p_{1,a} p_{3,b} p_{4,c} \Gamma^{i_3 i_4}\big]\,, \tag{4.2}$$

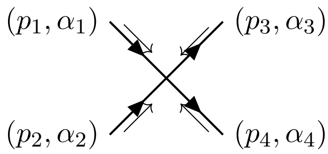
where in the superstring case we should omit the last ε^{abc} term.

[25] In what follows in discussing superstring we will redefine the notation $\mathrm{D} \to D$.

[26] Since Γ_a matrices satisfy the 2d Clifford algebra they can be identified with 2d Dirac matrices in 32×32 representation.

The quartic fermion vertex from (3.14) may be represented as

$$V_{\alpha_3\alpha_4}{}^{\alpha_1\alpha_2}(p_1,p_2,p_3,p_4) =$$



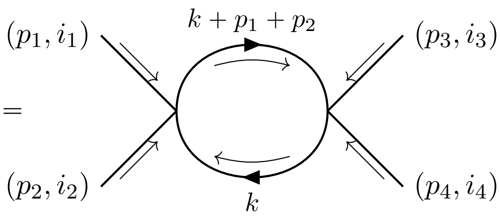
$$= \tfrac{i}{2}\big[(\not p_1)_{\alpha_4}{}^{\alpha_2}(\not p_2)_{\alpha_3}{}^{\alpha_1} + (\not p_1)_{\alpha_4}{}^{\alpha_1}(\not p_2)_{\alpha_3}{}^{\alpha_2} - (\not p_1)_{\alpha_3}{}^{\alpha_2}(\not p_2)_{\alpha_4}{}^{\alpha_1} - (\not p_1)_{\alpha_3}{}^{\alpha_1}(\not p_2)_{\alpha_4}{}^{\alpha_2}\big], \tag{4.3}$$

where it is understood that all $\not p$ factors contain also the projector P_-.

4.1 1-loop order

The fermionic loop contribution to 4-scalar scattering amplitude is given by

$$\mathrm{D_F}(1,2,3,4) =$$



$$= \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\left[V^{i_3i_4}(k+p_1+p_2,-k,p_3,p_4)\frac{-iP_-(\not k+\not p_1+\not p_2)}{(k+p_1+p_2)^2}V^{i_1i_2}(k,-k-p_1-p_2,p_1,p_2)\frac{-iP_-\not k}{k^2}\right], \tag{4.4}$$

where $V^{i_1i_2}$ was given in (4.2). The trace is proportional to $\mathrm{tr}\,P_- = n_F = 16$; we also include the factor $\frac{1}{2}$ as spinors are Majorana, and (-1) to account for the Fermi statistics so that effectively $\mathrm{tr}\,P_- \to -\frac{1}{2}n_F = -8$. Expanding the above expression for $d = 2-2\varepsilon$ we find that resulting 1-loop contribution to the $D = 10$ superstring scattering amplitude has a divergent part

$$\mathcal{M}^{(1)\,i_1i_2i_3i_4}_{\mathrm{F},\frac{1}{\varepsilon}} = -\tfrac{1}{24\pi\,\varepsilon}\,stu\,(\delta^{i_1i_2}\delta^{i_3i_4}+\delta^{i_1i_3}\delta^{i_2i_4}+\delta^{i_1i_4}\delta^{i_2i_3})\,. \tag{4.5}$$

This vanishes directly in $d=2$ where $stu=0$. Combined with the bosonic loop contribution (2.12) evaluated at $D=10$ that corresponds to the coefficient of the evanescent counterterm in (2.13) being changed to

$$\mathrm{c} = D-8+4 \overset{D\to 10}{=} 6\,. \tag{4.6}$$

The finite part of the fermion loop contribution for $t=0$ is given by

$$A^{(1)}_{\mathrm{F,fin}}(s) = -C^{(1)}_{\mathrm{F,fin}}(s) = -\tfrac{1}{48\pi}s^3\,, \qquad\qquad B^{(1)}_{\mathrm{F,fin}} = 0\,. \tag{4.7}$$

Added to the bosonic loop contribution (1.8) with $D=10$, the total 1-loop amplitude in the superstring case is given by (1.10) [30], i.e.

$$A^{(1)}_{\mathrm{fin}} = -C^{(1)}_{\mathrm{fin}} = -\tfrac{1}{192\pi}(D-26+4)\,s^3 \overset{D\to 10}{=} \tfrac{1}{16\pi}\,s^3\,, \qquad\qquad B^{(1)}_{\mathrm{fin}} = i\,\tfrac{1}{16}\,s^3\,. \tag{4.8}$$

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As in the bosonic case, one can satisfy the integrability requirement $A^{(1)} = C^{(1)} = 0$ by adding the counterterm (1.9) with [30][27]

$$\mathrm{b} = D - 26 + 4 \overset{D \to 10}{=} -12\,. \tag{4.9}$$

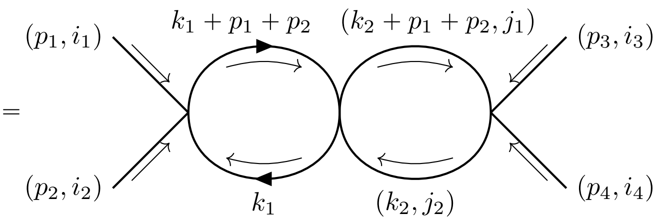
4.2 2-loop order

We get the same 2-loop diagram topologies as in (2.16) where now there are additional contributions from diagrams with internal fermionic lines. We will discuss these in turn.

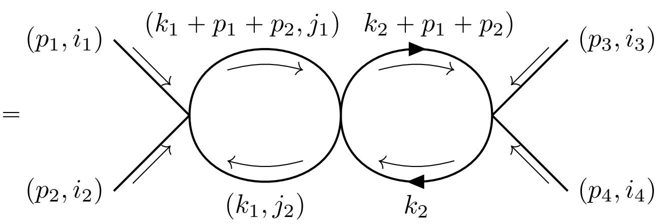
4.2.1 Double-bubble diagrams

There are 3 such diagrams — two with one bosonic and one fermionic loop (related in an obvious way by exchanging $12 \leftrightarrow 34$), and the one with two fermionic loops:

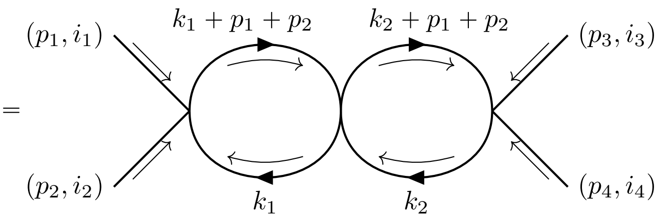
$(\mathrm{DB}_{a,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} =$ (4.10)



$(\mathrm{DB}_{b,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} =$ (4.11)



$(\mathrm{DB}_{c,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} =$ (4.12)



Explicitly, they are given by

$$(\mathrm{DB}_{a,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} = \frac{1}{2i}\int \widetilde{dk_1}\,\widetilde{dk_2}\,\frac{1}{k_2^2(k_2+p_1+p_2)^2} V^{j_1 j_2 i_3 i_4}_{p_1+p_2+k_2,-k_2,p_3,p_4}$$
$$\mathrm{tr}\left[V^{i_1 i_2}_{k_1,-k_1-p_1-p_2,p_1,p_2} \frac{P_-(\not{k}_1)}{k_1^2} V^{j_1 j_2}_{k_1+p_1+p_2,-k_1,-k_2-p_1-p_2,k_2} \frac{P_-(\not{k}_1+\not{p}_1+\not{p}_2)}{(k_1+p_1+p_2)^2}\right], \tag{4.13}$$

[27]As was noticed in [30], if one defines the GS scattering amplitude as a "double-dimensional" [56] limit of the amplitude on S^1-compactified M2 brane, then one gets directly that $A^{(1)} = C^{(1)} = 0$, implying that the definition of the GS path integral via the M2 brane one automatically provides the required measure factors or local counterterms, at least to the 1-loop order.

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$$(\mathrm{DB}_{b,\mathrm{F}})^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} = \frac{1}{2i}\int \widetilde{dk_1}\,\widetilde{dk_2}\,\frac{1}{k_1^2(k_1+p_1+p_2)^2}V^{i_1i_2j_1j_2}_{p_1,p_2,-k_1-p_1-p_2,k_1}$$
$$\mathrm{tr}\left[V^{i_3i_4}_{k_2+p_1+p_2,-k_2,p_3,p_4}\frac{P_-(\not{k}_2+\not{p}_1+\not{p}_2)}{(k_2+p_1+p_2)^2}V^{j_1j_2}_{k_2,-k_2-p_1-p_2,k_1+p_1+p_2,-k_1}\frac{P_-(\not{k}_2)}{k_2^2}\right], \tag{4.14}$$

$$(\mathrm{DB}_{c,\mathrm{F}})^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} = \frac{1}{i}\int \widetilde{dk_1}\,\widetilde{dk_2}\Big[S(k_1+p_1+p_2)V^{i_1i_2}_{k_1,-k_1-p_1-p_2,p_1,p_2}S(k_1)\Big]_{\alpha_2}^{\ \ \alpha_1}$$
$$\Big[S(k_2)V^{i_3i_4}_{k_2+p_1+p_2,-k_2,p_3,p_4}S(k_2+p_1+p_2)\Big]_{\alpha_4}^{\ \ \alpha_3}V_{\alpha_1\alpha_3}{}^{\alpha_2\alpha_4}(k_1+p_1+p_2,k_2), \tag{4.15}$$

where the vertices V were given in (4.2), (4.3) and we used the notation $\widetilde{dk} = \frac{d^dk}{(2\pi)^d}$ and $S(p) = \frac{P_-\not{p}}{p^2}$.

Introducing $q = k_1 + p_1 + p_2$ and

$$U \equiv S(k_1+p_1+p_2)V^{i_1i_2}_{k_1,-k_1-p_1-p_2,p_1,p_2}S(k_1), \qquad W \equiv S(k_2)V^{i_3i_4}_{k_2+p_1+p_2,-k_2,p_3,p_4}S(k_2+p_1+p_2), \tag{4.16}$$

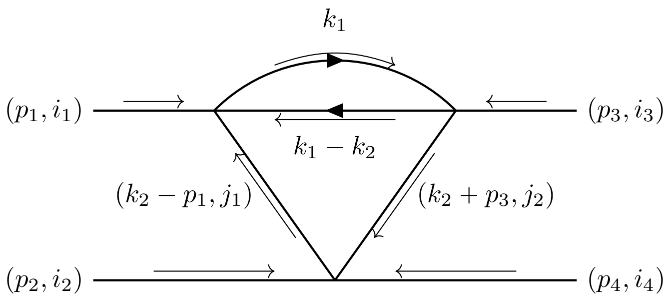
we have for the integrand in (4.15) (cf. (4.3))

$$\begin{aligned}&\tfrac{i}{2}\,U_{\alpha_2}{}^{\alpha_1}W_{\alpha_4}{}^{\alpha_3}[(\not{q})_{\alpha_3}{}^{\alpha_4}(\not{k}_2)_{\alpha_1}{}^{\alpha_2}+(\not{q})_{\alpha_3}{}^{\alpha_2}(\not{k}_2)_{\alpha_1}{}^{\alpha_4}-(\not{q})_{\alpha_1}{}^{\alpha_4}(\not{k}_2)_{\alpha_3}{}^{\alpha_2}-(\not{q})_{\alpha_1}{}^{\alpha_2}(\not{k}_2)_{\alpha_3}{}^{\alpha_4}]\\ &=\tfrac{i}{2}[\mathrm{tr}(U\not{k}_2)\,\mathrm{tr}(W\not{q})+\mathrm{tr}(U\not{q}W\not{k}_2)-\mathrm{tr}(U\not{k}_2W\not{q})-\mathrm{tr}(U\not{q})\,\mathrm{tr}(W\not{k}_2)].\end{aligned} \tag{4.17}$$

4.2.2 Wine-glass diagrams

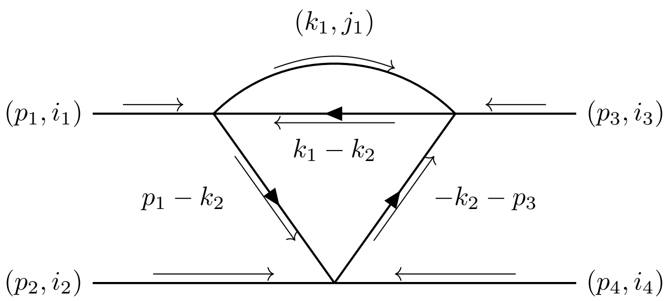
There are 3 diagrams of this topology: one with 2 fermionic propagators and two with 3 fermionic propagators (these are related by $(p_1, i_1) \leftrightarrow (p_3, i_3)$ exchange):

