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L^p-L^q estimates for transition semigroups associated to dissipative stochastic systems

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ABSTRACT

In a separable Hilbert space, we study supercontractivity and ultracontractivity properties for a transition semigroup associated with a stochastic partial differential equation. This is done in terms of exponential integrability of Lipschitz functions and some logarithmic Sobolev-type inequalities with respect to invariant measures. The abstract characterization results concerning the improvement of summability can be applied to transition semigroups associated to stochastic reaction-diffusion equations.

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1. Introduction

Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and induced norm $\|\cdot\|_H$. We are interested in studying the stochastic partial differential equation

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + RdW(t), & t > 0, \\ X(0) = x \in H, \end{cases} \tag{1.1}$$

where $\{W(t)\}_{t\geq 0}$ is a H-cylindrical Wiener process defined on a normal filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ (see e.g. [21, Section 4.1.2] for details on cylindrical Wiener processes in infinite dimension), $F : \mathrm{Dom}(F) \subseteq H \to H$ is a regular enough function, $A : \mathrm{Dom}(A) \subseteq H \to H$ is a linear closed operator and $R : H \to H$ is a linear bounded, self-adjoint and positive operator. Under suitable assumptions, problem (1.1) has a unique generalized mild solution $\{X(t,x)\}_{t\geq 0}$ that allows us to consider the transition semigroup $\{P(t)\}_{t\geq 0}$ associated to (1.1) defined by

$$(P(t)\varphi)(x) := \mathbb{E}\left[\varphi(X(t,x))\right] = \int_\Omega \varphi(X(t,x)(\omega))\mathbb{P}(d\omega)$$

for every $t > 0$, $x \in H$ and any Borel bounded function $\varphi : H \to \mathbb{R}$. Moreover, assuming some additional hypotheses on A, F and R, it is possible to prove existence and uniqueness of a probability invariant measure ν for $\{P(t)\}_{t\geq 0}$, namely a Borel probability measure on H such that the equality

$$\int_H P(t)\varphi d\nu = \int_H \varphi d\nu, \qquad t > 0, \tag{1.2}$$

holds true for every bounded and continuous function $\varphi : H \to \mathbb{R}$. By the invariance of ν, the semigroup $\{P(t)\}_{t\geq 0}$ is uniquely extendable to a strongly continuous semigroup $\{P_p(t)\}_{t\geq 0}$ in $L^p(H,\nu)$ for every $p \geq 1$. Such semigroups are consistent, i.e., for every $f \in L^p(H,\nu) \cap L^q(H,\nu)$ it holds that $P_p(t)f = P_q(t)f$. For this reason, we will omit the subscript p, if no confusion can arise, and we continue to denote them as $\{P(t)\}_{t\geq 0}$.

The best known, and most studied, of these properties is the case $1 < p < q < +\infty$ and $\|P(t)\|_{\mathcal{L}(L^p(H,\nu),L^q(H,\nu))} \leq 1$ for any $t \geq \bar{t} = \bar{t}(p,q) > 0$. In this case $\{P(t)\}_{t\geq 0}$ is said to be hypercontractive and lots of results concerning this property are well known both in finite and infinite dimension (see [3,6,9,24–26,30,40]). Classical results in this direction concern with the connection between the hypercontractivity property, logarithmic Sobolev inequality (see [24–26]), and estimates for Gaussian tails of Lipschitz functions (see [7, Chapter 5] and [31, Sections 2.2 and 2.3]).

Here, we are interested in investigating connections of this type in infinite dimension in the case of stronger improving summability properties and in characterizing them in terms of functional inequalities and integrability properties of exponential functions with respect to the invariant measure ν.

Indeed, in a "scale" of summability improving properties, the next in line, stronger than the hypercontractivity, is the supercontractivity which means that for any $t > 0$, the semigroup $P(t)$ belongs to $\mathcal{L}(L^p(H,\nu), L^q(H,\nu))$ for any $1 < p < q < \infty$ and $\|P(t)\|_{\mathcal{L}(L^p(H,\nu),L^q(H,\nu))} \leq 1$ for any p, q and any positive t. In finite dimension, there exists a quite large literature on this topic, see for instance [1,5,33,39]. In [5] the authors prove that if $H = \mathbb{R}^n$, the supercontrativity property of $\{P(t)\}_{t\geq 0}$ is equivalent to a stronger version (or defective, see [31, Section 2.4]) of the logarithmic Sobolev inequality and to exponential integrability of Lipschitz functions. In infinite dimension a relation between defective logarithmic Sobolev inequality and the exponential integrability of Lipschitz functions is known, see [31, Section 2.4]. However, to the best of our knowledge, the characterization result in [5] is confined to the finite dimensional case. The goal of this paper is to improve some results in [7, Chapter 5] and [31, Chapter 2] in our setting to obtain an infinite dimensional version of the characterization results as in [5]. In a theoretical framework, we are also able to characterize the ultraboundedness of $\{P(t)\}_{t\geq 0}$, i.e. $\|P(t)\|_{\mathcal{L}(L^p(H,\nu)),L^\infty(H,\nu))} < +\infty$ for any $t > 0$ and $p > 1$.

We underline that the abstract results of this paper are applicable to the transition semigroup $P(t)$ associated to the generalized mild solution of the well-studied stochastic reaction-diffusion equation

$$\begin{cases} dX(t) = \left[\frac{\partial^2}{\partial \xi^2} X(t) - b(X(t))\right] dt + dW(t), & t > 0, \\ X(t)(0) = X(t)(1), & t > 0, \\ X(0) = x \in L^2([0,1], \lambda) \end{cases} \tag{1.3}$$

where λ is the Lebesgue measure on $[0,1]$ and $b : \mathbb{R} \to \mathbb{R}$ is a smooth enough polynomial function, see Section 7.3 for details and [14,17] for further results on this problem. In particular, we point out that the characterization results complement the hypercontractivity result obtained in [17] and the ultraboundness result obtained in [19]. In particular in [19] it is showed that $P(t)(L^2(H,\nu)) \subseteq L^\infty(H,\nu)$. Here in Example 7.3, thanks to Proposition 6.1 we show that

$$P(t)(L^p(H,\nu)) \subseteq L^\infty(H,\nu),$$

for every $p > 1$.

We underline that it is natural to wonder whether it is possible to take advantage of the results in finite dimension to extend them to the infinite dimensional setting, letting the dimension to infinity. It must be emphasized that this technique does not work directly in our situation and the results obtained in this paper are not a trivial extension to those proved in finite dimension. Indeed, even if the semigroup $\{P(t)\}_{t\geq 0}$ can be approximated in some sense by a sequence of finite dimensional semigroups $\{P_n(t)\}$, $n \in \mathbb{N}$, we cannot relate the improving summability properties of $\{P(t)\}$ in terms of those of $\{P_n(t)\}$ and viceversa. This is due essentially to the fact that in many cases, the relation between the invariant measure ν associated to $\{P(t)\}_{t\geq 0}$ and the sequence of the invariant measures

ν_n associated to $\{P_n(t)\}$ is not known. This is, for instance the reason why it is not possible to prove the analogue of [5, Theorem 3.1] in our setting. We conclude observing that a classical tool that fails in the transition from the finite dimension to the infinite one, and that in the first case allows to prove L^p-L^q estimates, is the use of Lyapunov functions and some related estimates that depend explicitly on the dimension and that cannot be passed to the limit (see Remark 5.10).

The paper is organized in the following way. In Section 2 we collect our standing assumptions and recall some basic results. In Sections 3 we prove that the operator

$$R\nabla : C_b^1(H) \subseteq L^2(H,\nu) \to L^2(H,\nu;H),$$

where ∇ is the Fréchet gradient and $C_b^1(H)$ is the set of bounded and Fréchet differentiable functions with bounded and continuous derivative, is closable. This property allows us to define the Sobolev space $W_R^{1,2}(H,\nu)$ as the domain of its closure. We point out that such kind of results has been already proved using more restrictive conditions in [10,17].

In Section 4, we prove a logarithmic Sobolev inequality associated to ν that thanks to [26, Theorem 3.7] yields the hypercontractivity of $\{P(t)\}_{t\geq 0}$. Furthermore, we prove a less standard Fernique-type result for ν, both with respect to the norm of H and with respect to the norm $\|R^{-1}\cdot\|_H$ (see Theorems 4.5 and 4.7). In Section 5, we study the supercontractivity of $\{P(t)\}_{t\geq 0}$ and its connections with stronger logarithmic Sobolev-type inequalities and exponential integrability of Lipschitz functions with respect to ν. We stress that the results contained in Theorems 5.6 and 5.7 are stronger than those in [31, Section 2.4]. This is due to the explicit estimates for the coefficients in (5.13). Further in Theorem 5.9, assuming $\nu(R(H)) = 1$, we characterize the supercontractivity of $\{P(t)\}_{t\geq 0}$.

In Section 6 we prove a characterization result for the ultraboundeness of $\{P(t)\}_{t\geq 0}$ (see Theorem 6.1) and we provide sufficient conditions on F (Hypotheses 6.2) that guarantee such a property (see Proposition 6.4). Finally, Section 7 is devoted to provide explicit examples of stochastic partial differential equations whose coefficients satisfy our assumptions and to which our results apply.

Notations

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Banach spaces equipped with the norms $\|\cdot\|_{\mathcal{K}_1}$ and $\|\cdot\|_{\mathcal{K}_2}$, respectively. Let H be a Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle_H$ and associated norm $\|\cdot\|_H$.

We denote by $\mathcal{L}(\mathcal{K}_1;\mathcal{K}_2)$ the space of linear and continuous maps from $\mathcal{K}_1$ to $\mathcal{K}_2$, if $\mathcal{K}_1 = \mathcal{K}_2$ we write $\mathcal{L}(\mathcal{K}_1)$.