$(\mathrm{W}_{a,\mathrm{F}})^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} =$



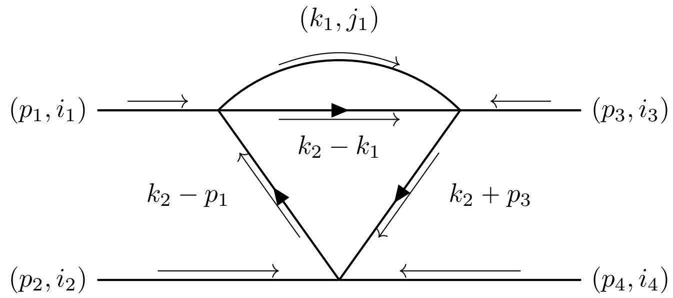
(4.18)

$(\mathrm{W}_{b,\mathrm{F}})^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} =$



(4.19)

$$(\mathrm{W}_{c,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} = \qquad (4.20)$$



The corresponding momentum integrals are

$$\begin{aligned}(\mathrm{W}_{a,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} &= \tfrac{1}{i} \int \widetilde{dk_1}\, \widetilde{dk_2} \frac{V^{i_2 j_1 j_2 i_4}_{p_2,-k_2+p_1,k_2+p_3,p_4}}{(k_2+p_3)^2 (k_2-p_1)^2} \\ &\quad \times \operatorname{tr}\left[S(k_1) V^{i_1 j_1}_{k_1-k_2,-k_1,p_1,k_2-p_1} S(k_1-k_2) V^{i_3 j_2}_{k_1,-k_1+k_2,p_3,-k_2-p_3}\right],\end{aligned} \tag{4.21}$$

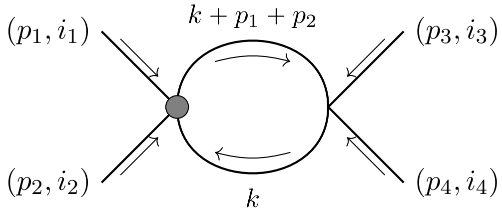
$$\begin{aligned}(\mathrm{W}_{b,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} &= \tfrac{1}{i} \int \widetilde{dk_1}\, \widetilde{dk_2} \frac{1}{k_1^2} \operatorname{tr}\Big[S(k_1-k_2) V^{i_3 j_1}_{-k_2-p_3,k_2-k_1,p_3,k_1} \\ &\quad \times S(-k_2-p_3) V^{i_2 i_4}_{p_1-k_2,k_2+p_3,p_2,p_4} S(p_1-k_2) V^{i_1 j_1}_{k_1-k_2,k_2-p_1,p_1,-k_1}\Big],\end{aligned} \tag{4.22}$$

$$\begin{aligned}(\mathrm{W}_{c,\mathrm{F}})^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} &= \tfrac{1}{i} \int \widetilde{dk_1}\, \widetilde{dk_2} \frac{1}{k_1^2} \operatorname{tr}\Big[S(k_2-k_1) V^{i_1 j_1}_{k_2-p_1,k_1-k_2,p_1,-k_1} S(k_2-p_1) \\ &\quad \times V^{i_2 i_4}_{k_2+p_3,-k_2+p_1,p_2,p_4} S(k_2+p_3) V^{i_3 j_1}_{k_2-k_1,-k_2-p_3,p_3,k_1}\Big].\end{aligned} \tag{4.23}$$

4.2.3 Counterterm diagram with X^4 vertex from fermionic loop

Next, let us consider possible counterterm diagrams corresponding to the third topology in (2.16). In the case when the loop in that diagram is bosonic, the counterterm X^4 vertex may correspond to the 1-loop diagram from either bosonic or fermionic loop.

The case of the bosonic 1-loop counterterm (2.13) was already discussed in section 2. Let us now consider the case of the 1-loop counterterm vertex originating from the fermionic loop diagram (4.4). The corresponding 2-loop order contribution is represented by



(4.24)

where the dashed circle corresponds to the counterterm canceling the pole in (4.5) due to a fermionic loop, i.e. the evanescent operator (2.13) with $\mathrm{c} \equiv \mathrm{c}_{\mathrm{F}} = 4$ (i.e. the contribution of 4 to total c in (4.6). This gives[28]

$$\Delta\mathcal{M}^{i_1i_2i_3i_4}=\Big\{\tfrac{i}{1536(d^2-1)\pi}s^3\Big[(d-2)[-4+d(D-4)]\,(t^2+u^2) \\ +2[d^2(D-4)-2(d+2)(D-2)]tu\Big]\delta^{i_1i_2}\delta^{i_3i_4}-\tfrac{i(d-2)}{384(d-1)\pi}s^5\left(\delta^{i_1i_4}\delta^{i_2i_3}+\delta^{i_1i_3}\delta^{i_2i_4}\right)\Big\}G_{1,1}(s). \tag{4.25}$$

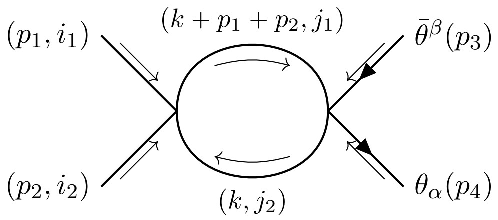
There is also a similar contribution from the diagram like (4.24) with the counterterm vertex on the right.

4.2.4 Counterterm diagram with $X^2\theta^2$ vertex

When expanding around $d=2$, the above 2-loop diagrams with internal fermions (both of double-bubble and wine-glass type) have divergent sub-diagrams corresponding also to the 1-loop $XX\to\bar\theta\theta$ process. We thus need to determine the associated counterterms to be inserted into the 1-loop diagrams of third type in (2.16) with the loop being the fermionic one. This is needed to cancel various 1-loop sub-divergences in diagrams (4.10)–(4.12) and (4.18)–(4.20) associated with sub-diagrams with two bosonic and two fermionic legs.

Let us first find the counterterm that is required to cancel the pole in the 1-loop $XX\to\bar\theta\theta$ amplitude. This amplitude may receive several contributions. One is from the bosonic loop diagram[29]

$(\mathrm{D_{BB}})_\alpha{}^\beta =$



$$=\frac{1}{2i}\int\frac{d^dk}{(2\pi)^d}\frac{V^{i_1i_2j_1j_2}_{p_1,p_2,-k-p_1-p_2,k}\,(V^{j_1j_2})_\alpha{}^\beta(p_3,p_4,k+p_1+p_2,-k)}{k^2(k+p_1+p_2)^2}\,. \tag{4.26}$$

For notational simplicity we will suppress the trivial $\delta^{i_1i_2}$ factor in the expressions for these amplitudes in what follows.

The two-fermion counterterm is a matrix with spinor indices, which we may decompose in a basis of Dirac matrices. From the perspective of the 2d worldsheet, this basis contains $\{I,\Gamma^a\}$. On dimensional and locality grounds, or simply because the two-fermion vertices are linear in the 2d Dirac matrices, the identity matrix cannot appear in the decomposition of this matrix. Thus, the counterterm must be of the form $\not{v}\equiv v_a\Gamma^a$ where v_a is some 2d vector. To simplify the calculations we take a trace of its product with an auxiliary matrix $p_5^a\Gamma_a$, and extract the vector v_a from the resulting scalar expression.

Computing $\mathrm{tr}(D_{\rm BB}\not{p}_5)$ we get (cf. (2.10))

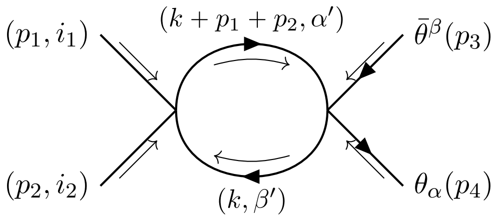
$$\mathrm{tr}(\mathrm{D_{BB}}\not{p}_5)=\tfrac{i}{8\,(d^2-1)}\,s^2\Big[-4\widehat{D}[t(p_1\cdot p_5-p_2\cdot p_5)+s(p_1\cdot p_5+p_3\cdot p_5)] \\ +2s(\widehat{D}-6)(d-2)(p_1\cdot p_5+p_2\cdot p_5+2p_3\cdot p_5) \\ +s(\widehat{D}-4)(d-2)^2(p_1\cdot p_5+p_2\cdot p_5+2p_3\cdot p_5)\Big]G_{1,1}(s)\,. \tag{4.27}$$

[28]Here we specify to the string case and set $\widehat{D}=D-2$.

[29]Diagram (4.26) cancels the bosonic bubble sub-divergence of (4.10) and its left-right flipped version cancels the sub-divergence of (4.11).

Another contribution is from the diagram with the fermionic loop[30]

$(\mathrm{D}_{\mathrm{FF}})_\alpha{}^\beta =$



$$= \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{[S(k+p_1+p_2)V^{i_1i_2}(k,-k-p_1-p_2,p_1,p_2)S(k)]_{\alpha'}{}^{\beta'}V_{\alpha\beta'}{}^{\beta\alpha'}(p_3,k+p_1+p_2)}{k^2(k+p_1+p_2)^2}. \tag{4.28}$$

Let us define

$$q = k + p_1 + p_2, \qquad \mathcal{V}^{i_1i_2} = S(k+p_1+p_2)V^{i_1i_2}(k,-k-p_1-p_2,p_1,p_2)S(k). \tag{4.29}$$

Considering like in (4.27) the combination $\mathrm{D}_{\mathrm{FF}}\not{p}_5$ and using (4.3) we have

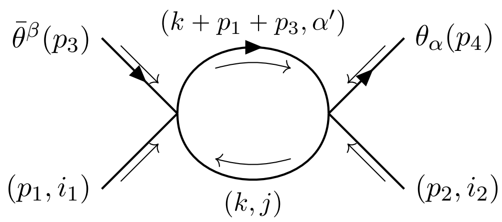
$$\begin{aligned}
&(\mathcal{V}^{i_1i_2})_{\alpha'}{}^{\beta'}V_{\alpha\beta'}{}^{\beta\alpha'}(p_3,q)(\not{p}_5)_\beta{}^\alpha \\
&= \tfrac{i}{2}(\mathcal{V}^{i_1i_2})_{\alpha'}{}^{\beta'}(\not{p}_5)_\beta{}^\alpha\big[(\not{p}_3)_{\beta'}{}^{\alpha'}(\not{q})_\alpha{}^\beta + (\not{p}_3)_{\beta'}{}^\beta(\not{q})_\alpha{}^{\alpha'} - (\not{p}_3)_\alpha{}^{\alpha'}(\not{q})_{\beta'}{}^\beta - (\not{p}_3)_\alpha{}^\beta(\not{q})_{\beta'}{}^{\alpha'}\big] \\
&= \tfrac{i}{2}\Big(\,\mathrm{tr}[\mathcal{V}^{i_1i_2}\not{p}_3]\,\mathrm{tr}[\not{p}_5\not{q}] + \mathrm{tr}[\mathcal{V}^{i_1i_2}\not{p}_3\not{p}_5\not{q}] - \mathrm{tr}[\mathcal{V}^{i_1i_2}\not{q}\not{p}_5\not{p}_3] - \mathrm{tr}[\mathcal{V}^{i_1i_2}\not{q}]\,\mathrm{tr}[\not{p}_3\not{p}_5]\Big)\,.
\end{aligned} \tag{4.30}$$

Thus after the loop integration

$$\begin{aligned}
\mathrm{tr}(\mathrm{D}_{\mathrm{FF}}\not{p}_5) = -\tfrac{i}{d^2-1}\,s^2\,\Big[&2t(p_1\cdot p_5 - p_2\cdot p_5) + 2s(p_1\cdot p_5 + p_3\cdot p_5) \\
&+ (s+2t)(p_1\cdot p_5 - p_2\cdot p_5)(d-2)\Big]G_{1,1}(s)\,.
\end{aligned} \tag{4.31}$$

The third possible diagram has one bosonic and one fermionic propagator in the loop[31]

$(\mathrm{D}_{\mathrm{FB}})_\alpha{}^\beta =$



$$= \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{(V^{i_1,j})_{\alpha'}{}^\beta(p_3,-k-p_1-p_3,p_1,k)S_{\beta'}{}^{\alpha'}(k+p_1+p_3)(V^{i_2,j})_\alpha{}^{\beta'}(k+p_1+p_3,p_4,p_2,-k)}{k^2(k+p_1+p_3)^2}. \tag{4.32}$$

In this case

$$\begin{aligned}
\mathrm{tr}(\mathrm{D}_{\mathrm{FB}}\not{p}_5) =& \tfrac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k+p_1+p_3)^2} \\
&\times\mathrm{tr}\Big[\not{p}_5V^{i_2,j}(k+p_1+p_3,p_4,p_2,-k)S(k+p_1+p_2)V^{i_1,j}(p_3,-k-p_1-p_3,p_1,k)\Big]
\end{aligned} \tag{4.33}$$

$$= -\tfrac{i}{8(d-1)}t^2\Big[(2(d+2)s+(5d-2)t)p_1\cdot p_5 - 4dtp_2\cdot p_5 + (2(d+2)s+(d-2)t)p_3\cdot p_5\Big]G_{1,1}(t).$$

[30]The diagram (4.28) and its left-right flipped version cancel the sub-divergences of (4.12).