We denote by $\mathcal{B}(\mathcal{K}_1)$ the family of the Borel subsets of $\mathcal{K}_1$. $B_b(\mathcal{K}_1;\mathcal{K}_2)$ is the set of the bounded and Borel measurable functions from $\mathcal{K}_1$ to $\mathcal{K}_2$. If $\mathcal{K}_2 = \mathbb{R}$ we simply write $B_b(\mathcal{K}_1)$. $C_b(\mathcal{K}_1;\mathcal{K}_2)$ (BUC$(\mathcal{K}_1;\mathcal{K}_2)$, respectively) is the space of bounded and continuous

(uniformly continuous, respectively) functions from $\mathcal{K}_1$ to $\mathcal{K}_2$. If $\mathcal{K}_2 = \mathbb{R}$ we write $C_b(\mathcal{K}_1)$ ($\mathrm{BUC}(\mathcal{K}_1)$, respectively). $C_b(\mathcal{K}_1;\mathcal{K}_2)$ and $\mathrm{BUC}(\mathcal{K}_1;\mathcal{K}_2)$ are Banach spaces if endowed with the norm

$$\|f\|_\infty = \sup_{x\in\mathcal{K}_1} \|f(x)\|_{\mathcal{K}_2}.$$

By $L^1((0,+\infty))$ we denote the set of all measurable functions $f : (0,+\infty) \to \mathbb{R}$ with the usual Lebesgue norm.

Let X be a Banach space and let $A : \mathrm{Dom}(A) \subseteq X \to X$ be a linear operator. If Y is a Banach space continuously embedded in X, we call part of A in Y the operator $A_Y : \mathrm{Dom}(A_Y) \subseteq Y \to Y$ defined as

$$\begin{aligned} \mathrm{Dom}(A_Y) &:= \{y \in \mathrm{Dom}(A) \cap Y \mid Ay \in Y\}; \\ A_Y y &:= Ay, \qquad y \in \mathrm{Dom}(A_Y). \end{aligned}$$

If $A : \mathrm{Dom}(A) \subseteq H \to H$ is a linear operator, we denote with $\mathcal{FC}_b^k(H)$ the space of real-valued, continuous, cylindrical along $\mathrm{Dom}(A^*)$ and bounded functions on H whose derivatives up to order k are continuous and bounded. More precisely,

$$\mathcal{FC}_b^k(H) := \left\{ f : H \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{there exist } n \in \mathbb{N},\ \varphi \in C_b^k(\mathbb{R}^n) \text{ and } a_1,\dots,a_n \in \mathrm{Dom}(A^*) \\ \text{such that } f(x) = \varphi(\langle x,a_1\rangle_H,\dots,\langle x,a_n\rangle_H) \text{ for any } x \in H \end{array} \right\}.$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let H be a separable Hilbert space. Let $\xi : (\Omega,\mathcal{F},\mathbb{P}) \to (H,\mathcal{B}(H))$ be a random variable, we denote by

$$\mathbb{E}[\xi] := \int_\Omega \xi(\omega)\mathbb{P}(d\omega)$$

the expectation of ξ with respect to $\mathbb{P}$. Let $\{Y(t)\}_{t\geq 0}$ be a $\mathcal{K}$-valued stochastic process, we say that $\{Y(t)\}_{t\geq 0}$ is continuous if the map $Y(\cdot) : [0,+\infty) \to H$ is continuous $\mathbb{P}$-a.s.

For any $T > 0$ and $p \geq 1$ the set $C_p([0,T];H)$ ($C_p((0,T];H)$, respectively) denotes the space of progressive measurable H-valued processes $\{Y(t)\}_{t\in[0,T]}$ belonging to $L^p(\Omega;C([0,T];H))$ ($L^p(\Omega;C((0,T];H)\cap L^\infty(0,T;H))$, respectively), endowed with the norm

$$\left\|\{Y(t)\}_{t\in[0,T]}\right\|_{C_p([0,T];H)} = \left\|\{Y(t)\}_{t\in[0,T]}\right\|_{C_p((0,T];H)} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} \|Y(t)\|_H^p\right]\right)^{1/p},$$

see [14, Section 6.2] for further details.

2. Basic hypotheses and preliminary results

In the whole paper, H will be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and induced norm $\|\cdot\|_H$. Here we state the basic assumptions on the coefficients of the stochastic partial differential equation

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + RdW(t), & t > 0, \\ X(0) = x \in H, \end{cases}$$

and we recall some essential results about its solvability and the definition of its associated transition semigroup.

Hypotheses 2.1.

(i) There exists a separable Banach space E, continuously and densely embedded in H, such that $E \subseteq \mathrm{Dom}(F)$ and $F(E) \subseteq E$ where $F : \mathrm{Dom}(F) \subseteq H \to H$;

(ii) $R \in \mathcal{L}(H)$ is a linear bounded, self-adjoint and positive operator;

(iii) $A : \mathrm{Dom}(A) \subseteq H \to H$ is a self-adjoint operator that generates a strongly continuous semigroup e^{tA} on H and A_E (the part of A in E) generates an analytic semigroup e^{tA_E} on E. There exists $\zeta_A \in \mathbb{R}$ such that $A - \zeta_A \mathrm{Id}_H$ is dissipative in H and $A_E - \zeta_A \mathrm{Id}_E$ is dissipative in E;

(iv) the stochastic convolution process $\{W_A(t)\}_{t \geq 0}$, defined as

$$W_A(t) := \int_0^t e^{(t-s)A} R dW(s), \qquad t \geq 0, \tag{2.1}$$

belongs to $C_p([0,T], E)$ for any $T > 0$ and $p \geq 1$ and

$$\sup_{t \geq 0} \mathbb{E}\left[\|W_A(t)\|_E^p\right] < +\infty; \tag{2.2}$$

(v) there exists $\zeta_F \in \mathbb{R}$ such that $F - \zeta_F \mathrm{Id}_H$ is m-dissipative in H and $F_{|_E} - \zeta_F \mathrm{Id}_E$ is dissipative in E. Moreover, $F_{|_E} : E \to E$ is Fréchet differentiable and there exist $m \in \mathbb{N}$ and $C > 0$ such that for any $x \in E$

$$\begin{aligned} \|F_{|_E}(x)\|_E &\leq C(1 + \|x\|_E^{2m+1}); \\ \|\mathcal{D}F_{|_E}(x)\|_{\mathcal{L}(E;E)} &\leq C(1 + \|x\|_E^{2m}); \\ \|\mathcal{D}F_{|_E}(x)h\|_H &\leq C(1 + \|x\|_E^{2m})\|h\|_H \qquad h \in E; \end{aligned} \tag{2.3}$$

(vi) $\zeta := \zeta_A + \zeta_F < 0$.

We want to underline that Hypothesis 2.1(vi) is not a standard requirement assumed in order to get existence results of a solution to (1.1) but it will be used to deduce the asymptotic behavior of the semigroup that we are going to define.

The assumptions on R allow to prove that the space $H_R := R(H)$ is a separable Hilbert space with respect to the norm $\|x\|_R := \|R^{-1}x\|_H$ induced by the inner product

$$\langle x, y\rangle_R := \langle R^{-1}x, R^{-1}y\rangle_H, \qquad x, y \in H_R. \tag{2.4}$$

Furthermore H_R is a Borel measurable space, continuously embedded in H and, as it is easy to prove, it holds that

$$\|h\| \le \|R\|_{\mathcal{L}(H)}\|h\|_{H_R}, \tag{2.5}$$

for any $h \in H_R$ (see [4,10–13], [27, Theorem 15.1] and [32, Appendix C] for further details). We need to define a family of functions that we will use throughout the paper. We say that a function $g : H \to \mathbb{R}$ is H_R-Lipschitz if

$$|g(x+h) - g(x)| \le L\|h\|_R, \tag{2.6}$$

for any $x \in H$, $h \in H_R$ and some $L > 0$. The best constant in (2.6) is denoted $\mathrm{Lip}_R(g)$. We denote by $\mathrm{Lip}_{H_R}(H)$ the set of H_R-Lipschitz functions from H to $\mathbb{R}$.

Recall that, if $\mathrm{Dom}(F) = H$ and $x \in H$, a mild solution of (1.1) is a H-valued stochastic process $\{X(t,x)\}_{t\ge 0}$ satisfying

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + W_A(t), \qquad t \ge 0,\ \mathbb{P}\text{-a.s.}, \tag{2.7}$$

where $\{W_A(t)\}_{t\ge 0}$ is defined in (2.1). On the other hand, we say that $\{\widetilde{X}(t,x)\}_{t\ge 0}$ is a generalized mild solution of (1.1) if

(i) for any $x \in E$ there exists a E-valued stochastic process $\{X(t,x)\}_{t\ge 0}$ satisfying (2.7);
(ii) for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ converging to x in H and for any $T > 0$ it holds

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \|X(t,x_n) - \widetilde{X}(t,x)\|_H\right] = 0, \tag{2.8}$$

where $\{X(t,x_n)\}_{t\ge 0}$ is a mild solution of (1.1) with initial datum x_n.

We point out that, if $x \in E$ the generalized mild solution of (1.1) actually coincide with the mild solution. In view of this fact, when there is no confusion, we will use the same notation for both of them.

In the following theorem we collect some known results concerning existence, uniqueness and regularity of mild and generalized mild solutions to (1.1) whose proof can be found in [2, Proposition 2.5] and [8, Theorem 3.9, Corollary 3.10 and Proposition 3.13].