[31]The diagram (4.32) cancels the sub-divergence of the “Wine-glass” diagrams (4.18), (4.19).

There is also a similar diagram obtained by swapping p_1 and p_2 or $t \to u$ (we will denote its contribution as $\mathrm{D_{BF}}$).

Summing up the above expressions and setting $d = 2 - 2\varepsilon$ we get for the pole part in the total $XX \to \bar{\theta}\theta$ amplitude

$$\mathrm{tr}[(\mathrm{D_{BB}} + \mathrm{D_{FF}} + \mathrm{D_{FB}} + \mathrm{D_{BF}})\not{p}_5] = -\tfrac{1}{12\pi\varepsilon}(D-4)\, s\, (u\, p_1 \cdot p_5 + t\, p_2 \cdot p_5 - s\, p_3 \cdot p_5) + \mathcal{O}(\varepsilon^0). \tag{4.34}$$

The $\mathrm{D_{BB}}$, etc., matrices contain an implicit factor P_-. Using that $\mathrm{tr}\, P_- = 16$ we then get for the pole part

$$\mathrm{D}_{\mathrm{tot},\frac{1}{\varepsilon}} = \tfrac{1}{192\pi\varepsilon}\,(D-4)\, s\,\big(-u\not{p}_1 - t\not{p}_2 + s\not{p}_3\big)\,. \tag{4.35}$$

This is a matrix with spinor indices (it is also proportional to $\delta^{i_1 i_2}$ which we suppress here).

In general, at the L-loop order, in addition to the bosonic counterterms (2.31) we may have also the ones containing fermions, e.g., $T^{1-L}\sum_{n,k}\int d^d\sigma\, \partial^{dL}(\partial X)^{2n}\,(\bar{\theta}\partial\theta)^k$.[32] In particular, for $L = 1$ and $n = k = 1$ we may have a counterterm of the following symbolic structure in $d = 2$

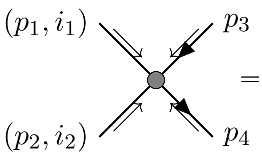
$$\frac{1}{\varepsilon}\int d^2\sigma\, \partial^2\partial X^i \partial X^i\, \bar{\theta}\partial\theta\,. \tag{4.36}$$

A covariant counterterm that contains (4.36) in its expansion is (cf. (C.6) and (C.7))

$$R_{ab}\,\bar{\theta}\Gamma^a\partial^b\theta = -(K^iK^i)_{ab}\,\bar{\theta}\Gamma^a\partial^b\theta = -\partial_a\partial^c X^i\partial_b\partial_c X^i\,\bar{\theta}\Gamma^a\partial^b\theta + \dots\,, \tag{4.37}$$

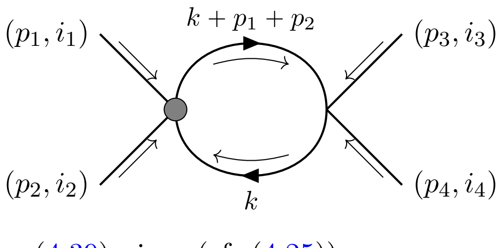
where we used the on-shell conditions $\not{\partial}\theta = 0,\ \partial^2 X = 0$.[33]

We are now ready to compute the analog of the 4-scalar counterterm diagram (4.24) containing the above $X^2\theta^2$ counterterm vertex that cancels the pole in (4.35)



$$= -\tfrac{i}{192\pi\varepsilon}(D-4)\, p_1 \cdot p_2\, (p_2 \cdot p_3\, \not{p}_1 + p_1 \cdot p_3\, \not{p}_2)\, \delta^{i_1 i_2}\,, \tag{4.38}$$

and the fermionic loop



$$. \tag{4.39}$$

Evaluation of the diagram (4.39) gives (cf. (4.25))

$$\Delta\mathcal{M}^{i_1 i_2 i_3 i_4} = \tfrac{i}{3072\pi(d^2-1)}\,(D-4)\, s^3\Big[(d-2)(t^2+u^2) - 2d\, tu\Big]\, G_{1,1}(s)\,\delta^{i_1 i_2}\delta^{i_3 i_4}. \tag{4.40}$$

A similar diagram with the counterterm vertex on the right obtained by interchanging $1234 \leftrightarrow 3412$ gives the equivalent contribution.

[32] Here we use that ∂X and $\bar{\theta}\partial\theta$ that appear in (3.1), (3.10) are dimensionless.

[33] The tree level contribution corresponding to (4.36) is $-p_1 \cdot p_2(p_2 \cdot p_3\, \not{p}_1 + p_1 \cdot p_3\, \not{p}_2) = \frac{1}{4}s(-u\not{p}_1 - t\not{p}_2)$, which has the same structure as (4.35) after using that $\not{p}_3$ term does not contribute on-shell.

4.2.5 Total 2-loop contribution

Let us now sum up the contributions of the above diagrams with fermions. Let us denote by DB the sum of the double-bubble expressions in (4.10), (4.11), (4.12), by W — the sum of wine-glass diagram expressions (4.18), (4.19), (4.20) and by CT and CT′ the contributions of the counterterm diagrams in (4.39) and (4.24) plus their reflected versions. We should also sum over appropriate crossed versions of these diagrams.

Keeping first s, t, u generic it is enough to find the A-amplitude as B and C can be found from it by crossing as in (1.2). We find, in particular, that the double-pole contributions are given by[34]

$$A^{(2)}_{\mathrm{DB},\frac{1}{\varepsilon^2}} = -\frac{D}{576\pi^2\varepsilon^2}\, s^2 tu, \quad A^{(2)}_{\mathrm{W},\frac{1}{\varepsilon^2}} = \frac{17}{1152\pi^2\varepsilon^2}\, s^2 tu, \quad A^{(2)}_{\mathrm{CT},\frac{1}{\varepsilon^2}} = \frac{D-4}{2304\pi^2\varepsilon^2}\, s^2 tu, \quad A^{(2)}_{\mathrm{CT}',\frac{1}{\varepsilon^2}} = \frac{D}{576\pi^2\varepsilon^2}\, s^2 tu, \tag{4.41}$$

so that each of them vanishes due to the kinematic condition $stu = 0$ in $d = 2$.

For the $t = 0$ choice of the 2d kinematics we get for the single pole and finite contributions of these diagrams (we again include $4\pi e^{-\gamma_E}$ into μ^2)

$$\begin{aligned}
&A^{(2)}_{\mathrm{DB},\frac{1}{\varepsilon}} = -\frac{D}{1152\pi^2\varepsilon}\, s^4, \qquad A^{(2)}_{\mathrm{W},\frac{1}{\varepsilon}} = \frac{41}{4608\pi^2\varepsilon}\, s^4, \qquad A^{(2)}_{\mathrm{CT},\frac{1}{\varepsilon}} = \frac{D-4}{4608\pi^2\varepsilon}\, s^4, \qquad A^{(2)}_{\mathrm{CT}',\frac{1}{\varepsilon}} = \frac{D-12}{1152\pi^2\varepsilon}\, s^4,\\
&A^{(2)}_{\mathrm{DB,fin}} = \frac{D}{576\pi^2}\, s^4 \log\left(-\frac{s}{\mu^2}\right) - \frac{8D-6}{1728\pi^2}\, s^4,\\
&A^{(2)}_{\mathrm{W,fin}} = -\frac{1}{96\pi^2}\, s^4 \log\left(\frac{s}{\mu^2}\right) + \frac{17}{2304\pi^2}\, s^4 \log\left(-\frac{s}{\mu^2}\right) + \frac{935}{27648\pi^2}\, s^4,\\
&A^{(2)}_{\mathrm{CT,fin}} = -\frac{D-4}{4608\pi^2}\, s^4 \log\left(-\frac{s}{\mu^2}\right) + \frac{D-4}{1728\pi^2}\,,\\
&A^{(2)}_{\mathrm{CT}',\mathrm{fin}} = \frac{1}{192\pi^2}\, s^4 \log\left(\frac{s}{\mu^2}\right) - \frac{D-6}{1152\pi^2}\, s^4 \log\left(-\frac{s}{\mu^2}\right) + \frac{5D-72}{3456\pi^2}\, s^4\,.
\end{aligned} \tag{4.42}$$

The sum of these expressions gives the following result for the total fermionic contribution to the 2-loop amplitude

$$\begin{aligned}
A^{(2)}_{\mathrm{F}}(s) &= \frac{1}{4608\pi^2\,\varepsilon}\,(D-11)\, s^4 + \frac{1}{1536\pi^2}\,(D-10)\, s^4 \log(-\tfrac{s}{\mu^2})\\
&\quad - \frac{1}{27648\pi^2}\,(72D-391)\, s^4 - i\,\frac{1}{192\pi}\, s^4\,, \qquad (4.43)\\
C^{(2)}_{\mathrm{F}}(s) &= \frac{1}{4608\pi^2\,\varepsilon}\,(D-11)\, s^4 + \frac{1}{1536\pi^2}\,(D-10)\, s^4 \log(-\tfrac{s}{\mu^2})\,,\\
&\quad - \frac{1}{27648\pi^2}\,(72D-391)\, s^4 + i\big[\frac{1}{192\pi} + \frac{1}{1536\pi}\,(D-10)\big] s^4\,, \qquad (4.44)\\
B^{(2)}_{\mathrm{F}}(s) &= -\frac{1}{192\pi^2\varepsilon}\, s^4 - \frac{1}{1152\pi^2}\, s^4\,, \qquad (4.45)
\end{aligned}$$

where we used that $\log s = \log(-s) + i\pi$. Note that while in the bosonic case (2.21), (2.22) the $\log s$ terms cancelled for any D, here in the superstring case that happens only in the special case of $D = 10$. Also, the condition (1.5), i.e. $A(s) = C^*(-s)$ holds only for $D = 10$.

The total 2-loop amplitude in the superstring case is found by summing together the expressions in (2.21), (2.22) and (4.43)–(4.45) evaluated for $D = 10$. The result was given

[34] As already mentioned, this is due to the absence of 1-loop divergences in $d = 2$ and thus absence of sub-divergences in 2-loop diagrams.

in (1.16)–(1.18), i.e.

$$A^{(2)}(s) = [C^{(2)}(-s)]^* = -\tfrac{1}{1536\pi^2\,\varepsilon}\, s^4 + \tfrac{145}{9216\pi^2}\, s^4 + i\tfrac{1}{64\pi}s^4\,, \tag{4.46}$$

$$B^{(2)}(s) = -\tfrac{1}{128\pi^2\varepsilon}\, s^4 + \tfrac{25}{768\pi^2}\, s^4 - \tfrac{1}{192}s^4\,. \tag{4.47}$$

As in the bosonic case [35], one can check that these 2-loop GS string contributions are consistent with 2-particle unitarity.

Like in the bosonic case, the poles here that are not accompanied by $\log s$ terms come from the evanescent counterterm contributions, so they are artifacts of the dimensional regularization and should be subtracted. They can be cancelled out by adding the counterterm (cf. (1.14), (2.24), (2.25))

$$\tfrac{1}{64\pi^2\,\varepsilon}\int d^2\sigma\,\sqrt{-h}\,\big[\,\mathrm{tr}(K^iK^j)^2 - \tfrac{5}{12}\,\mathrm{tr}(K^iK^i)^2\big]. \tag{4.48}$$

In addition, like in the $D \neq 26$ bosonic case, we need to add the contribution of the PPS 1-loop counterterm (1.9) with the coefficient given by (4.9) that was required for the integrability of the $D = 10$ GS string at the 1-loop level [30]. The corresponding 2-loop contribution containing 4-vertex from this counterterm cancels the imaginary terms in $A^{(2)}$ and $C^{(2)}$ (see appendix D.4). Then the remaining real s^4 terms can be cancelled by a real counterterm of the same form as in (1.14), (4.48). As a result, we end up with the amplitudes in (1.15), i.e. $A^{(2)} = C^{(2)} = 0$, $B^{(2)} = -\frac{1}{192}s^4$, as required for the 2-loop integrability of the $D = 10$ GS string.

5 2-loop amplitude on supermembrane

Like in the bosonic membrane case, in the M2 brane case there are no 1-loop divergences [30] so the only 2-loop contributions come from the first two genuine 2-loop diagrams in (2.16). This makes the computation effectively more straightforward than in the GS string case.

5.1 1-loop order

To get the 1-loop amplitude we need to add to the bosonic loop contribution (1.19) the contribution of the fermionic loop with the vertices (4.1) including the ε^{abc} term in (4.2). After summing over crossings the fermionic loop result is found to be

$$A_{\rm F}^{(1)}(s,t,u) = \tfrac{1}{1024}\,[(-s)^{3/2}\,(s^2 - 8tu) + 4(-t)^{3/2}\,t(t+2s) + 4(-u)^{3/2}\,u(u+2s)], \tag{5.1}$$

$$B_{\rm F}^{(1)}(s,t,u) = A_{\rm F}^{(1)}(t,s,u), \qquad C_{\rm F}^{(1)}(s,t,u) = A_{\rm F}^{(1)}(u,t,s). \tag{5.2}$$

Adding this to (1.19) we get for the total A amplitude the following expression [30]

$$A^{(1)}(s,t,u) = -\tfrac{1}{64}tu\,[(-s)^{3/2} + (-t)^{3/2} + (-u)^{3/2}]\,. \tag{5.3}$$

Thus the total 1-loop amplitude has a simple form proportional to the tree level one[35]

$$\mathcal{M}^{(1)}(s,t,u) = \tfrac{1}{32}\big[(-s)^{3/2} + (-t)^{3/2} + (-u)^{3/2}\big]\,\mathcal{M}^{(0)}(s,t,u)\,, \tag{5.4}$$

$$\mathcal{M}^{(0)i_1i_2i_3i_4}(s,t,u) = -\tfrac{1}{2}\big(tu\,\delta^{i_1i_2}\delta^{i_3i_4} + su\,\delta^{i_1i_3}\delta^{i_2i_4} + st\,\delta^{i_1i_4}\delta^{i_2i_3}\big)\,. \tag{5.5}$$

[35] As in any odd number of dimensions the 1-loop amplitude is finite. A similar computation of 4-particle scattering in the case of the supersymmetric D3-brane action ($d = 4$) gives 1-loop divergence $\sim \frac{1}{\varepsilon}(s^2 + t^2 + u^2)$ times tree-level amplitude [61].

5.2 2-loop order

The fermionic double-bubble diagrams (4.10), (4.11), (4.12) give (after sum over crossing) a finite contribution which for $D = 11$ is given by

$$A^{(2)}_{\rm DB_F} = -\tfrac{3}{65536}\, s\big(5s^4 + 20s^3t + 16s^2t^2 - 8st^3 - 4t^4\big)\,. \tag{5.6}$$

Each of the three wine-glass diagrams (4.18), (4.19), (4.20) should be summed over 6 crossed configurations. As a result, we get an expression containing a UV pole part (accompanied by the corresponding $\log s$, etc., terms). In dimension $D = 11$ it reads

$$A^{(2)}_{{\rm W_F},\frac{1}{\varepsilon}} = \tfrac{1}{80640\pi^2\,\varepsilon}\, s\big(8s^4 + 14s^3t - 113s^2t^2 - 254st^3 - 127t^4\big)\,. \tag{5.7}$$

Adding this to the bosonic membrane pole contribution in (2.29) we conclude that the total UV pole part of the $D = 11$ M2 brane 2-loop amplitude is given by

$$A^{(2)}_{\frac{1}{\varepsilon}} = -\tfrac{1}{256\pi^2\,\varepsilon}\, s\,t^2u^2\,. \tag{5.8}$$

Thus despite its maximal target space supersymmetry the S-matrix of M2 brane theory is not UV finite starting from 2-loop order.

The total expression for the finite part of the M2 brane amplitude (including the wine-glass, double-bubble (5.6) and also the bosonic finite part in (2.30) evaluated at $D = 11$) is given by

$$\begin{aligned} A^{(2)}_{\rm fin} = {}& \tfrac{1}{384}tu\Big(\tfrac{1}{\pi^2}\big[s^3\log(-\tfrac{s}{\mu^2}) + t^3\log(-\tfrac{t}{\mu^2}) + u^3\log(-\tfrac{u}{\mu^2})\big] + \tfrac{9}{16}stu\Big)\\ &+ \tfrac{1}{80640\pi^2}\, s\big(6s^4 + 124s^3t - 1833s^2t^2 - 3914st^3 - 1957t^4\big)\,, \end{aligned} \tag{5.9}$$

where we absorbed π and $e^{-\gamma_E}$ factors into μ^2.[36]

Let us note that the coefficients in the finite $\frac{1}{\pi^2}$ term in the second line of (5.9) are potentially scheme-dependent. They are sensitive to how one treats contractions of the four ε^{abc} symbols in 2-loop diagrams in dimensional regularization. We may first group them in pairs and then use the relation (B.5) for each pair. Particular choices of ordering in pairs may differ by order $\varepsilon = -\frac{1}{2}(d-3)$ terms and this may lead to different finite parts when multiplied by $\frac{1}{\varepsilon}$ pole. The expression in (5.9) was found by using the symmetric average over the three pairing orders.[37]

[36] Note that since $s^3 + t^3 + u^3 = 3stu$ the total coefficient of the $\log\mu^2$ term differs from the coefficient of the pole term in (5.8) by factor of 2, as expected in the 2-loop contribution (see footnote 17).

[37] In more detail, in the wine-glass diagrams (4.19), (4.20) there is just one four ε^{abc} contraction. Introducing the notation $\varepsilon(p,q,r) \equiv \varepsilon^{abc}p_aq_br_c$, we find such quartic contraction where one factor comes from the Dirac trace (B.4) (we will indicate this it by underlining it) and the other three from the last term in (4.2). This gives

$$\underline{\varepsilon}(k_1-k_2,-k_2-p_3,-k_2+p_1)\,\varepsilon(k_1-k_2,p_1,-k_1)\,\varepsilon(-k_2+p_1,p_2,-p_1-p_2-p_3)\,\varepsilon(-k_2-p_3,p_3,k_1) \equiv \underline{\varepsilon}\;\varepsilon'\;\varepsilon''\;\varepsilon'''.$$

We may group these factors in pairs with arbitrary coefficients α,β as $\alpha\,[\underline{\varepsilon}\varepsilon'][\varepsilon''\varepsilon'''] + \beta\,[\underline{\varepsilon}\varepsilon''][\varepsilon'\varepsilon'''] + \gamma\,[\underline{\varepsilon}\varepsilon'''][\varepsilon'\varepsilon'']$ where $\gamma = 1-\alpha-\beta$. We find that the dependence on α in $A^{(2)}$ drops out, so that there is just one scheme-dependence parameter β. Then the analog of the second line in (5.9) is found to be

$$\tfrac{1}{40320\pi^2}(7-12\beta)\,s^5 + \tfrac{1}{80640\pi^2}\,stu\big[4s^2(-37+18\beta) - tu(1965-24\beta)\big].$$

The "symmetric" choice of $\beta = \frac{1}{3}$ leads to the expression in the second line of (5.9).

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Acknowledgments

We thank S. Dubovsky for sharing some details about the results in [35]. Part of this work was done while AAT was attending the workshop "Higher-d Integrability" in Favignana, Italy (02-13.06.25) and he thanks the organizers for the hospitality. MB is supported by the INFN grant GAST. RR is supported by the U.S. Department of Energy (DOE) under award number DE-SC00019066. The work of AAT is supported by the STFC grant ST/T000791/1.

A Expansion of M2 brane and GS string actions to fourth order

Here we will present details of the derivation of the expansions in (3.7)–(3.9) and (3.13), (3.14). The starting point is the covariant M2 brane action in (3.1), (3.2). In the static gauge we fix $X^a = \sigma^a$, $a = 0, 1, 2$ and expand in powers of $D-3$ fields X^i. The expansion of $\sqrt{-h}$ was given in (2.3) and we are interested in similar terms involving θ. We have (see (3.3))

$$L_1 = -\sqrt{-g} = -\sqrt{-h}\Big[1 - i\partial_a X_\mu \bar\theta\Gamma^\mu\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_\mu\partial_a\theta\,\bar\theta\Gamma^\mu\partial^a\theta - \tfrac{1}{2}(\partial_a X_\mu\bar\theta\Gamma^\mu\partial^a\theta)^2 + \tfrac{1}{2}\partial_a X_\mu\bar\theta\Gamma^\mu\partial_b\theta\,\partial^b X_\nu\bar\theta\Gamma^\nu\partial^a\theta + \tfrac{1}{2}\partial_a X_\mu\bar\theta\Gamma^\mu\partial_b\theta\,\partial^a X_\nu\bar\theta\Gamma^\nu\partial^b\theta + \ldots\Big], \tag{A.1}$$

where $\mu = (a, i)$ and $\partial_a X^b = \delta^b_a$. Hence ($\partial\!\!\!/ \equiv \Gamma^a\partial_a$)

$$L_1 = -\sqrt{-h} + i\bar\theta\partial\!\!\!/\theta + i\,\partial_a X_i\bar\theta\Gamma^i\partial^a\theta + i(\tfrac{1}{2}\eta^{ab}\partial^c X^i\partial_c X^i - \partial^a X^i\partial^b X^i)\bar\theta\Gamma_a\partial_b\theta + \tfrac{1}{2}\bar\theta\Gamma_b\partial_a\theta\,\bar\theta\Gamma^b\partial^a\theta + \tfrac{1}{2}\bar\theta\Gamma_i\partial_a\theta\,\bar\theta\Gamma^i\partial^a\theta + \tfrac{1}{2}(\bar\theta\partial\!\!\!/\theta)^2 - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\,\bar\theta\Gamma^b\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\,\bar\theta\Gamma^a\partial^b\theta + \ldots. \tag{A.2}$$

Similarly, expanding the integrand $L_2 = L_{2,1} + L_{2,2}$ of the WZ term in (3.2) we get (using (3.6))

$$\begin{aligned} L_{2,1} &= -\tfrac{i}{2}\varepsilon^{abc}\bar\theta\Gamma_{\mu\nu}\partial_a\theta\,\Pi^\mu_b\Pi^\nu_c = -\tfrac{i}{2}\varepsilon^{abc}\bar\theta\Gamma_{bc}\partial_a\theta - i\varepsilon^{abc}\bar\theta\Gamma_{bi}\partial_a\theta\partial_c X^i - \varepsilon^{abc}\bar\theta\Gamma_{bi}\partial_a\theta\bar\theta\Gamma^i\partial_c\theta \\ &\quad - \tfrac{1}{2}\varepsilon^{abc}(\bar\theta\Gamma_{bd}\partial_a\theta\bar\theta\Gamma^d\partial_c\theta - \bar\theta\Gamma_{cd}\partial_a\theta\bar\theta\Gamma^d\partial_b\theta) - \tfrac{i}{2}\varepsilon^{abc}\bar\theta\Gamma_{ij}\partial_a\theta\partial_b X^i\partial_c X^j + \ldots \\ &= -i\bar\theta\Gamma^\star\partial\!\!\!/\theta - i\bar\theta\Gamma^\star\Gamma_i\Gamma^{ac}\partial_a\theta\partial_c X^i - \tfrac{i}{2}\varepsilon^{abc}\bar\theta\Gamma_{ij}\partial_a\theta\partial_b X^i\partial_c X^j \\ &\quad - \varepsilon^{abc}\bar\theta\Gamma_{bi}\partial_a\theta\bar\theta\Gamma^i\partial_c\theta - \tfrac{1}{2}\varepsilon^{abc}(\bar\theta\Gamma_{bd}\partial_a\theta\bar\theta\Gamma^d\partial_c\theta - \bar\theta\Gamma_{cd}\partial_a\theta\bar\theta\Gamma^d\partial_b\theta) + \ldots, \end{aligned} \tag{A.3}$$

$$L_{2,b} = \tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{\mu\nu}\partial_a\theta(\partial_b X^\mu - i\bar\theta\Gamma^\mu\partial_b\theta)\bar\theta\Gamma^\nu\partial_c\theta = \tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{bd}\partial_a\theta\,\bar\theta\Gamma^d\partial_c\theta + \tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{bi}\partial_a\theta\,\bar\theta\Gamma^i\partial_c\theta + \ldots$$