Theorem 2.2. *Assume Hypotheses 2.1 hold true and let $T > 0$ and $p \geq 1$. The following statements hold true.*

(i) *For any $x \in E$, (1.1) has a unique mild solution $\{X(t,x)\}_{t\geq 0}$ belonging to $C_p([0,T];H) \cap C_p((0,T];E)$. Moreover it is Gateaux differentiable as a function from E to $C_p([\varepsilon,T];E)$ for any $0 < \varepsilon < T$ and, for any $x \in E$, its Gateaux derivative $\{\mathcal{D}_G X(t,x)h\}_{t\geq 0}$ is the unique mild solution of*

$$\begin{cases} dY_x(t) = [A + \mathcal{D}F(X(t,x))]Y_x(t)dt, \quad t > 0, \\ Y_x(0) = h \in E \end{cases} \tag{2.9}$$

i.e.

$$\mathcal{D}_G X(t,x)h = e^{tA}h + \int_0^t e^{(t-s)A}\mathcal{D}F(X(s,x))\mathcal{D}_G X(s,x)h ds, \qquad x,h \in E,\ t > 0. \tag{2.10}$$

Further

$$\|\mathcal{D}_G X(t,x)h\|_E \leq e^{\zeta t}\|h\|_E, \qquad x,h \in E,\ t > 0, \quad \mathbb{P}\text{-}a.s. \tag{2.11}$$

(ii) *For any $x \in H$, (1.1) has a unique generalized mild solution $\{X(t,x)\}_{t\geq 0}$ belonging to $C_p([0,T];H)$ that is Gateaux differentiable as a function from E to $C_p([0,T];H)$. Further, its Gateaux derivative $\{\mathcal{D}_G X(t,x)h\}_{t\geq 0}$ satisfies*

$$\|\mathcal{D}_G X(t,x)h\|_H \leq e^{\zeta t}\|h\|_H, \qquad x \in E,\ h \in H,\ t > 0, \quad \mathbb{P}\text{-}a.s. \tag{2.12}$$

(iii) *For every $p > 0$ there exists a real positive random variable K_p such that $\mathbb{E}[K_p] < +\infty$ and*

$$\|X(t,x)\|_{\mathcal{O}}^p \leq K_p\,(1 + \|x\|_{\mathcal{O}}^p), \qquad x \in \mathcal{O}, \quad \mathbb{P}\text{-}a.s., \tag{2.13}$$

being either $\mathcal{O} = H$ or $\mathcal{O} = E$.

We emphasize that, unless explicitly stated otherwise, all (in)equalities in the preceding theorem and the subsequent discussion are to be understood as holding $\mathbb{P}$-almost surely. Moreover, whenever we refer to uniqueness, we specifically mean pathwise uniqueness.

Theorem 2.2 allows to define the transition semigroup $\{P(t)\}_{t\geq 0}$ ($P(t)$ in short) defined as

$$(P(t)\varphi)(x) := \mathbb{E}[\varphi(X(t,x))], \qquad x \in H,\ t \geq 0,\ \varphi \in B_b(H) \tag{2.14}$$

and, more precisely, assumption (2.2) and Hypothesis 2.1(vi) guarantee the existence of a unique probability invariant measure ν for $P(t)$, i.e. a probability measure ν on H such that

$$\int_H P(t)\varphi d\nu = \int_H \varphi d\nu$$

for any $t > 0$ and $\varphi \in C_b(H)$.

Remark 2.3. Note that, as proved in [8, Theorem 3.19], Hypotheses 2.1 imply that

(i) $\nu(E) = 1$;
(ii) for any $p \geq 1$, it holds that

$$\int_E \|x\|_E^p \nu(dx) < +\infty;$$

(iii) for any $\varphi \in C_b(\mathcal{O})$ and $x \in \mathcal{O}$, being $\mathcal{O}$ either E or H,

$$\lim_{t\to+\infty} (P(t)\varphi)(x) = \int_{\mathcal{O}} \varphi d\nu. \tag{2.15}$$

3. Closability of the gradient along H_R

The aim of this section is to introduce suitable Sobolev spaces. The results that we prove extend those in [17] (where the authors consider the case when F is dissipative and R is the identity operator), as well as the results contained in [10] (where the domain of F is the whole space H). To this aim we need some a priori estimates on $\|\mathcal{D}_G X(t,x)h\|_R$ uniformly with respect to $x \in E$. This can be obtained under a further compatibility condition between A and R together with either the Lipschitzianity of F or some stronger dissipativity assumption on $F_{|_E}$.

Here we assume the following additional hypotheses:

Hypotheses 3.1. Besides Hypotheses 2.1, assume that $E \cap H_R$ is dense in H_R and for any $t > 0$, $e^{tA}(H) \subseteq R(H)$. Moreover there exist $\gamma \in [0,1)$, $w > 0$ and $M > 0$ such that

$$\|e^{tA}x\|_R \leq Me^{-wt}t^{-\gamma}\|x\|_H, \qquad t > 0,\ x \in H. \tag{3.1}$$

Note that Hypotheses 3.1 are satisfied, for instance, if $R = (-A)^{-\beta}$ for some $\beta \in [0,1)$ and A generates an analytic semigroup of negative type in H. In this case, estimate (3.1) is proved in [34, Proposition 2.1.1].

Lemma 3.2. *Assume Hypotheses 3.1 hold true. There exists a real positive random variable K depending on m, γ, w and ζ such that $\mathbb{E}[K] < +\infty$ and*

$$\|\mathcal{D}_G X(t,x)h\|_R \leq K(1+\|x\|_E^{2m})e^{-\min\{w,|\zeta|\}t}\max\{t^{-\gamma}, t^{1-\gamma}\}\|h\|_H, \quad \mathbb{P}\text{-a.s.} \tag{3.2}$$

for any $x \in E$, $h \in H_R$ and $t > 0$.

Proof. Let $h \in E \cap H_R$. Using Hypotheses 3.1, we infer that

$$\|\mathcal{D}_G X(t,x)h\|_R \leq \frac{Me^{-wt}}{t^\gamma}\|h\|_H + \int_0^t \frac{Me^{-w(t-s)}}{(t-s)^\gamma}\|\mathcal{D}F(X(s,x))\mathcal{D}_G X(s,x)h\|_H \, ds \tag{3.3}$$

for any $x \in E$, $h \in H$ and $t > 0$. Thanks to (2.3), (2.12) and (2.13) we obtain

$$\begin{aligned}\int_0^t \frac{Me^{-w(t-s)}}{(t-s)^\gamma}&\|\mathcal{D}F(X(s,x))\mathcal{D}_G X(s,x)h\|_H ds \\ &\leq C_1(1+\|x\|_E^{2m})\|h\|_H \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\gamma}e^{\zeta s}ds \\ &\leq C_1(1+\|x\|_E^{2m})C_2\max\{1, t^{1-\gamma}\}e^{-\min\{w,|\zeta|\}t}\|h\|_H, \end{aligned} \tag{3.4}$$

for some real positive random variable C_1 such that $\mathbb{E}[C_1] < +\infty$ and a positive constant C_2. Thus, taking (3.3) and (3.4) into account we get (3.2) for every $h \in E \cap H_R$. Finally recalling that $E \cap H_R$ is dense in H_R we obtain the statement. □

The following result is a straightforward consequence of the previous lemma and (2.5), but it will be fundamental at various points in the paper.

Corollary 3.3. *Under the assumptions of Lemma 3.2, if Hypotheses 2.1*(v) *are satisfied with $m = 0$ then*

$$\|\mathcal{D}_G X(t,x)h\|_R \leq Ke^{-\min\{w,|\zeta|\}t}\max\{t^{-\gamma}, t^{1-\gamma}\}\|R\|_{\mathcal{L}(R)}\|h\|_R =: \theta(t)\|h\|_R, \quad \mathbb{P}\text{-a.s.} \tag{3.5}$$

for any $x \in E$, $h \in H_R$ and some positive constant K that depends only on the Lipschitz constant of F.

Now, we assume a stronger dissipativity condition on F to get a more precise decay estimate on $\|\mathcal{D}_G X(t,x)h\|_R$.

Hypotheses 3.4. Beside Hypotheses 3.1, assume that, $\mathcal{D}F_{|_E}(x)(H_R \cap E) \subseteq H_R$ for any $x \in E$ and there exists $\zeta_R \in \mathbb{R}$ such that

$$\big\langle [A_R + \mathcal{D}F_{|E}(x)]h, h\big\rangle_R \leq \zeta_R \|h\|_R^2, \qquad x \in E, h \in \mathrm{Dom}(A_R), \qquad \mathbb{P}\text{-a.s.} \tag{3.6}$$

where A_R denotes the part of A in $H_R \cap E$.

Here, using Hypotheses 3.4 we prove an alternative estimate for $\|\mathcal{D}_G X(t,x)h\|_R$.

Proposition 3.5. *If Hypotheses 3.4 hold true, then*

$$\|\mathcal{D}_G X(t,x)h\|_R \leq e^{\zeta_R t}\|h\|_R, \tag{3.7}$$

for any $x \in E$, $h \in H_R$ and $t > 0$.

Proof. It is not restrictive to assume that $\{\mathcal{D}_G X(t,x)h\}_{t\geq 0}$ is a strict solution of (2.9). Indeed, otherwise we proceed as in [8, Proposition 3.6] or [14, Proposition 6.2.2] approximating $\{\mathcal{D}_G X(t,x)h\}_{t\geq 0}$ by means of sequences of more regular processes using the Yosida approximations of A_R. Moreover, estimates (2.11) and (3.2) yield that $\mathcal{D}_G X(t,x)h \in E \cap H_R$ for any $t > 0$, $x \in E$ and $h \in E \cap H_R$. Now, let us fix x, h as before. Multiplying (2.9) by $\mathcal{D}_G X(t,x)h$ and using estimate (3.6), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|\mathcal{D}_G X(s,x)h\|_R^2 &= \langle [A_R + \mathcal{D}F(X(s,x))]\mathcal{D}_G X(s,x)h, \mathcal{D}_G X(s,x)h\rangle_R \\ &\leq \zeta_R \|\mathcal{D}_G X(s,x)h\|_R^2\end{aligned}$$

whence, integrating from 0 to t with respect to s, we get (3.7) for every $h \in E \cap H_R$. The claim follows by the fact that $E \cap H_R$ is dense in H_R. □

Now we define the functional spaces which will play a crucial role in this paper. The following notion of differentiability first appeared in [24] (see also [28]).

Definition 3.6. Let $R \in \mathcal{L}(H)$ be a linear bounded, self-adjoint and positive operator. We say that a function $\Phi : H \to \mathbb{R}$ is H_R-differentiable at $x \in H$ if there exists $L_x \in \mathcal{L}(H_R; \mathbb{R})$ such that

$$\lim_{\|h\|_R \to 0} \frac{|\Phi(x+h) - \Phi(x) - L_x h|}{\|h\|_R} = 0$$

(see (2.4)). If L_x exists, then it is unique and we set $\mathcal{D}_R\Phi(x) := L_x$. We say that Φ is H_R-differentiable if Φ is H_R-differentiable at every $x \in H$.

Note that if $R = \mathrm{Id}_H$, then the H_R-differentiability coincides with the standard Fréchet differentiability and in this case we will drop the subscript R.

Remark 3.7. Let $f : H \to \mathbb{R}$ be a H_R-differentiable function. Since H_R is a Hilbert space, by the Riesz representation theorem, for every $x \in H$ there exists a unique $l_x \in H_R$ such that

$$\mathcal{D}_R f(x)h = \langle l_x, h\rangle_R, \qquad h \in H_R.$$

We call l_x the H_R-gradient of f at $x \in H$ and we denote it by $\nabla_R f(x)$. If $R = \mathrm{Id}_H$ then ∇_R is the classical Fréchet gradient and we simply write ∇.

The notion of H_R-differentiability is a sort of Fréchet differentiability along the directions of H_R and it has been already considered in various papers (see, for example, [4,11,10,15,16]). The proof of the following result follows the same arguments used in the proof of [11, Proposition 17] or in [13, Theorem 2.21].

Proposition 3.8. *Assume Hypothesis 2.1*(ii) *holds true. If $f : H \to \mathbb{R}$ is a Fréchet differentiable function with continuous derivative operator, then f is also H_R-differentiable with continuous H_R-derivative operator and, for every $x \in H$, it holds $\nabla_R f(x) = R^2 \nabla f(x)$.*

It is well-known that $P(t)$ extends to a strongly continuous semigroup in $L^p(H,\nu)$ denoted by $P_p(t)$. By [8, Theorem 1.1], the infinitesimal generator $N_2 : \mathrm{Dom}(N_2) \subseteq L^2(H,\nu) \to L^2(H,\nu)$ of $P_2(t)$ is the closure in $L^2(H,\nu)$ of the operator $\mathcal{N}_0$ defined on smooth cylindrical functions $\varphi \in \mathcal{FC}_b^2(H)$ as follows

$$[\mathcal{N}_0\varphi](x) := \frac{1}{2}\mathrm{Tr}[R^2\nabla^2\varphi(x)] + \langle x, A^*\nabla\varphi(x)\rangle_H + \langle F(x), \nabla\varphi(x)\rangle_H, \qquad x \in E. \tag{3.8}$$

We remark that since $\nu(E) = 1$, then the value of $\mathcal{N}_0\varphi$ outside of E is not important for our calculations.

Lemma 3.9. *Assume Hypotheses 2.1 hold true. For any $\varphi, \psi \in \mathcal{FC}_b^2(H)$ it holds*

$$[\mathcal{N}_0(\varphi\psi)](x) = \varphi(x)[\mathcal{N}_0\psi](x) + \psi(x)[\mathcal{N}_0\varphi](x) + \langle \nabla_R\varphi(x), \nabla_R\psi(x)\rangle_R, \qquad x \in E. \tag{3.9}$$

Furthermore, for any $\psi \in \mathrm{Dom}(N_2)$, it holds

$$\int_H \psi N_2\psi d\nu = -\frac{1}{2}\int_H \|\nabla_R\psi\|_R^2 d\nu. \tag{3.10}$$

Proof. The facts that $\varphi\psi$ belongs to $\mathcal{FC}_b^2(H)$ and that formula (3.9) holds true, are immediate. In order to prove (3.10) we start by recalling that since ν is an invariant measure and formula (3.9) holds true, for any $\varphi \in \mathcal{FC}_b^2(H) \subseteq \mathrm{Dom}(N_2)$

$$0 = \int_H N_2(\varphi^2)d\nu = \int_H \big(2\varphi N_2\varphi + \|\nabla_R\varphi\|_R^2\big)d\nu. \tag{3.11}$$

Since $\mathcal{FC}_b^2(H)$ is a core for N_2, by (3.11) and the Young inequality, it follows that the map

$$\nabla_R : \mathcal{F}C_b^2(H) \subseteq \mathrm{Dom}(N_2) \to L^2(H,\nu;H_R), \qquad \varphi \mapsto \nabla_R\varphi,$$

is continuous and, consequently, it can be extended to functions $\varphi \in \mathrm{Dom}(N_2)$ (endowed with the graph norm). Now (3.10) follows by a standard density argument. □

The next result is a technical lemma about the behavior of the gradient of the semigroup $P_2(t)$ which will be useful to prove the closability of the gradient operator in $L^2(H,\nu)$.

Lemma 3.10. *Assume Hypotheses 2.1 hold true. For any $\varphi \in \mathcal{F}C_b^1(H)$ it holds that*

$$\int_H |P_2(t)\varphi|^2 d\nu + \int_0^t \int_H \|\nabla_R P_2(s)\varphi\|_R^2 d\nu ds = \int_H |\varphi|^2 d\nu. \tag{3.12}$$

Proof. Let $\varphi \in \mathcal{F}C_b^2(H)$, then

$$\frac{d}{ds} P_2(s)\varphi = N_2 P_2(s)\varphi, \qquad s > 0. \tag{3.13}$$

Multiplying both sides of (3.13) by $P_2(s)\varphi$, integrating on H with respect to ν, and taking into account (3.10), we find

$$\int_H \frac{d}{ds}|P_2(s)\varphi|^2 d\nu = -\int_H \|\nabla_R P_2(s)\varphi\|_R^2 d\nu. \tag{3.14}$$

The thesis follows integrating (3.14) with respect to s from 0 to t. By a standard approximation argument (3.14) holds true for $\varphi \in \mathcal{F}C_b^1(H)$. □

The next proposition is the main result of this section and allows us to define the Sobolev space $W_R^{1,2}(H,\nu)$ if either estimate (3.5) or estimate (3.7) hold true. This is the case, for instance, when either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 hold true.

Proposition 3.11. *If either Hypotheses 3.1 hold true with $m = 0$ or Hypothesis 3.4 is satisfied, then the operator $\nabla_R : \mathcal{F}C_b^1(H) \subseteq L^2(H,\nu) \to L^2(H,\nu;H_R)$ is closable.*

Proof. We will prove the claim only in the case when Hypotheses 3.1 with $m = 0$ hold true, as the case in which Hypotheses 3.4 hold true follows using similar arguments. To this aim, we consider $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(H)$ a sequence such that

$$\begin{aligned} L^2(H,\nu)\text{-}\lim_{n\to+\infty} \varphi_n &= 0; \\ L^2(H,\nu;H_R)\text{-}\lim_{n\to+\infty} \nabla_R\varphi_n &= \Psi. \end{aligned} \tag{3.15}$$

From here onwards, for the remainder of the proof, we will assume that $t \in (0,1)$. By (3.12), the continuity of the operator $P_2(t)$ and (3.15), we have

$$\lim_{n\to+\infty} \int_0^t \int_H \|\nabla_R P_2(s)\varphi_n\|_R^2 d\nu ds = \lim_{n\to+\infty} \left(\int_H |\varphi_n|^2 d\nu - \int_H |P_2(t)\varphi_n|^2 d\nu \right) = 0. \quad (3.16)$$

We claim that

$$\lim_{n\to+\infty} \int_0^t \int_H \|\nabla_R P_2(s)\varphi_n\|_R^2 d\nu ds = \int_0^t \int_H \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx) ds \quad (3.17)$$

where $(\mathcal{D}_G X(s,x))^*$ denotes the adjoint of $\mathcal{D}_G X(s,x)$ in $\mathcal{L}(H_R)$. Indeed by [11, Corollary 3.11]

$$\nabla_R P_2(t)\varphi_n(x) = \mathbb{E}[(\mathcal{D}_G X(t,x))^* \nabla_R \varphi_n(X(t,x))].$$

We note that, by (3.5) for every $x \in E$ and $t \geq 0$ we have $\mathcal{D}_G X(t,x) \in \mathcal{L}(H_R)$. Moreover, since H_R is a Hilbert space, $\|\mathcal{D}_G X(t,x)\|_{\mathcal{L}(H_R)} = \|(\mathcal{D}_G X(t,x))^*\|_{\mathcal{L}(H_R)}$. Hence $(\mathcal{D}_G X(t,x))^*$ satisfies (3.5). By the fact that $\nu(E) = 1$, the invariance of ν (see (1.2)) and (2.5), we can estimate

$$\begin{aligned}
&\int_0^t \int_H \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \nabla_R \varphi_n(X(s,x))] - \mathbb{E}[(\mathcal{D}_G X(t,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx) ds \\
&\quad = \int_0^t \int_E \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \nabla_R \varphi_n(X(s,x))] - \mathbb{E}[(\mathcal{D}_G X(t,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx) ds \\
&\quad \leq \|R\|_{\mathcal{L}(H)} \int_0^t \int_E \theta(s) \mathbb{E}\left[\|\nabla_R \varphi_n(X(s,x)) - \Psi(X(s,x))\|_R^2 \right] \nu(dx) ds \\
&\quad \leq \|R\|_{\mathcal{L}(H)} \int_0^t \int_E \theta(s) P_2(s) \left[\|\nabla_R \varphi_n - \Psi\|_R^2 \right](x) \nu(dx) ds \\
&\quad \leq \frac{\mathbb{E}[K] \|R\|_{\mathcal{L}(H)} \Gamma(1-\gamma)}{(\min\{\omega, |\zeta|\})^{1-\gamma}} \int_E \|\nabla_R \varphi_n - \Psi\|_R^2 d\nu, \qquad (3.18)
\end{aligned}$$

where γ, ω and ζ are introduced in Hypotheses 3.1, θ, K are defined in formula (3.5) and Γ is the gamma function, namely $\Gamma(z) := \int_0^{+\infty} r^{z-1} e^{-r} dr$. So taking the limit as n approaches infinity in (3.18) we obtain

$$0 \leq \lim_{n\to+\infty} \int_0^t \int_H \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \nabla_R \varphi_n(X(s,x))] - \mathbb{E}[(\mathcal{D}_G X(s,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx)ds$$
$$\leq \frac{\mathbb{E}[K]\|R\|_{\mathcal{L}(H)}\Gamma(1-\gamma)}{(\min\{\omega,|\zeta|\})^{1-\gamma}} \lim_{n\to+\infty} \int_H \|\nabla_R \varphi_n - \Psi\|_R^2 d\nu = 0,$$

which implies that (3.17) holds. Combining (3.16) and (3.17) we get

$$\int_0^t \int_H \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx)ds = 0.$$