Thus (see (3.6))

$$L = L_1 + L_2 = L_B + L_{\theta^2} + L_{X\theta^2} + L_{X^2\theta^2} + L_{\theta^4} + \ldots, \tag{A.4}$$

$$L_B = -\sqrt{-h}, \qquad L_{\theta^2} = i\bar\theta(1-\Gamma^\star)\partial\!\!\!/\theta, \qquad L_{X\theta^2} = i\partial_a X_i\bar\theta\Gamma^i\partial^a\theta + i\partial_a X^i\bar\theta\Gamma^\star\Gamma_i\Gamma^{ab}\partial_b\theta, \tag{A.5}$$

$$L_{X^2\theta^2} = \tfrac{i}{2}\partial_a X^i\partial^a X^i\bar\theta\partial\!\!\!/\theta - i\partial_a X^i\partial_b X^i\bar\theta\Gamma^a\partial^b\theta - \tfrac{i}{2}\partial_a X^i\partial_b X^j\varepsilon^{abc}\bar\theta\Gamma_{ij}\partial_c\theta, \tag{A.6}$$

$$L_{\theta^4} = -\varepsilon^{abc}\bar\theta\Gamma_{bi}\partial_a\theta\bar\theta\Gamma^i\partial_c\theta - \tfrac{1}{2}\varepsilon^{abc}(\bar\theta\Gamma_{bd}\partial_a\theta\bar\theta\Gamma^d\partial_c\theta - \bar\theta\Gamma_{cd}\partial_a\theta\bar\theta\Gamma^d\partial_b\theta) + \tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{b\nu}\partial_a\theta\,\bar\theta\Gamma^\nu\partial_c\theta + \tfrac{1}{2}\bar\theta\Gamma_b\partial_a\theta\bar\theta\Gamma^b\partial^a\theta + \tfrac{1}{2}\bar\theta\Gamma_i\partial_a\theta\bar\theta\Gamma^i\partial^a\theta + \tfrac{1}{2}(\bar\theta\partial\!\!\!/\theta)^2 - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\bar\theta\Gamma^b\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\bar\theta\Gamma^a\partial^b\theta. \tag{A.7}$$

Here (A.5), (A.6) are the same as in [30] where θ^4 terms were not included as they do not contribute to the bosonic S-matrix in 1-loop approximation.

We can rearrange L_{θ^4} as follows

$$\begin{aligned} L_{\theta^4} &= -\tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{bi}\partial_a\theta\bar\theta\Gamma^i\partial_c\theta - \tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{bd}\partial_a\theta\bar\theta\Gamma^d\partial_c\theta + \tfrac{1}{2}\bar\theta\Gamma_i\partial_a\theta\bar\theta\Gamma^i\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\bar\theta\Gamma^b\partial^a\theta + \tfrac{1}{2}(\bar\theta\partial\!\!\!/\theta)^2 \\ &= \tfrac{1}{2}\bar\theta\Gamma^\star\Gamma_i\Gamma^{ac}\partial_a\theta\bar\theta\Gamma^i\partial_c\theta - \tfrac{1}{2}\varepsilon^{abc}\bar\theta\Gamma_{bd}\partial_a\theta\bar\theta\Gamma^d\partial_c\theta + \tfrac{1}{2}\bar\theta\Gamma_i\partial_a\theta\bar\theta\Gamma^i\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\bar\theta\Gamma^b\partial^a\theta + \tfrac{1}{2}(\bar\theta\partial\!\!\!/\theta)^2 . \end{aligned} \tag{A.8}$$

Using that $\varepsilon^{abc}\Gamma_{cd} = \Gamma^\star(\delta^a_d\Gamma^b - \delta^b_d\Gamma^a)$ we get

$$L_{\theta^4} = \tfrac{1}{2}\bar\theta\Gamma^\star\Gamma_i\Gamma^{ac}\,\partial_a\theta\,\bar\theta\Gamma^i\partial_c\theta + \tfrac{1}{2}\bar\theta\Gamma_i\partial_a\theta\,\bar\theta\Gamma^i\partial^a\theta - \tfrac{1}{2}\bar\theta(1-\Gamma^\star)\Gamma_a\partial_b\theta\,\bar\theta\Gamma^b\partial^a\theta + \tfrac{1}{2}\bar\theta(1-\Gamma^\star)\partial\!\!\!/\theta\,\bar\theta\partial\!\!\!/\theta\,. \tag{A.9}$$

Fixing the κ-symmetry gauge as in (3.5) we conclude that $L_{X\theta^2}$ in (A.5) vanishes. Using that $[\Gamma^\star,\Gamma^a]=0$ and $\{\Gamma^\star,\Gamma^i\}=0$, the terms in the first line in (A.9) vanish because of $\bar\theta = \bar\theta P_-$, $P_-\Gamma_i = \Gamma_i P_+$ and $P_+\theta = 0$. We thus end up with the expressions in (3.7), (3.8), (3.9) (after the rescaling $\theta \to \frac{1}{\sqrt{2}}\theta$).

In the GS string case (3.10), (3.11) the volume part L_1 is the same, i.e. we get again the expansion in (A.2). For the WZ part we have (after using that in the static gauge $\partial_a X^b = \delta^a_b$)

$$\begin{aligned} L_2 &= -i\,\varepsilon^{ab}\,\bar\theta\Gamma_\mu\Gamma_2\,\partial_a\theta\,(\partial_b X^\mu - \tfrac{i}{2}\bar\theta\Gamma^\mu\partial_b\theta) \\ &= -i\,\varepsilon^{ab}\,\bar\theta\Gamma_b\Gamma_2\,\partial_a\theta - i\,\varepsilon^{ab}\,\bar\theta\Gamma^i\Gamma_2\,\partial_a\theta\,\partial_b X^i - \tfrac{1}{2}\varepsilon^{ab}\,\bar\theta\Gamma_c\Gamma_2\,\partial_a\theta\,\bar\theta\Gamma^c\partial_b\theta - \tfrac{1}{2}\varepsilon^{ab}\,\bar\theta\Gamma^i\Gamma_2\,\partial_a\theta\,\bar\theta\Gamma^i\partial_b\theta. \end{aligned} \tag{A.10}$$

Fixing the κ-symmetry gauge as in (3.12) we have again $\bar\theta = \bar\theta P_-$, $P_-\theta = 0$; since $[P_-,\Gamma_a] = [P_-,\Gamma_2] = 0$, $P_\pm\Gamma^i = \Gamma^i P_\mp$ we get

$$L_2 = -i\,\varepsilon^{ab}\,\bar\theta\Gamma_b\Gamma_2\,\partial_a\theta - \tfrac{1}{2}\varepsilon^{ab}\,\bar\theta\Gamma_c\Gamma_2\,\partial_a\theta\,\bar\theta\Gamma^c\partial_b\theta = i\,\bar\theta\partial\!\!\!/\theta - \tfrac{1}{2}\bar\theta\,\Gamma^b\partial_c\theta\,\bar\theta\Gamma^c\partial_b\theta + \tfrac{1}{2}(\bar\theta\,\partial\!\!\!/\theta)^2, \tag{A.11}$$

where we used that $\varepsilon^{ab}\Gamma_b\Gamma_2 = -\Gamma^a\Gamma^\star$, $\varepsilon^{ab}\Gamma_c\Gamma_2 = \Gamma^\star(\delta^a_c\Gamma^b - \delta^b_c\Gamma^a)$. Combining this with (A.2) (where jot 8pt $\bar\theta\Gamma^i\partial_a\theta\,\bar\theta\Gamma^i\partial^a\theta$ vanishes due to the P_- projection) we find

$$\begin{aligned} L &= 2i\bar\theta\partial\!\!\!/\theta + i(\tfrac{1}{2}\eta^{ab}\partial_c X^i\partial_c X^i - \partial^a X^i\partial^b X^i)\bar\theta\Gamma_a\partial_b\theta \\ &\quad + \tfrac{1}{2}\bar\theta\Gamma_b\partial_a\theta\,\bar\theta\Gamma^b\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\,\bar\theta\Gamma^b\partial^a\theta - \tfrac{1}{2}\bar\theta\Gamma_a\partial_b\theta\,\bar\theta\Gamma^a\partial^b\theta - \tfrac{1}{2}\bar\theta\,\Gamma^b\partial_c\theta\,\bar\theta\Gamma^c\partial_b\theta + (\bar\theta\,\partial\!\!\!/\theta)^2 \\ &= 2i\bar\theta\partial\!\!\!/\theta + i(\tfrac{1}{2}\eta^{ab}\partial_c X^i\partial_c X^i - \partial^a X^i\partial^b X^i)\bar\theta\Gamma_a\partial_b\theta - \bar\theta\Gamma_a\partial_b\theta\,\bar\theta\Gamma^b\partial^a\theta + (\bar\theta\,\partial\!\!\!/\theta)^2\,. \end{aligned} \tag{A.12}$$

This leads to (3.13), (3.14) after the rescaling $\theta \to \frac{1}{\sqrt{2}}\theta$.

B Some useful relations

In $D = 11$, the Majorana fermions have 32 real components. After κ-symmetry gauge fixing we are left with $n_F = 16$ real components. The number of physical 3d fermionic degrees of freedom is further halved by the Dirac equation $\partial\!\!\!/\theta = 0$, i.e it is given by $\frac{1}{2}n_F = 8$. This is same as the number $\widehat{D} = D - 3 = 8$ of scalar X^i bosonic degrees of freedom. We have for the spinor traces

$$\mathrm{tr}\, I = 2n_F, \qquad \mathrm{tr}\, P_- = n_F\,. \tag{B.1}$$

Considering a loop of Majorana fermions, we have an extra $-1/2$ factor so that effectively we get $\operatorname{tr} P_- \to -\frac{1}{2}n_F = -8$. Some other basic traces are

$$\operatorname{tr}(P_-\Gamma_a\Gamma_b) = \eta_{ab}\ \operatorname{tr} P_-, \qquad \operatorname{tr}(P_-\Gamma_a\Gamma_b\Gamma_c\Gamma_d) = (\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc})\ \operatorname{tr} P_-, \tag{B.2}$$

$$\operatorname{tr}(P_-\Gamma_{a_1}\dots\Gamma_{a_n}\Gamma^{ij}) = 0, \qquad \operatorname{tr}(P_-\Gamma_a\Gamma_b\Gamma^{ij}\Gamma^{k\ell}) = \eta_{ab}(\delta^{i\ell}\delta^{jk} - \delta^{ik}\delta^{j\ell})\ \operatorname{tr} P_-. \tag{B.3}$$

Using the definition of P_- in (3.5) we have

$$\operatorname{tr}(P_-\Gamma_a\Gamma_b\Gamma_c) = \varepsilon_{abc}\operatorname{tr} P_- \ . \tag{B.4}$$

Similarly, $\operatorname{tr}(P_-\Gamma_a\Gamma_b\Gamma_c\Gamma^{i_1i_2}\Gamma^{i_3i_4}\Gamma^{i_5i_6})$ factorizes into $\operatorname{tr} P_-\ \varepsilon_{abc}$ times a combination of δ-symbols.

We use also that

$$\epsilon^{abc}\epsilon^{a'b'c'} = -\begin{vmatrix} \eta^{aa'} & \eta^{ab'} & \eta^{ac'} \\ \eta^{ba'} & \eta^{bb'} & \eta^{bc'} \\ \eta^{ca'} & \eta^{cb'} & \eta^{cc'} \end{vmatrix}, \tag{B.5}$$

which implies, in particular, that $(\varepsilon^{abc}p_{1,a}p_{2,b}p_{3,c})^2 = \frac{1}{4}stu$.

Some basic momentum integrals are

$$\frac{1}{P_1^{\alpha_1}\dots P_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\dots+\alpha_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)}\int_0^1 dx_1\dots dx_n\,\delta\left(1-\sum_{i=1}^n x_i\right)\frac{x_1^{\alpha_1-1}\dots x_n^{\alpha_n-1}}{(\sum_{i=1}^n x_iP_i)^{\alpha_1+\dots+\alpha_n}}, \tag{B.6}$$

$$\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2+M^2)^n} = \frac{i\Gamma\left(n-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(n)}(M^2)^{\frac{d}{2}-n}, \tag{B.7}$$

$$\int\frac{d^dp}{(2\pi)^d}\frac{p^ap^b}{(p^2+M^2)^n} = \frac{i\Gamma\left(n-1-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(n)}(M^2)^{\frac{d}{2}+1-n}\frac{1}{2}\eta^{ab}, \tag{B.8}$$

$$\int\frac{d^dp}{(2\pi)^d}\frac{p^ap^bp^cp^d}{(p^2+M^2)^n} = \frac{i\Gamma\left(n-2-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(n)}(M^2)^{\frac{d}{2}+2-n}\frac{1}{4}(\eta^{ab}\eta^{cd}+\eta^{ac}\eta^{bd}+\eta^{ad}\eta^{bc}). \tag{B.9}$$

C Geometrical relations for induced metric

C.1 Extrinsic curvature identities

The induced metric on a brane in flat target space is defined as $h_{ab} = \eta_{\mu\nu}\partial_aX^\mu\partial_bX^\nu$. The tangent space is spanned by $\{\partial_aX^\mu\}$. The normal bundle has the basis $\{(n^i)^\mu\}$ such that

$$n^i_\mu(n^j)^\mu = \delta^{ij}, \qquad n^i_\mu\partial_aX^\mu = 0, \qquad i,j = 1,\dots,D-d\,. \tag{C.1}$$

Let ∇_a be the covariant derivative with respect to the induced metric so that $\nabla_ah_{bc} = 0$. Then $\nabla_a\partial_bX^\mu$ is in the normal bundle so that we can introduce the extrinsic curvature K as its coefficients

$$\nabla_a\partial_bX^\mu = K^i_{ab}(n^i)^\mu, \qquad K^i_{ab} = K^i_{ba}\,. \tag{C.2}$$

In the static gauge $X^a = \sigma^a$ we have $h_{ab} = \eta_{ab} + \partial_aX^i\partial_bX^i$ and the tangent vectors are $\partial_aX^\mu = (\delta^b_a, \partial_aX^i)$. The normal basis vectors can be chosen as

$$n^i_\mu = (-\partial_aX^i, \delta^i_j) + O((\partial X)^2)\,. \tag{C.3}$$

Then from (C.2) we get that[38]

$$\nabla_a\partial_b X^i = K^j_{ab}(n^j)^i = K^i_{ab} + \mathcal{O}((\partial X)^2), \qquad\qquad K^i_{ab} = \partial_a\partial_b X^i + \mathcal{O}((\partial X)^3). \tag{C.4}$$

In the case of the flat target space, we have the Gauss-Codazzi relation[39]

$$R_{abcd} = K^i_{ac}K^i_{bd} - K^i_{ad}K^i_{bc}. \tag{C.5}$$

Let us discuss some relations for K and its derivatives when expanding them in powers of X^i assumed to be subject to the free equations of motion. Let us denote by $\approx$ the equality up to terms that vanish on-shell (i.e. for $\partial^2 X = 0$) or are of higher order in X. From (C.2) we have

$$K^i \equiv \eta^{ab}K^i_{ab} \approx 0. \tag{C.6}$$

Contracting two indices in (C.5) we get[40]

$$R_{ab} = K^i K^i_{ab} - K^i_{ac}K^i{}_b{}^c \approx -K^i_{ac}K^i{}_b{}^c\,. \tag{C.7}$$

Using that $0 = \nabla_a h_{cd} = \nabla_a(\eta_{cd} + \partial_c X^i\partial_d X^i) = K^i_{ac}\partial_d X^i + K^i_{ad}\partial_c X^i$ and contracting this with η^{ac} and using (C.6) gives

$$K^i_{ab}\partial^a X^i \approx 0. \tag{C.8}$$

Considering

$$\nabla_a K^i_{bc} - \nabla_b K^i_{ac} = [\nabla_a, \nabla_b]\partial_c X^i = R_{dcab}\partial^d X^i, \tag{C.9}$$

and contracting this with η^{bc} and using (C.6), (C.7) we get

$$\nabla^a K^i_{ab} = R_{bc}\partial^c X^i \approx K^j_{bd}\,K^j{}_c{}^d\,\partial^c X^i. \tag{C.10}$$

C.2 Expansion of scalar curvature density

Given

$$L = \sqrt{-h}\,R^{(d)}\,, \qquad h_{ab} = \eta_{ab} + \mathrm{h}_{ab}, \qquad \mathrm{h}_{ab} = \partial_a X^i\partial_b X^i\,, \tag{C.11}$$

let us consider the expansion of L in powers of h_{ab}. The linear term in h_{ab} is a total derivative: $L_1 = \partial_a\partial_b \mathrm{h}^{ab} - \Box\,\mathrm{h}$, $\mathrm{h} = \eta^{ab}\mathrm{h}_{ab}$. The quadratic term is

$$L_2 = -\tfrac{1}{4}\big(\partial^c\mathrm{h}^{ab}\,\partial_c\mathrm{h}_{ab} - 2\partial_a\mathrm{h}^{ab}\,\partial^c\mathrm{h}_{bc} + 2\partial_a\mathrm{h}\,\partial_b\mathrm{h}^{ab} - \partial^a\mathrm{h}\partial_a\mathrm{h}\big)\,. \tag{C.12}$$

In the special case of $d = 2$ this can be shown to be a total derivative. Indeed, in 2 dimensions we have an identity following from the fact that the total antisymmetrization

[38] Corrections to the leading term in K come from the covariant derivative in (C.2) and also from Gram-Schmidt orthogonalization procedure. The contributions from the connection are found using that

$$h^{ab} = \eta^{ab} - \partial^a X^i\partial^b X^i + \mathcal{O}(X^4), \qquad \Gamma^c_{ab} = \tfrac{1}{2}(\partial_b h_{ca} + \partial_a h_{cb} - \partial_c h_{ab}) + O(X^4) = \partial^c X^j\partial_{ab}X^j + \mathcal{O}(X^4).$$

[39] In $d = 2$ one has $R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R$ and $R_{ab} = \frac{1}{2}g_{ab}R$, while in $d = 3$ where the Weyl tensor vanishes

$$R_{abcd} = g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac} - \tfrac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

[40] We also have the identity $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$ which implies relations between the derivatives and products of K.

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of 3 indices vanishes. In particular (here the repeated indices are contracted with η_{ab} and we use that $[ab'c] = 0$)

$$\begin{aligned}
0 &= \partial_c \mathrm{h}_{ab}\partial_{c'}\mathrm{h}_{a'b'}[\delta_{aa'}\delta_{bb'}\delta_{cc'} - \delta_{ac'}\delta_{bb'}\delta_{ca'} + \delta_{ac'}\delta_{bc}\delta_{b'a'} - \delta_{ab}\delta_{cc'}\delta_{a'b'} - \delta_{aa'}\delta_{bc}\delta_{b'c'} + \delta_{ab}\delta_{a'c}\delta_{b'c'}] \\
&= \partial_c \mathrm{h}_{ab}\partial_c \mathrm{h}_{ab} - \partial_c \mathrm{h}_{ab}\partial_a \mathrm{h}_{bc} + \partial_c \mathrm{h}_{ac}\partial_a \mathrm{h}_{a'a'} - \partial_c \mathrm{h}_{aa}\partial_c \mathrm{h}_{a'a'} - \partial_c \mathrm{h}_{ac}\partial_{b'}\mathrm{h}_{ab'} + \partial_c \mathrm{h}_{aa}\partial_{b'}\mathrm{h}_{cb'} \\
&= (\partial_a \mathrm{h}_{bc})^2 - \partial_c \mathrm{h}_{ab}\partial_a \mathrm{h}_{bc} + 2\partial_c \mathrm{h}_{ac}\partial_a \mathrm{h} - \partial_c \mathrm{h}\partial_c \mathrm{h} - \partial_c \mathrm{h}_{ac}\partial_{b'}\mathrm{h}_{ab'}. \qquad \text{(C.13)}
\end{aligned}$$

Using this in the first term in (C.12) we get

$$\begin{aligned}
L_2\big|_{d=2} &= -\tfrac{1}{4}[\partial_c \mathrm{h}_{ab}\partial_a \mathrm{h}_{bc} - 2\partial_c \mathrm{h}_{ac}\partial_a \mathrm{h} + \partial_c \mathrm{h}\partial_c \mathrm{h} + \partial_c \mathrm{h}_{ac}\partial_{b'}\mathrm{h}_{ab'} - 2\partial_a \mathrm{h}_{ab}\partial_c \mathrm{h}_{bc} + 2\partial_a \mathrm{h}\partial_b \mathrm{h}_{ab} - \partial_a \mathrm{h}\partial_a \mathrm{h}] \\
&= -\tfrac{1}{4}[\partial_c(\mathrm{h}_{ab}\partial_a \mathrm{h}_{bc}) - \partial_a(\mathrm{h}_{ab}\partial_c \mathrm{h}_{bc})] = \tfrac{1}{4}\partial_a[\mathrm{h}_{ab}\partial_c \mathrm{h}_{bc} - \mathrm{h}_{bc}\partial_c \mathrm{h}_{ab}]. \qquad \text{(C.14)}
\end{aligned}$$

In fact, $L\big|_{d=2}$ is a total derivative to all orders in h_{ab} (this is an integrand of the Euler number invariant in 2d).

For general d one finds from (C.12) using the expression for h_{ab} in (C.11) and dropping total derivative terms and terms that vanish on the equation of motion $\partial^2 X^i = 0$

$$L_2 = -\partial^a X^i \partial^b \partial^c X^i \; \partial_b X^j \partial_a \partial_c X^j \,. \qquad \text{(C.15)}$$

This agrees with the expression used, e.g., in [32, 35].

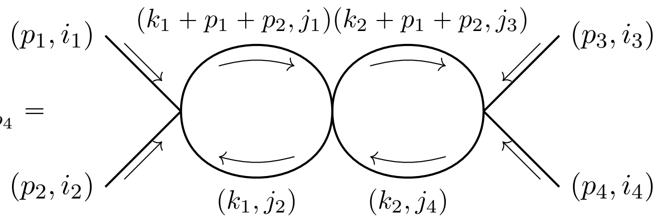
D Details of computation of 2-loop diagrams

Here we present some details of the computation of the three types of diagrams in (2.16) in the bosonic brane case using IBP and tensor reduction implemented in FIRE [50]. One can treat the string $d = 2$ and membrane $d = 3$ cases in parallel keeping d generic till the final expansion stage.