So for a.e. $s \in (0,t)$ (with respect to the Lebesgue measure) it holds

$$\int_H \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx) = 0. \tag{3.19}$$

Thus, let $A \subseteq (0,t)$ be the set of measure zero in which (3.19) fails to hold. By the monotone convergence theorem we infer that

$$\begin{aligned} 0 &= \int_H \|\mathbb{E}[(\mathcal{D}_G X(s,x))^* \Psi(X(s,x))]\|_R^2 \nu(dx) \\ &= \sum_{i=1}^{+\infty} \int_H |\mathbb{E}[\langle (\mathcal{D}_G X(s,x))^* \Psi(X(s,x)), h_i\rangle_R]|^2 \nu(dx) \\ &= \sum_{i=1}^{+\infty} \int_H |\mathbb{E}[\langle \Psi(X(s,x)), \mathcal{D}_G X(s,x) h_i\rangle_R]|^2 \nu(dx) \end{aligned}$$

for any $s \in (0,t) \setminus A$, being $\{h_i \,|\, i \in \mathbb{N}\}$ an orthonormal basis of H_R. So, for such s and any $i \in \mathbb{N}$

$$\int_H |\mathbb{E}[\langle \Psi(X(s,x)), \mathcal{D}_G X(s,x) h_i\rangle_R]|^2 \nu(dx) = 0.$$

Thus for any $s \in (0,t) \setminus A$ and $i \in \mathbb{N}$

$$\begin{aligned} 0 \leq & \|P_2(s)(\langle \Psi(\cdot), h_i\rangle_R)\|_{L^2(H,\nu)} = \|\mathbb{E}[\langle \Psi(X(s,\cdot)), h_i\rangle_R]\|_{L^2(H,\nu)} \\ = & \|\mathbb{E}[\langle \Psi(X(s,\cdot)), h_i\rangle_R]\|_{L^2(H,\nu)} - \|\mathbb{E}[\langle \Psi(X(s,\cdot)), \mathcal{D}_G X(s,\cdot) h_i\rangle_R]\|_{L^2(H,\nu)} \\ \leq & \|\mathbb{E}[\langle \Psi(X(s,\cdot)), h_i\rangle_R] - \mathbb{E}[\langle \Psi(X(s,\cdot)), \mathcal{D}_G X(s,\cdot) h_i\rangle_R]\|_{L^2(H,\nu)} \\ = & \|\mathbb{E}[\langle \Psi(X(s,\cdot)), h_i - \mathcal{D}_G X(s,\cdot) h_i\rangle_R]\|_{L^2(H,\nu)}. \end{aligned} \tag{3.20}$$

The continuity of $s \mapsto \mathcal{D}_G X(s,\cdot)$ and the dominated convergence theorem allow us to conclude that the right hand side in (3.20) vanishes as s approaches zero from the right. Consequently, taking the limit as $s \to 0^+$ in (3.20) we can conclude that

$$\|\langle \Psi(\cdot), h_i\rangle_R\|_{L^2(H,\nu)} = 0, \qquad i \in \mathbb{N}.$$

Standard argument allows to conclude that $\Psi(x) = 0$ for ν-a.e $x \in H$ and this proves the closability of $\nabla_R : \mathcal{F}C_b^1(H) \subseteq L^2(H,\nu) \to L^2(H,\nu;H_R)$. □

The previous result allows us to define the Sobolev space $W_R^{1,2}(H,\nu)$ as the domain of the closure of the operator $\nabla_R : \mathcal{F}C_b^1(H) \subseteq L^2(H,\nu) \to L^2(H,\nu;H_R)$.

4. Hypercontractivity and exponential integrability

As already recalled, the semigroup $P(t)$ extends to a bounded and strongly continuous semigroup in $L^p(H,\nu)$ for any $p \geq 1$. In this section we are interested in proving the exponential integrability with respect to the invariant measure ν of H_R-Lipschitz functions. This can be obtained by means of some logarithmic Sobolev inequalities which are connected to the hypercontractivity of the semigroup $P(t)$ in $L^p(H,\nu)$ spaces. Using formula (2.14), the chain rule in [11, Corollary 21] and estimates (3.5) and (3.7) it can be proved that the semigroup satisfies the following gradient estimate:

$$\|\nabla_R P(t)\varphi\|_R^2 \leq \psi(t) P(t)\|\nabla_R \varphi\|_R^2, \qquad \varphi \in \mathcal{F}C_b^1(H),\ t > 0, \tag{4.1}$$

where $\psi(t) = (\theta(t))^2$ (see (3.5)) if Hypotheses 3.1 are satisfied with $m = 0$ whereas $\psi(t) = e^{2\zeta_R t}$ if Hypotheses 3.4 are satisfied. In this section, in addition to Hypotheses 2.1 we assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$.

In these cases, thanks to estimate (4.1), the asymptotic behavior of $P(t)$ as $t \to +\infty$ (see (2.15)) and the fact that $\psi \in L^1((0,\infty))$, applying the classical idea of Deuschel and Stroock (see [22]) we can prove a logarithmic Sobolev inequality with respect to the invariant measure ν.

Theorem 4.1. *Assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$. For $p \geq 1$ and $\varphi \in C_b^1(H)$, the following inequality holds:*

$$\int_H |\varphi|^p \ln|\varphi|^p d\nu - \left(\int_H |\varphi|^p d\nu\right) \ln\left(\int_H |\varphi|^p d\nu\right) \leq p^2 C \int_H |\varphi|^{p-2} \|\nabla_R \varphi\|_R^2 \chi_{\{\varphi \neq 0\}} d\nu \tag{4.2}$$

where

$$C := \|\psi\|_{L^1((0,\infty))} = \begin{cases} K\Big(\frac{r(1-\gamma,\min\{\omega,|\zeta|\})}{\min\{\omega,|\zeta|\}^{1-\gamma}} + \frac{e^{\min\{\omega,|\zeta|\}}}{\min\{\omega,|\zeta|\}}\Big), & \text{Hypotheses 3.1 with } m=0, \\ \frac{1}{2|\zeta_R|}, & \text{Hypotheses 3.4 with } \zeta_R < 0. \end{cases} \tag{4.3}$$

Here K is the constant introduced in (3.5) and r is the lower incomplete Gamma function, i.e., $r(s,x) := \int_0^x \xi^{s-1}e^{-\xi}d\xi$, for $x > 0$ and $s \in \mathbb{C}$ with positive real part. Further, if $\varphi \in W_R^{1,2}(H,\nu)$ then

$$\int_H |\varphi|^2 \ln|\varphi|^2 d\nu - \left(\int_H |\varphi|^2 d\nu\right) \ln\left(\int_H |\varphi|^2 d\nu\right) \le 4C \int_H \|\nabla_R \varphi\|_R^2 \chi_{\{\varphi\neq 0\}} d\nu. \tag{4.4}$$

Proof. The proof is quite similar to that in [10, Theorem 1.9] and we provide it for the sake of completeness.

We begin by establishing (4.2) for a function φ in the space $\mathcal{F}C_b^1(H)$, provided that there exists a positive constant c satisfying the condition $c \le \varphi \le 1$. To this aim we consider the function

$$[0,\infty) \ni t \mapsto G(t) := \int_H (P(t)\varphi^p) \ln(P(t)\varphi^p) d\nu,$$

which is well defined thanks to the contractivity and the positivity preserving of $P(t)$. By a standard argument involving the invariance of ν, (4.1) and the fact that $P(t)\|\nabla_R \varphi^p\|_R \le [P(t)(\|\nabla_R \varphi^p\|_R^2 \varphi^{-p})]^{1/2}(P(t)\varphi^p)^{1/2}$ we have

$$G'(t) \ge -\psi(t)p^2 \int_H \varphi^{p-2}\|\nabla_R \varphi\|_R^2 d\nu.$$

Integrating the latter inequality w.r. to t from 0 to $+\infty$ and by taking (2.15) into account, we get

$$\int_H \varphi^p \ln \varphi^p d\nu - \left(\int_H \varphi^p d\nu\right) \ln\left(\int_H \varphi^p d\nu\right) \le p^2 \|\psi\|_{L^1((0,+\infty))} \int_H \varphi^{p-2}\|\nabla_R \varphi\|_R^2 d\nu.$$

The general case follows by standard approximation arguments and the Fatou lemma. □

A classical consequence of the logarithmic Sobolev inequality (4.4) is the Poincaré inequality (see for instance [6, Theorem 5.2] for a proof in finite dimension and [10, Theorem 1.10] for a proof in infinite dimension).

Corollary 4.2. *Assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$. Then, for any $\varphi \in W_R^{1,2}(H,\nu)$*

$$\|\varphi - m_\nu(\varphi)\|_R^2 \le C\|\nabla_R \varphi\|_R^2,$$

where C is the same constant in (4.3) and $m_\nu(\varphi)$ denotes the average of φ with respect to ν, i.e.,

$$m_\nu(\varphi) := \int_H \varphi d\nu, \qquad \varphi \in L^1(H,\nu).$$

The formulation of the logarithmic Sobolev inequality given by Gross in [26] is

$$\int_H |f|^p \ln|f| d\sigma - \|f\|_{L^p(H,\sigma)}^p \ln\|f\|_{L^p(H,\sigma)} \le c_p \int_H f|f|^{p-2}(-\mathcal{L}f) d\sigma \tag{4.5}$$

where c_p is a positive constant and $\mathcal{L}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ in $L^p(H,\sigma)$, for a suitable probability measure σ. In our case $\sigma = \nu$, $\mathcal{L} = \mathcal{N}_0$ and $T(t) = P(t)$. Note that, writing (3.10) with $\psi = f^{\frac{p}{2}}$ for a smooth enough positive function f and using the action of $\mathcal{N}_0$ (see (3.8)) on smooth functions we get

$$\int_H f^{p-2}|\nabla f|^2 d\nu = -c_p \int_H f^{\frac{p}{2}} \mathcal{N}_0(f^{\frac{p}{2}}) d\nu = -c_p \int_H f^{p-1}\mathcal{N}_0 f d\nu$$

for some positive constant c_p, whence the inequalities (4.2) and (4.5) coincide. The general case can be obtained by approximation (see [26, Section 4] for details). Therefore, applying [26, Theorem 3.7] we deduce the hypercontractivity of the semigroup $P(t)$, i.e., for any fixed $p, q \in (1,+\infty)$ with $p < q$, there exists $t_0 = t_0(p,q) > 0$ such that $P(t)$ maps $L^p(H,\nu)$ into $L^q(H,\nu)$ for any $t \ge t_0$ and

$$\|P(t)\|_{\mathcal{L}(L^p(H,\nu);L^q(H,\nu))} \le 1, \qquad t \ge t_0.$$

Theorem 4.3. *Assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$. For every $t > 0$, $q \in (1,+\infty)$ and $p \le (q-1)e^{t/C} + 1$ it holds*

$$\|P(t)\varphi\|_{L^p(H,\nu)} \le \|\varphi\|_{L^q(H,\nu)}, \qquad \varphi \in L^q(H,\nu). \tag{4.6}$$

where C is defined in (4.3).

Thanks to [26, Theorem 3.12], one can show that the logarithmic Sobolev inequality (4.2) and the hypercontractivity of $P(t)$ are equivalent.

Remark 4.4. If $A = -\mathrm{Id}_H$ and $F \equiv 0$ in (1.1) then Hypotheses 3.4 is verified with $\zeta_R = -1$ and so the constant C in (4.2) is equal to $1/2$. Moreover estimate (4.6) holds for every $t > 0$ and $p \le C(t,q) := (q-1)e^{-2t} + 1$. We stress that, in this framework, $C(t,q)$ is optimal as explained in [25, Remark 3.4].

A less standard result concerns the exponential integrability of Lipschitz functions along H_R with respect to ν.

Theorem 4.5. *Assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$ and that $\nu(H_R) = 1$. Any function $g \in \mathrm{Lip}_{H_R}(H)$ with $\mathrm{Lip}_R(g) \leq 1$ belongs to $L^1(H, \nu)$ and for any $t > 0$ it holds*

$$\nu(\{x \in H \,|\, g(x) \geq m_\nu(g) + t\}) \leq e^{-\frac{t^2}{16\sqrt{2}C}}$$

where C is defined in (4.3). Further, if $\alpha < (16\sqrt{2}C)^{-1}$, then

$$\int_{H_R} e^{\alpha g^2} d\nu < \infty. \tag{4.7}$$

Proof. First of all, let us observe that (4.2) can be easily proved for any $\varphi \in C^1_{b,H_R}(H)$, i.e. the space of bounded and H_R-differentiable functions from H into $\mathbb{R}$. We claim that

$$\tau \int_H g e^{\tau g} d\nu - m_\nu(e^{\tau g}) \ln\left(m_\nu(e^{\tau g})\right) \leq 4\sqrt{2}C\tau^2 m_\nu(e^{\tau g}), \tag{4.8}$$

for any $\tau > 0$ and any bounded $g \in \mathrm{Lip}_{H_R}(H)$ with $\mathrm{Lip}_R(g) \leq 1$. To this aim, let us fix τ and g as above and let g_ε be the Lasry–Lions approximations along H_R of g (see Appendix A). Since g_ε belongs to $C^1_{b,H_R}(H)$ and $\|\nabla_R g_\varepsilon\|_R \leq 4\sqrt{2}\mathrm{Lip}_R(g) \leq 4\sqrt{2}$, by (4.2) with $\varphi = e^{\tau g_\varepsilon}$ and $p = 1$ we get

$$\tau \int_H g_\varepsilon e^{\tau g_\varepsilon} d\nu - m_\nu(e^{\tau g_\varepsilon}) \ln\left(m_\nu(e^{\tau g_\varepsilon})\right) \leq C\tau^2 \int_H e^{\tau g_\varepsilon} \|\nabla_R g_\varepsilon\|_R^2 d\nu \leq 4\sqrt{2}C\tau^2 m_\nu(e^{\tau g_\varepsilon}). \tag{4.9}$$

Since g_ε converges pointwise to g and $|g_\varepsilon| \leq \|g\|_\infty$ (Proposition 7.2), by (4.9) and the dominated convergence theorem we get the claim. Now, introducing

$$H(\tau) := \begin{cases} m_\nu(g), & \tau = 0; \\ \tau^{-1} \ln\left(m_\nu(e^{\tau g})\right), & \tau > 0, \end{cases}$$

and using (4.8) we get $H'(\tau) \leq 4\sqrt{2}C$, whence integrating from 0 to t we obtain

$$H(t) - H(0) \leq 4\sqrt{2}Ct \iff m_\nu(e^{tg}) \leq e^{4\sqrt{2}Ct^2 + t m_\nu(g)}. \tag{4.10}$$

The Chernoff bound (see [36, Introduction of Section 4.2]) and (4.10) yield

$$\nu(\{x \in H \,|\, g(x) \geq m_\nu(g) + s\}) \leq \inf_{t \geq 0} \left(e^{-t(s+m_\nu(g))} m_\nu(e^{tg}) \right)$$
$$\leq \inf_{t \geq 0} \left(e^{-ts + 4\sqrt{2}Ct^2} \right) = e^{-\frac{s^2}{16\sqrt{2}C}}, \qquad s \in \mathbb{R}. \tag{4.11}$$

We now consider the general case. Let g be as in the statement and let $\psi_n \in C^1_b(\mathbb{R})$ be an odd increasing function such that $\psi_n(t) = t$ for $t \in [0, n]$, $\psi_n(t) = n+1$ if $t \geq n+2$ and $\psi_n'(t) \in [0,1]$ for any $t \in \mathbb{R}$. Then, the function $g_n = \psi_n \circ g$ belongs to $\mathrm{Lip}_{H_R}(H)$, satisfies $\mathrm{Lip}_R(g_n) \leq 1$ and is bounded as well. By the previous step, applying (4.11) to $-|g_n|$ we deduce

$$\nu\left(\{x \in H \,|\, |g_n(x)| \leq m_\nu(|g_n|) - t\}\right) \leq e^{-\frac{t^2}{16\sqrt{2}C}}.$$

Choosing $t_0 > 0$ such that $e^{-t_0^2/(16\sqrt{2}C)} \leq 1/2$, m such that $\nu(\{x \in H \,|\, |g(x)| \geq m\}) < 1/2$ and using that $|g_n| \leq |g|$ we deduce that $\|g_n\|_{L^1(H,\nu)} = m_\nu(|g_n|) \leq m + t_0$ for any $n \in \mathbb{N}$. Indeed, by contradiction, if $m_\nu(|g_n|) > m + t_0$, we have

$$\begin{aligned}
\nu(\{x \in H \,|\, |g(x)| < m\}) &\leq \nu(\{x \in H \,|\, |g_n(x)| < m\}) \\
&= \nu(\{x \in H \,|\, |g_n(x)| + t_0 < m + t_0\}) \\
&\leq \nu(\{x \in H \,|\, |g_n(x)| + t_0 < m_\nu(|g_n|)\}) \\
&= \nu(\{x \in H \,|\, |g_n(x)| < m_\nu(|g_n|) - t_0\}) \leq 1/2
\end{aligned}$$

which yields a contradiction with $\nu(\{x \in H \,|\, |g(x)| \geq m\}) < 1/2$, as ν is a probability measure. Hence $\sup_n \|g_n\|_{L^1(H,\nu)} \leq m + t_0$ and, being $|g_n| \to |g|$ as $n \to +\infty$ almost surely, by the Fatou lemma, g belongs to $L^1(H,\nu)$. By the dominated convergence theorem, $g_n \to g$ in $L^1(H,\nu)$ as $n \to +\infty$ and hence in measure, i.e., for any $\varepsilon > 0$

$$\lim_{n \to +\infty} \nu(\{x \in H \,|\, |g_n(x) - g(x)| \geq \varepsilon\}) = 0. \tag{4.12}$$

From (4.11) and (4.12) for g_n we infer that (4.11) holds true for g as well.

We now prove (4.7). First of all, let us observe that for any $\alpha > 0$, the layer cake formula implies

$$\int_H e^{\alpha(g(x))^2} \nu(dx) \leq 1 + \int_1^\infty \nu\left(\left\{ x \in H \,\middle|\, e^{\alpha(g(x))^2} > s \right\} \right) ds$$
$$= 1 + \int_1^\infty \nu\left(\left\{ x \in H \,\middle|\, (g(x))^2 > \frac{\ln s}{\alpha} \right\} \right) ds.$$

Now, performing the change of variables $s = e^{\alpha(t + m_\nu(g))^2}$ we get

$$\int_H e^{\alpha(g(x))^2}\nu(dx) \le 1 + 2\alpha \int_{\mathbb{R}} \nu(\{g > m_\nu(g) + t\})e^{\alpha(m_\nu(g)+t)^2}(m_\nu(g) + t)dt.$$

Thus, using estimate (4.11) and choosing $\alpha < (16\sqrt{2}C)^{-1}$ we conclude the proof. □

The following is a Fernique-type result for the measure ν.

Corollary 4.6. *Assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$ and that $\nu(H_R) = 1$. The map $x \mapsto e^{\lambda\|x\|_R^2}$ belongs to $L^1(H_R, \nu)$ for any $\lambda < (16\sqrt{2}C)^{-1}$ where C is defined in (4.3).*

Using the fact that $\nabla_R\varphi = R^2\nabla\varphi$ for every $\varphi \in C_b^1(H)$ (see Proposition 3.8), the proof of Theorem 4.5 can be repeated similarly if we replace a H_R-Lipschitz function with a Lipschitz continuous function without assuming that $\nu(H_R) = 1$. This fact allows us to obtain the following result.

Theorem 4.7. *Assume either Hypotheses 3.1 with $m = 0$ or Hypotheses 3.4 with $\zeta_R < 0$. Any function $g \in \mathrm{Lip}(H)$ with $\mathrm{Lip}(g) \le 1$ belongs to $L^1(H, \nu)$ and for any $t > 0$ it holds*

$$\nu(\{x \in H \mid g(x) \ge m_\nu(g) + t\}) \le e^{-\frac{t^2}{16\sqrt{2}C'}}$$

where $C' = \|R\|_{\mathcal{L}(H)}C$ and C is defined in (4.3). Further, if $\alpha < (16\sqrt{2}C')^{-1}$

$$\int_H e^{\alpha g^2} d\nu < \infty. \tag{4.13}$$

In particular $x \mapsto e^{\lambda\|x\|^2}$ belongs to $L^1(H, \nu)$ for any $\lambda < (16\sqrt{2}C')^{-1}$.