D.1 Double-bubble diagram

The double-bubble diagram gives the following integral expression (the vertex V is given in (2.4))

$\mathrm{DB}^{i_1 i_2 i_3 i_4}_{p_1,p_2,p_3,p_4} =$



$$= \frac{1}{4i}\int \widetilde{dk_1}\,\widetilde{dk_2}\,\frac{V^{i_1 i_2 j_1 j_2}_{p_1,p_2,-p_1-p_2-k_1,k_1} V^{j_1 j_2 j_3 j_4}_{p_1+p_2+k_1,-k_1,-p_1-p_2-k_2,k_2} V^{j_3 j_3 i_3 i_4}_{p_1+p_2+k_2,-k_2,-p_3,-p_4}}{k_1^2 k_2^2 (k_1+p_1+p_2)^2 (k_2+p_1+p_2)^2}, \qquad \text{(D.1)}$$

where $\widetilde{dk} = \frac{d^dk}{(2\pi)^d}$. Application of IBP and tensor reduction implemented in FIRE gives

$$\mathrm{DB}^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} = A^{(2)}_{\mathrm{DB}}\delta^{i_1i_2}\delta^{i_3i_4} + B^{(2)}_{\mathrm{DB}}\delta^{i_1i_3}\delta^{i_2i_4} + C^{(2)}_{\mathrm{DB}}\delta^{i_1i_4}\delta^{i_2i_3}\,, \tag{D.2}$$

$$A^{(2)}_{\mathrm{DB}} = \frac{s^4}{2048(-1+d^2)^2}\Big[((-2+d)(24(-2+\widehat{D}) - d^2(-48+\widehat{D}^2) + d^3\,(48-12\widehat{D}+\widehat{D}^2)$$
$$- 4d(12-9\widehat{D}+\widehat{D}^2))\; s^2 - 16\widehat{D}^2st - 16\widehat{D}^2t^2)\Big]\,[G_{1,1}]^2,$$
$$B^{(2)}_{\mathrm{DB}} = -\frac{s^5((2-2d+d^2)s+2t)}{64(-1+d)^2}\,[G_{1,1}]^2, \qquad C^{(2)}_{\mathrm{DB}} = -\frac{s^5(-2ds+d^2s-2t)}{64(-1+d)^2}\,[G_{1,1}]^2\,,$$

where $G_{1,1}$ was given in (2.10). In the string case we set $d = 2-2\varepsilon$ and $\widehat{D} = D-2$. Expanding $[G_{1,1}]^2$ in ε we then get

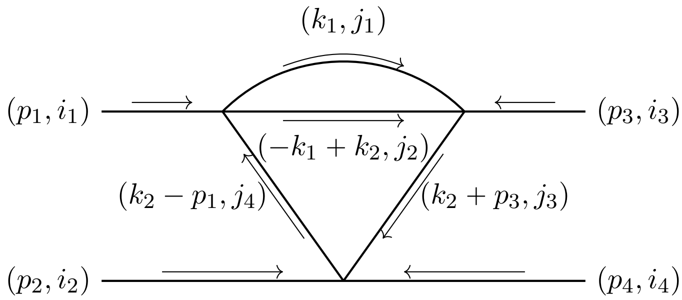
$$[G_{1,1}]^2 = -\frac{1}{4\pi^2}\left[\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}\left(\log\frac{-s}{4\pi} + \gamma_{\mathrm{E}}\right)\right]\frac{1}{s^2} + \text{finite}. \tag{D.3}$$

The contributions (D.2) should be summed over diagrams related by crossing.

D.2 Wine-glass diagram

For this diagram we get

$$\mathrm{W}^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} = \tag{D.4}$$



$$= \frac{1}{2i}\int \widetilde{dk_1}\,\widetilde{dk_2}\,\frac{V^{i_1j_1j_2j_4}_{p_1,-k_1,k_1-k_2,k_2-p_1}\,V^{j_1i_3j_3j_2}_{k_1,p_3,-k_2-p_3,-k_1+k_2}\,V^{i_2j_4j_3i_4}_{p_2,-k_2+p_1,k_2+p_3,p_4}}{k_1^2(k_1-k_2)^2(k_2+p_3)^2(k_2-p_1)^2}.$$

Application of FIRE gives

$$\mathrm{W}^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} = A^{(2)}_{\mathrm{W}}\delta^{i_1i_2}\delta^{i_3i_4} + B^{(2)}_{\mathrm{W}}\delta^{i_1i_3}\delta^{i_2i_4} + C^{(2)}_{\mathrm{W}}\delta^{i_1i_4}\delta^{i_2i_3}\,, \tag{D.5}$$

$$A^{(2)}_{\mathrm{W}} = Q(-3+d)t^5(-8(-1+d)(1+d)(-10+d(7+2d))s + (-2+d)(-12+d(1+d)(2+d))\widehat{D}s$$
$$-8(-1+d)(1+d)(-8+d(8+d))t - (-2+d)d(8+(-3+d)d)\widehat{D}t),$$
$$B^{(2)}_{\mathrm{W}} = Q(-3+d)t^4((-8(-1+d)(1+d)(2+d) + (-24+d(32+d(8-d(7+3d))))\widehat{D})s^2$$
$$+(-8(-1+d)(1+d)(2+d) + (-24+d(32+d(8-d(7+3d))))\widehat{D})st$$
$$+2(-2+d)(-1+d)(20+d(8+3d(-4+\widehat{D})-5\widehat{D})-6\widehat{D})t^2),$$
$$C^{(2)}_{\mathrm{W}} = -Q((-3+d)t^5((-8(-1+d)(1+d)(-10+d(7+2d))$$
$$+(-2+d)(-12+d(1+d)(2+d))\widehat{D})s + 2(-2+d)(-1+d)(-4(1+d)^2+(6+d+d^2)\widehat{D})t)),$$
$$Q \equiv \frac{1}{144(d^2-1)(3d-8)(3d-4)(3d-2)}\,G_{1,1,1,1}\,.$$

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The master integral here is

$$G_{1,1,1,1} = \int \widetilde{dk_1}\,\widetilde{dk_2} \frac{1}{k_1^2(k_1-k_2)^2(k_2+p_3)^2(k_2-p_1)^2}. \tag{D.6}$$

Integrating over k_1 using (2.10) we get

$$G_{1,1,1,1} = C_d \int \widetilde{dk} \frac{1}{(k^2)^{2-\frac{d}{2}}(k+p_3)^2(k-p_1)^2}, \qquad C_d = \tfrac{i}{(4\pi)^{d/2}} \tfrac{\Gamma\left(2-\frac{d}{2}\right)\left[\Gamma\left(\frac{d}{2}-1\right)\right]^2}{\Gamma(d-2)}. \tag{D.7}$$

Introducing Feynman parameters as in (B.6) with $x_3 = 1 - x_1 - x_2$ we have

$$G_{1,1,1,1} = C_d \frac{\Gamma\left(4-\frac{d}{2}\right)}{\Gamma\left(2-\frac{d}{2}\right)} \int [d^3x]\, x_1^{1-\frac{d}{2}} \int \widetilde{dk} \frac{1}{[x_1k^2 + x_2(k+p_3)^2 + x_3(k-p_1)^2]^{4-\frac{d}{2}}}. \tag{D.8}$$

Shifting k and using $p_i^2 = 0$ we have $x_1k^2 + x_2(k+p_3)^2 + x_3(k-p_1)^2 \to k^2 + 2x_2x_3p_1\cdot p_3$. Integrating over k using (B.7) and also over x_i we get

$$G_{1,1,1,1} = -\tfrac{1}{(4\pi)^d} \tfrac{\Gamma(4-d)\Gamma\left(2-\frac{d}{2}\right)\left[\Gamma\left(\frac{d}{2}-1\right)\right]^2[\Gamma(d-3)]^2}{\Gamma(d-2)\Gamma\left(\frac{3}{2}d-4\right)} \, (2p_1\cdot p_3)^{d-4}. \tag{D.9}$$

Expanded for $d = 2 - 2\varepsilon$ this gives

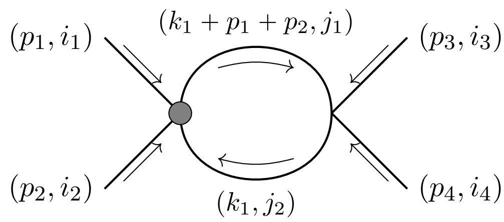
$$G_{1,1,1,1} = \tfrac{3}{2(4\pi)^2} \left[\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon}\left(\log\frac{-t}{4\pi} + \gamma_E - \tfrac{1}{2}\right) + \ldots\right] \frac{1}{t^2} + \text{finite}. \tag{D.10}$$

The sum over crossing contains 6 diagrams where the labels (1,3) are replaced by all distinct ordered pairs.

D.3 Evanescent counterterm diagram

The third diagram in (2.16) contains one vertex from the 1-loop evanescent counterterm (2.13) with the vertex E in (2.14)[41]

$$\mathrm{CT}^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} =$$



(D.11)

$$= -\tfrac{1}{2i} \int \widetilde{dk_1} \, \frac{E^{i_1j_1j_2i_2}_{p_1,-k_1-p_1-p_2,k_1,p_2} \; V^{j_1i_3i_4j_2}_{k_1+p_1+p_2,p_3,p_4,-k_1}}{k_1^2(k_1+p_1+p_2)^2}.$$

There is also a similar diagram with the counterterm vertex on the right, i.e. with the vertices E and V interchanged. Application of FIRE gives for the sum of these two diagrams

$$\mathrm{CT}^{i_1i_2i_3i_4}_{p_1,p_2,p_3,p_4} = A^{(2)}_{\mathrm{CT}}\delta^{i_1i_2}\delta^{i_3i_4} + B_{\mathrm{CT}}\delta^{i_1i_3}\delta^{i_2i_4} + C_{\mathrm{CT}}\delta^{i_1i_4}\delta^{i_2i_3}, \tag{D.12}$$

$$A^{(2)}_{\mathrm{CT}} = \frac{i\kappa s^3((-2+d)(-4+d(-2+\widehat{D}))s^2 + 8(2+\widehat{D})st + 8(2+\widehat{D})t^2)}{32(-1+d^2)} G_{1,1}, \tag{D.13}$$

$$B^{(2)}_{\mathrm{CT}} = C^{(2)}_{\mathrm{CT}} = -\frac{i(-2+d)\kappa s^5}{8(-1+d)} G_{1,1}, \tag{D.14}$$

[41] As in the expressions above we do not explicitly include normalization $\mu^{2\varepsilon}$ factor (cf. footnote 17).

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where κ is the coefficient in (2.13) and $G_{1,1}$ is given by (2.10). Summing over crossing one gets for the corresponding contribution to the pole part of the 2-loop amplitude in the string case ($d = 2 - 2\varepsilon$, $\widehat{D} = D - 2$)

$$\mathcal{M}^{(2)\, i_1 i_2 i_3 i_4}_{\mathrm{CT},\frac{1}{\varepsilon}}(p_1,p_2,p_3,p_4) = \frac{D}{24\pi}\frac{\kappa}{\varepsilon}\, stu\left[s\,\delta^{i_1 i_2}\delta^{i_3 i_4} + t\,\delta^{i_1 i_3}\delta^{i_2 i_4} + u\,\delta^{i_1 i_4}\delta^{i_2 i_3}\right]. \tag{D.15}$$

D.4 Finite PPS counterterm diagram

Similar result is found for the diagram with the insertion of the vertex V in (2.26) coming from the finite PPS counterterm (1.9) required to preserve the integrability of the bosonic S-matrix for any D. The corresponding contribution to the 2-loop diagram is

$$\begin{aligned}\mathcal{M}^{(2)\, i_1 i_2 i_3 i_4}_{\mathrm{PPS}} = \frac{ibs^3}{32(d^2-1)}\Big[&\tfrac{1}{2}((24-2d-17d^2+4(-2+d)(1+d)\widehat{D})s^2+24st+24t^2)\delta^{i_1 i_2}\delta^{i_3 i_4}\\ &+(2+d)s((-2+d)s-6t)\delta^{i_1 i_3}\delta^{i_2 i_4}+(2+d)s((4+d)s+6t)\delta^{i_1 i_4}\delta^{i_2 i_3}\Big]\,G_{1,1}(s).\end{aligned} \tag{D.16}$$

After summing over crossing, the expansion around $d = 2$ of the corresponding $A^{(2)}_{\mathrm{PPS}}(s,t,u)$ coefficient of $\delta^{i_1 i_2}$ part reads

$$\begin{aligned}A^{(2)}_{\mathrm{PPS}} =& -\tfrac{7b}{8\pi}s^2 t(s+t)\left(\tfrac{1}{\varepsilon}+\gamma_{\mathrm{E}}-\log(4\pi)\right)+\tfrac{b}{48\pi}(-78s^3t-70s^2t^2+16st^3+8t^4+s^4(6D-37))\\ &+\tfrac{b}{8\pi}s^2(-2s^2+st+t^2)\log(-s)-\tfrac{b}{4\pi}st^3\log(-t)+\tfrac{b}{4\pi}s(s+t)^3\log(s+t).\end{aligned} \tag{D.17}$$

Setting $t = 0$ one finds that it contributes only to the finite part of the 2-loop bosonic string amplitude according to (2.28):

$$A^{(2)}_{\mathrm{PPS}}(s,0,-s) = \tfrac{b}{48\pi}s^4(6D-37)+\tfrac{b}{4\pi}s^4\left[\log s-\log(-s)\right] = \tfrac{1}{48\pi}\,b\,(6D-37)\,s^4 + i\tfrac{1}{4}\,b\,s^4. \tag{D.18}$$

A similar counterterm (1.9) is required in the $D = 10$ GS string case [30] where its coefficient is given by (4.9).[42]

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

[42] One may wonder if there is a supersymmetric partner of (1.9) with two X and two θ legs (cf. [62]) that may contribute extra 2-loop diagram with one counterterm $X^2\theta^2$ vertex and the fermionic loop. If this happens, the only change should be a real rational contribution to the amplitude as the imaginary parts of A and C in (4.46) are already cancelled by the contribution (2.28) of the bosonic PPS counterterm.