5. Supercontractivity and related results

In this section we are interested in proving some characterizations for the semigroup $P(t)$ to be supercontractive, i.e. for any $1 < p < q < +\infty$ and $t > 0$

$$\|P(t)\|_{\mathcal{L}(L^p(H,\nu);L^q(H,\nu))} \le c_{p,q}(t),$$

for some function $c_{p,q} : (0, +\infty) \to (0, +\infty)$ that blows up as $t \to 0^+$. Analogous results are known for semigroups associated to uniformly elliptic operators with unbounded coefficients in finite dimension, see for example [5]. In this section we assume Hypotheses 3.4 restricting our attention to the dissipative case. Indeed, as it is well known already in finite dimension, the Ornstein–Uhlenbeck semigroup associated to the one dimensional operator

$$(L\varphi)(\xi) = \Delta\varphi(\xi) - \xi\varphi'(\xi), \qquad \xi \in \mathbb{R},\ \varphi \in C_b^2(\mathbb{R}),$$

is not supercontractive with respect to its invariant measure, which is the Gaussian measure $\mu(dx) = (2\pi)^{-1/2}e^{-x^2/2}dx$, as proved in [37].

Finite dimensional results cannot be directly used in the infinite dimensional case. However, by employing a finite approximation procedure alongside Harnack inequalities, we will obtain the essential tools needed to derive our results in the infinite dimensional case.

We will use the Harnack inequality

$$|(P(t)\varphi)(x+h)|^p \le \big[(P(t)|\varphi|^p)(x)\big] \exp\left(\frac{p(e^{2\zeta_R t}-1)}{2\zeta_R(p-1)t^2}\|h\|_R^2\right), \tag{5.1}$$

which was proved in [4]. It holds for any $\varphi \in B_b(H)$, $t > 0$, $x \in H$, $h \in H_R$, and $p > 1$, under the additional assumption specified in [4, Hypotheses 3.2].

Hypotheses 5.1. Assume that Hypotheses 3.4 hold. Moreover, suppose that H_R is dense in H, and that A and R are simultaneously diagonalizable with respect to an orthonormal basis $\{e_k \mid k \in \mathbb{N}\}$ of H_R.

To begin with, we first construct the finite dimensional sequence of semigroups which approximates $P(t)$. We start by recalling the definition of the Yosida approximation procedure for the function F. For any $\delta > 0$ and $x \in H$, we let $J_\delta(x) \in \mathrm{Dom}(F)$ be the unique solution of

$$y - \delta(F(y) - \zeta_F y) = x,$$

where ζ_F was introduced in Hypothesis 2.1(v). The existence of $J_\delta(x)$, for every $x \in H$ and $\delta > 0$, is guaranteed by [20, Proposition 5.3.3]. We define $F_\delta : H \to H$ as

$$F_\delta(x) := F(J_\delta(x)), \qquad x \in H,\ \delta > 0.$$

Now, we set

$$\overline{\zeta}_F := \begin{cases} |\zeta_F|^{-1} & \text{if } \zeta_F \neq 0, \\ +\infty & \text{if } \zeta_F = 0. \end{cases}$$

We recall that for any $\delta \in (0, \overline{\zeta}_F)$ and $\mathcal{O} \in \{H, E, H_R\}$, the function $F_\delta : \mathcal{O} \to \mathcal{O}$ is Lipschitz continuous and the function $F_\delta - \zeta_F \mathrm{Id}_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}$ is dissipative and satisfies Hypothesis 2.1(v). Hence, by Theorem 2.2, for every $\delta \in (0, \overline{\zeta}_F)$ and $x \in H$, the problem

$$\begin{cases} dX_\delta(t) = \big[AX_\delta(t) + F_\delta(X_\delta(t))\big]dt + RdW(t), \quad t > 0; \\ X_\delta(0) = x, \end{cases}$$

has a unique mild solution $\{X_\delta(t,x)\}_{t\ge 0}$. Consequently the semigroup

$$(P_\delta(t)\varphi)(x) := \mathbb{E}[\varphi(X_\delta(t,x))], t > 0,$$

is well-defined for any $\varphi \in B_b(H)$. The proof of the following proposition can be found in [4, Proposition 4.3].

Proposition 5.2. *Assume Hypotheses 5.1 hold true. For any $T > 0$, $\varphi \in C_b(H)$ and $x \in E$*

$$\lim_{\delta\to 0} \sup_{t\in[0,T]} \|X_\delta(t,x) - X(t,x)\|_E = 0, \qquad \mathbb{P}\text{-}a.s.;$$
$$\lim_{\delta\to 0} |(P_\delta(t)\varphi)(x) - (P(t)\varphi)(x)| = 0, \qquad t > 0. \tag{5.2}$$

Now we introduce the finite dimensional approximation procedure we will use throughout this section. For any $n \in \mathbb{N}$ we define

$$H_n := \text{span}\{e_1, \dots, e_n\},$$

where $\{e_k \,|\, k \in \mathbb{N}\}$ is the orthonormal basis given by Hypotheses 5.1. Further, let $\pi_n : H \to H_n$ be the orthogonal projection in H. For any $n \in \mathbb{N}$ and $\delta \in (0, \overline{\zeta}_F)$ we define $A_n : H \to H_n$, $R_n : H \to H_n$ and $F_{\delta,n} : H \to H_n$ by

$$A_n := \pi_n A\pi_n (= A\pi_n), \qquad R_n := \pi_n R\pi_n (= R\pi_n) \qquad \text{and} \qquad F_{\delta,n} := \pi_n F_\delta \pi_n.$$

Now, fixed $n \in \mathbb{N}$ and $\delta \in (0, \overline{\zeta}_F)$ we consider

$$\begin{cases} dX_n(t) = [A_n X_n(t) + F_{\delta,n}(X_n(t))]dt + R_n dW_n(t), & t > 0, \\ X_n(0) = x \in H_n. \end{cases} \tag{5.3}$$

Here $W_n(t) := \pi_n W(t) = \sum_{k=1}^n \langle W(t), e_k\rangle_H e_k$.

It is straightforward to check that A_n, R_n and $F_{\delta,n}$ satisfy Hypotheses 2.1. Moreover, being R an injective operator it follows that R_n is bijective, hence $RH_n = H_n$ for any $n \in \mathbb{N}$. Moreover, fixed $\delta \in (0, \overline{\zeta}_F)$, $n \in \mathbb{N}$ and $x \in H_n$, by Theorem 2.2 we can deduce existence and uniqueness of a mild solution $\{X_{\delta,n}(t,x)\}_{t\geq 0}$ of (5.3) and consequently well-posedness for the associated transition semigroup defined for $f \in B_b(H_n)$ as

$$(P_{\delta,n}(t)f)(x) := \mathbb{E}[f(X_{\delta,n}(t,x))], \qquad t > 0,\ x \in H_n. \tag{5.4}$$

We recall that for any $n \in \mathbb{N}$, $\varphi \in C_b^2(H_n)$ and $x \in H_n$ we have

$$\lim_{t\to 0} \frac{(P_{\delta,n}(t)\varphi)(x) - \varphi(x)}{t} = \mathcal{N}_{\delta,n}\varphi(x),$$

where

$$\mathcal{N}_{\delta,n}\varphi(x) = \frac{1}{2}\text{Tr}[R_n^2 D^2\varphi(x)] + \langle A_n x + F_{\delta,n}(x), \nabla\varphi(x)\rangle_H, \tag{5.5}$$

see [8, Section 4.1] or [23]. The proof of the following proposition can be found in [4, Proposition 3.3].

Proposition 5.3. *Assume Hypotheses 5.1 hold true. For any $\delta \in (0, \overline{\zeta}_F)$, $f \in C_b(H)$, $t \geq 0$ and $x \in H_{n_0}$, for some $n_0 \in \mathbb{N}$, it holds*

$$\lim_{n \to +\infty} P_{\delta,n}(t)f(x) = P_\delta(t)f(x). \tag{5.6}$$

Proposition 5.4. *Assume Hypotheses 5.1 hold true with $\zeta_R < 0$. For any $t > 0$, $f \in C_b^1(H)$ and $x \in H$ it holds*

$$P(t)(|f|^2 \ln |f|^2)(x) \leq P(t)(|f|^2)(x) \ln P(t)(|f|^2))(x) + C(t)P(t)(\|\nabla_R f\|_R^2)(x). \tag{5.7}$$

Here $C : (0, +\infty) \to (0, +\infty)$ is defined as $C(t) := 3|\zeta_R|^{-1}(1 - e^{2\zeta_R t})$.

Proof. The proof of estimate (5.7) follows the same line of that of [5, Proposition 3.2]. However, for the sake of completeness we give a sketch of it reducing ourselves to the case when $f \in C_b^2(H)$ has positive infimum and $\sup_{x \in H} f(x) \leq 1$. This latter condition can be removed by applying (5.7) to the function $f\|f\|_\infty^{-1}$ observing that $P(t)c = c$ for any $t > 0$ and $c \in \mathbb{R}$. The general case can be obtained by the dominated convergence theorem writing (5.7) replacing f with $(f^2 + m^{-1})^{1/2}$ and letting $m \to +\infty$.