References

[1] M.B. Green and J.H. Schwarz, *Covariant Description of Superstrings*, *Phys. Lett. B* **136** (1984) 367 [INSPIRE].

[2] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and Eleven-Dimensional Supergravity*, *Phys. Lett. B* **189** (1987) 75 [INSPIRE].

[3] J. Simon, *Brane Effective Actions, Kappa-Symmetry and Applications*, *Living Rev. Rel.* **15** (2012) 3 [arXiv:1110.2422] [INSPIRE].

[4] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys. B* **533** (1998) 109 [hep-th/9805028] [INSPIRE].

[5] I. Bena, J. Polchinski and R. Roiban, *Hidden Symmetries of the $AdS_5 \times S^5$ Superstring*, *Phys. Rev. D* **69** (2004) 046002 [hep-th/0305116] [INSPIRE].

[6] A.B. Zamolodchikov, *Expectation value of composite field T anti-T in two-dimensional quantum field theory*, hep-th/0401146 [INSPIRE].

[7] F.A. Smirnov and A.B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [arXiv:1608.05499] [INSPIRE].

[8] A. Cavaglià, S. Negro, I.M. Szécsényi and R. Tateo, *$T\bar{T}$-deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [arXiv:1608.05534] [INSPIRE].

[9] V. Rosenhaus and M. Smolkin, *Integrability and renormalization under $T\bar{T}$*, *Phys. Rev. D* **102** (2020) 065009 [arXiv:1909.02640] [INSPIRE].

[10] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: Semiclassical partition function*, *JHEP* **04** (2000) 021 [hep-th/0001204] [INSPIRE].

[11] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *JHEP* **06** (2002) 007 [hep-th/0204226] [INSPIRE].

[12] T. Klose, T. McLoughlin, J.A. Minahan and K. Zarembo, *World-sheet scattering in $AdS_5 \times S^5$ at two loops*, *JHEP* **08** (2007) 051 [arXiv:0704.3891] [INSPIRE].

[13] R. Roiban, A. Tirziu and A.A. Tseytlin, *Two-loop world-sheet corrections in $AdS_5 \times S^5$ superstring*, *JHEP* **07** (2007) 056 [arXiv:0704.3638] [INSPIRE].

[14] R. Roiban and A.A. Tseytlin, *Strong-coupling expansion of cusp anomaly from quantum superstring*, *JHEP* **11** (2007) 016 [arXiv:0709.0681] [INSPIRE].

[15] S. Giombi et al., *Quantum $AdS_5 \times S^5$ superstring in the AdS light-cone gauge*, *JHEP* **03** (2010) 003 [arXiv:0912.5105] [INSPIRE].

[16] S. Giombi et al., *Generalized scaling function from light-cone gauge $AdS_5 \times S^5$ superstring*, *JHEP* **06** (2010) 060 [arXiv:1002.0018] [INSPIRE].

[17] L. Bianchi et al., *Two-loop cusp anomaly in ABJM at strong coupling*, *JHEP* **10** (2014) 013 [arXiv:1407.4788] [INSPIRE].

[18] N. Drukker, S. Giombi, A.A. Tseytlin and X. Zhou, *Defect CFT in the 6d (2,0) theory from M2 brane dynamics in $AdS_7 \times S^4$*, *JHEP* **07** (2020) 101 [arXiv:2004.04562] [INSPIRE].

[19] S. Giombi and A.A. Tseytlin, *Wilson Loops at Large N and the Quantum M2-Brane*, *Phys. Rev. Lett.* **130** (2023) 201601 [arXiv:2303.15207] [INSPIRE].

[20] M. Beccaria, S. Giombi and A.A. Tseytlin, *Instanton contributions to the ABJM free energy from quantum M2 branes*, *JHEP* **10** (2023) 029 [arXiv:2307.14112] [INSPIRE].

JHEP09(2025)191

- [21] M. Beccaria, S. Giombi and A.A. Tseytlin, *(2,0) theory on $S^5 \times S^1$ and quantum M2 branes*, Nucl. Phys. B **998** (2024) 116400 [arXiv:2309.10786] [INSPIRE].
- [22] M. Beccaria and A.A. Tseytlin, *Non-planar corrections to ABJM Bremsstrahlung function from quantum M2 brane*, J. Phys. A **58** (2025) 175401 [arXiv:2501.06858] [INSPIRE].
- [23] F.F. Gautason and J. van Muiden, *Localization of the M2-Brane*, Phys. Rev. Lett. **135** (2025) 101601 [arXiv:2503.16597] [INSPIRE].
- [24] F.F. Gautason and J. van Muiden, *Ensembles in M-theory and Holography*, arXiv:2505.21633 [INSPIRE].
- [25] N. Drukker, O. Shahpo and M. Trépanier, *Quantum holographic surface anomalies*, J. Phys. A **57** (2024) 085402 [arXiv:2311.14797] [INSPIRE].
- [26] N. Drukker and O. Shahpo, *Vortex loop operators and quantum M2-branes*, SciPost Phys. **17** (2024) 016 [arXiv:2312.17091] [INSPIRE].
- [27] D. Astesiano et al., *Wilson loops and spherical branes*, SciPost Phys. **18** (2025) 050 [arXiv:2407.07830] [INSPIRE].
- [28] S. Giombi, S.A. Kurlyand and A.A. Tseytlin, *Non-planar corrections in ABJM theory from quantum M2 branes*, JHEP **11** (2024) 056 [arXiv:2408.10070] [INSPIRE].
- [29] M. Beccaria, S.A. Kurlyand and A.A. Tseytlin, *On non-planar ABJM anomalous dimensions from M2 branes in $AdS_4 \times S^7/Z_k$* , JHEP **05** (2025) 187 [arXiv:2503.09360] [INSPIRE].
- [30] F.K. Seibold and A.A. Tseytlin, *Scattering on the supermembrane*, JHEP **08** (2024) 102 [arXiv:2404.09658] [INSPIRE].
- [31] F.K. Seibold and A.A. Tseytlin, *S-matrix on effective string and compactified membrane*, J. Phys. A **56** (2023) 485401 [arXiv:2308.12189] [INSPIRE].
- [32] S. Dubovsky, R. Flauger and V. Gorbenko, *Effective String Theory Revisited*, JHEP **09** (2012) 044 [arXiv:1203.1054] [INSPIRE].
- [33] S. Dubovsky, R. Flauger and V. Gorbenko, *Solving the Simplest Theory of Quantum Gravity*, JHEP **09** (2012) 133 [arXiv:1205.6805] [INSPIRE].
- [34] P. Cooper et al., *Looking for Integrability on the Worldsheet of Confining Strings*, JHEP **04** (2015) 127 [arXiv:1411.0703] [INSPIRE].
- [35] P. Conkey and S. Dubovsky, *Four Loop Scattering in the Nambu-Goto Theory*, JHEP **05** (2016) 071 [arXiv:1603.00719] [INSPIRE].
- [36] O. Aharony and Z. Komargodski, *The Effective Theory of Long Strings*, JHEP **05** (2013) 118 [arXiv:1302.6257] [INSPIRE].
- [37] A.M. Polyakov, *Quantum Geometry of Bosonic Strings*, Phys. Lett. B **103** (1981) 207 [INSPIRE].
- [38] J. Polchinski and A. Strominger, *Effective string theory*, Phys. Rev. Lett. **67** (1991) 1681 [INSPIRE].
- [39] A.A. Tseytlin and Z. Wang, *On world-sheet S-matrix of NSR string in static gauge*, Nucl. Phys. B **1018** (2025) 117057 [arXiv:2505.20262] [INSPIRE].
- [40] Z. Bern et al., *Evanescent Effects Can Alter Ultraviolet Divergences in Quantum Gravity without Physical Consequences*, Phys. Rev. Lett. **115** (2015) 211301 [arXiv:1507.06118] [INSPIRE].
- [41] Z. Bern, H.-H. Chi, L. Dixon and A. Edison, *Two-Loop Renormalization of Quantum Gravity Simplified*, Phys. Rev. D **95** (2017) 046013 [arXiv:1701.02422] [INSPIRE].

[42] W. Siegel, *Supersymmetric Dimensional Regularization via Dimensional Reduction*, Phys. Lett. B **84** (1979) 193 [INSPIRE].

[43] F. Paccanoni, P. Pasti and M. Tonin, *Some Remarks on the Consistency of Quantum Supermembranes*, Mod. Phys. Lett. A **4** (1989) 807 [INSPIRE].

[44] P.S. Howe and U. Lindstrom, *Kappa symmetric higher derivative terms in brane actions*, Class. Quant. Grav. **19** (2002) 2813 [hep-th/0111036] [INSPIRE].

[45] P.S. Howe, S.F. Kerstan, U. Lindstrom and D. Tsimpis, *The Deformed M2-brane*, JHEP **09** (2003) 013 [hep-th/0307072] [INSPIRE].

[46] G. Passarino and M.J.G. Veltman, *One Loop Corrections for e^+e^- Annihilation Into $\mu^+\mu^-$ in the Weinberg Model*, Nucl. Phys. B **160** (1979) 151 [INSPIRE].

[47] F.V. Tkachov, *A theorem on analytical calculability of 4-loop renormalization group functions*, Phys. Lett. B **100** (1981) 65 [INSPIRE].

[48] K.G. Chetyrkin and F.V. Tkachov, *Integration by parts: The algorithm to calculate β-functions in 4 loops*, Nucl. Phys. B **192** (1981) 159 [INSPIRE].

[49] S. Laporta, *High-precision calculation of multiloop Feynman integrals by difference equations*, Int. J. Mod. Phys. A **15** (2000) 5087 [hep-ph/0102033] [INSPIRE].

[50] A.V. Smirnov, *Algorithm FIRE — Feynman Integral REduction*, JHEP **10** (2008) 107 [arXiv:0807.3243] [INSPIRE].

[51] A.V. Smirnov and V.A. Smirnov, *FIRE4, LiteRed and accompanying tools to solve integration by parts relations*, Comput. Phys. Commun. **184** (2013) 2820 [arXiv:1302.5885] [INSPIRE].

[52] A.V. Smirnov, *FIRE5: A C++ implementation of Feynman Integral REduction*, Comput. Phys. Commun. **189** (2015) 182 [arXiv:1408.2372] [INSPIRE].

[53] A.V. Smirnov and F.S. Chukharev, *FIRE6: Feynman Integral REduction with modular arithmetic*, Comput. Phys. Commun. **247** (2020) 106877 [arXiv:1901.07808] [INSPIRE].

[54] A.V. Smirnov and M. Zeng, *FIRE 6.5: Feynman integral reduction with new simplification library*, Comput. Phys. Commun. **302** (2024) 109261 [arXiv:2311.02370] [INSPIRE].

[55] G. Goon, S. Melville and J. Noller, *Quantum corrections to generic branes: DBI, NLSM, and more*, JHEP **01** (2021) 159 [arXiv:2010.05913] [INSPIRE].

[56] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Superstrings in $D = 10$ from Supermembranes in $D = 11$*, Phys. Lett. B **191** (1987) 70 [INSPIRE].

[57] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Properties of the Eleven-Dimensional Super Membrane Theory*, Annals Phys. **185** (1988) 330 [INSPIRE].

[58] A. Achucarro, J.M. Evans, P.K. Townsend and D.L. Wiltshire, *Super p-Branes*, Phys. Lett. B **198** (1987) 441 [INSPIRE].

[59] R. Kallosh, *World volume supersymmetry*, Phys. Rev. D **57** (1998) 3214 [hep-th/9709069] [INSPIRE].

[60] R. Kallosh, *Quantization of p-branes, D-p-branes and M-branes*, Nucl. Phys. B Proc. Suppl. **68** (1998) 197 [hep-th/9709202] [INSPIRE].

[61] M. Shmakova, *One loop corrections to the D3-brane action*, Phys. Rev. D **62** (2000) 104009 [hep-th/9906239] [INSPIRE].

[62] Z. Yang, *Effective superstrings*, Nucl. Phys. B **377** (1992) 528 [hep-th/9109008] [INSPIRE].