Fix $n \in \mathbb{N}$, $\delta \in (0, \overline{\zeta}_F)$ and let us consider the semigroup $\{P_{\delta,n}(t)\}_{t \geq 0}$ given by (5.4) and a nonnegative function $f \in C_b^2(H_n)$ such that $\sup_{x \in H_n} f(x) \leq 1$ and $\inf_{x \in H_n} f(x) > \varepsilon$ for some $\varepsilon > 0$. For any fixed $t > 0$ we consider the function

$$g_{\delta,n}(r, x) := (P_{\delta,n}(r)f)(x) \qquad x \in H_n,\ r \in [0, t]. \tag{5.8}$$

The function $g_{\delta,n} : [0, T] \times H_n \to \mathbb{R}$ belongs to $C_b^{1,2}([0, T] \times H_n)$ and solves

$$\begin{cases} \frac{d}{dr} g_{\delta,n}(r, x) = \mathcal{N}_{\delta,n} g_{\delta,n}(r, x), & r \in (0, t); \\ g_{\delta,n}(0, x) = f(x), \end{cases}$$

where $\mathcal{N}_{\delta,n}$ is the operator defined in (5.5). Moreover $g_{\delta,n}(r, x) \geq \varepsilon$ for any $r \geq 0$ and $x \in H_n$ (see [33, Theorem 1.2.5]) and $\sup_{x \in H_n} g_{\delta,n}(r, x) \leq 1$, for any $r \in [0, t]$. For simplicity, we set $g(r) := g_{\delta,n}(r, \cdot)$ and we consider the function

$$G(r) := P_{\delta,n}(t - r)(g(r)^2 \ln g(r)^2)(x), \qquad r \in [0, t].$$

It is not difficult to show that, for every $r \in [0, t]$

$$G'(r) = P_{\delta,n}(t - r)\left[-\mathcal{N}_{\delta,n}(g(r)^2 \ln g(r)^2) + 2g(r)g'(r)(1 + \ln g(r)^2)\right](x). \tag{5.9}$$

By (5.5) and (5.8) we have $g'(r) = \mathcal{N}_{\delta,n} g(r)$ and

$$\mathcal{N}_{\delta,n}(g^2 \ln g^2) = 2g(1+\ln g^2)\mathcal{N}_{\delta,n}g + 2(3+\ln g^2)\|R_n\nabla_x g\|^2,$$

hence for any $x \in H_n$ and $r \in [0,t]$, (5.9) becomes

$$G'(r) = -P_{\delta,n}(t-r)\left[2(3+\ln(g(r))^2)\|R_n\nabla_x g(r)\|^2\right](x).$$

Recalling that $\varepsilon < \inf_{x\in H_n} g(x) \le \sup_{x\in H_n} g(x) \le 1$, we obtain

$$G'(r) \ge -6P_{\delta,n}(t-r)\left[\|R_n\nabla_x P_{\delta,n}(r)f\|^2\right](x), \qquad r\in[0,t].$$

Applying estimate (4.1) to $P_{\delta,n}(t)$ we deduce that

$$G'(r) \ge -6e^{2\zeta_R r}P_{\delta,n}(t)\left(\|R_n\nabla_x f\|^2\right)(x), \qquad r\in[0,t],$$

whence, integrating with respect to r from 0 and t, we get

$$\begin{aligned} G(t)-G(0) = \int_0^t G'(r)dr &\ge -6\left(\int_0^t e^{2\zeta_R r}dr\right)P_{\delta,n}(t)\left(\|R_n\nabla_x f\|^2\right)(x) \\ &\ge -3\zeta_R^{-1}(e^{2\zeta_R t}-1)P_{\delta,n}(t)\left(\|R_n\nabla_x f\|^2\right)(x), \end{aligned}$$

or, equivalently,

$$P_{\delta,n}(t)(f^2\ln f^2)(x) \le C(t)P_{\delta,n}(t)(\|R_n\nabla f\|^2)(x) + P_{\delta,n}(t)(f^2)(x)\ln\left(P_{\delta,n}(t)(f^2)(x)\right) \tag{5.10}$$

for any $t>0$, where $C(t) = 3|\zeta_R|^{-1}(1-e^{2\zeta_R t})$. Letting $n\to+\infty$ in (5.10) and taking into account (5.6) and Proposition 3.8 we obtain

$$P_\delta(t)(f^2\ln f^2)(x) \le C(t)P_\delta(t)(\|\nabla_R f\|_R^2)(x) + P_\delta(t)(f^2)(x)\ln P_\delta(t)(f^2)(x), \tag{5.11}$$

for any $\delta \in (0,\overline{\zeta}_F)$, $x \in \bigcup_{n\in\mathbb{N}} H_n$ and f as above. Moreover by the continuity of f and $\nabla_R f$ and by the density of $\bigcup_{n\in\mathbb{N}} H_n$ in H, inequality (5.11) holds true for any $x\in H$. Again for the continuity of f and $\nabla_R f$, and by using (5.2), letting $\delta\to 0$ we obtain (5.7) for any $x\in E$. Finally, the density of E in H, property (2.8) and the dominated convergence theorem allow to infer (5.7) for any $x\in H$. □

Remark 5.5. We point out that if, for any $t>0$ and $x\in H$, $\nu_{t,x}$ denotes the law of the random variable $X(t,x)$, then the inequality (5.7) can be reformulated as follows:

$$\int_H |f|^2\ln|f|^2 d\nu_{t,x} \le \left(\int_H |f|^2 d\nu_{t,x}\right)\ln\left(\int_H |f|^2 d\nu_{t,x}\right) + C(t)\int_H \|\nabla_R f\|_R^2 d\nu_{t,x}. \tag{5.12}$$

Observe that (5.12) is a logarithmic Sobolev inequality for the probability measure $\nu_{t,x}$.

To get a characterization for the supercontractivity of $P(t)$ we need to introduce a family of logarithmic Sobolev inequalities, called ε-logarithmic Sobolev inequalities, that have the form

$$\int_H f^2 \ln|f| d\nu - \left(\int_H f^2 d\nu\right) \ln \|f\|_{L^2(H,\nu)} \leq \varepsilon \int_H \|\nabla_R f\|_R^2 d\nu + \beta(\varepsilon)\|f\|^2_{L^2(H,\nu)}, \quad (5.13)$$

for any $f \in W^{1,2}_R(H,\nu)$, $\varepsilon > 0$ and some decreasing function $\beta : (0,+\infty) \to \mathbb{R}$ blowing up as $\varepsilon \to 0^+$.

Proposition 5.6. *Assume Hypotheses 5.1 hold true. If $\nu(H_R) = 1$ and the family of inequalities (5.13) is satisfied, then for any $g \in \mathrm{Lip}_{H_R}(H)$ with $\mathrm{Lip}_R(g) \leq 1$ and $t > 0$ it holds*

$$\nu(\{x \in H \,|\, g(x) \geq m_\nu(g) + t\}) \leq \inf_{\varepsilon > 0} \exp\left(-\frac{1}{2\varepsilon}[t + m_\nu(g) - \ln K - 2\beta(\varepsilon)]^2 - 2\beta(\varepsilon)\right),$$

for some positive constant K. Consequently, for any $\alpha > 0$

$$\int_{H_R} e^{\alpha g^2} d\nu < +\infty.$$

Proof. Let g be as in the statement and let g_n be the sequence of functions introduced in the proof of Theorem 4.5. Set

$$K_n(r) := \int_{H_R} e^{r g_n} d\nu, \qquad n \in \mathbb{N},\ r > 0.$$

Using estimate (4.7) and the inequality $|\psi_n(t)| \leq |t|$ which holds true for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we deduce

$$K_n(r) \leq K := \int_{H_R} e^{|g|} d\nu < +\infty, \qquad r \leq 1. \quad (5.14)$$

Now, applying (5.13) to the function $e^{\frac{r}{2} g_n}$ we obtain

$$r K_n'(r) - K_n(r) \ln K_n(r) \leq \frac{\varepsilon}{2} r^2 K_n(r) + 2\beta(\varepsilon) K_n(r)$$

whence, dividing by $r^2 K_n(r)$, we have

$$\left(\frac{1}{r} \ln K_n(r)\right)' \leq \frac{\varepsilon}{2} + \frac{2}{r^2}\beta(\varepsilon). \quad (5.15)$$

Integrating (5.15) from 1 to $v > 1$ we get

$$\frac{1}{v}\ln K_n(v) - \ln K_n(1) \leq \frac{\varepsilon}{2}(v-1) + 2\beta(\varepsilon)\left(1 - \frac{1}{v}\right),$$

and so by (5.14), for any $n \in \mathbb{N}$, $v > 1$ and $\varepsilon > 0$ we have

$$K_n(v) = \int\limits_{H_R} e^{vg_n} d\nu \leq \exp\left(\frac{\varepsilon}{2}v^2 + v\left[\ln K + 2\beta(\varepsilon)\right] - 2\beta(\varepsilon)\right). \tag{5.16}$$

Now, by the Chernoff bound (see [36, Introduction of Section 4.2]) and (5.16) we get

$$\begin{aligned}\nu(\{x \in H \,|\, g_n(x) \geq m_\nu(g_n) + s\}) &\leq \inf_{t\geq 0}\left(e^{-t(s+m_\nu(g_n))}m_\nu(e^{tg_n})\right)\\ &\leq \inf_{t\geq 1}\Big(\exp\Big(-t[s+m_\nu(g_n)] + \frac{\varepsilon}{2}t^2 + t\left[\ln K + 2\beta(\varepsilon)\right]\\ &\quad -2\beta(\varepsilon)\Big)\Big)\\ &= \exp\left(-\frac{1}{2\varepsilon}[s + m_\nu(g_n) - \ln K - 2\beta(\varepsilon)]^2 - 2\beta(\varepsilon)\right),\end{aligned} \tag{5.17}$$

for every $\varepsilon > 0$. Using a similar argument as the one used in the proof of Theorem 4.5 we obtain that (5.17) can be deduced for the function g too. By the Layer cake formula, for any $\alpha > 0$, we get

$$\begin{aligned}\int\limits_H e^{\alpha g^2} d\nu &\leq 1 + \int\limits_1^\infty \nu\Big(\Big\{x \in H \,\Big|\, e^{\alpha(g(x))^2} > s\Big\}\Big)ds\\ &= 1 + \int\limits_1^\infty \nu\left(\left\{x \in H \,\middle|\, |g(x)| > \sqrt{\frac{\ln s}{\alpha}}\right\}\right)ds.\end{aligned}$$

Performing the change of variables $s = e^{c(t+m_\nu(|g|))^2}$ we obtain

$$\int\limits_H e^{\alpha g^2} d\nu \leq 1 + 2\alpha \int\limits_{\mathbb{R}} \nu\{|g| > m_\nu(|g|) + t\} e^{\alpha(m_\nu(|g|)+t)^2}(t + m_\nu(|g|))dt$$

and using estimate (5.17) with g_n being replaced by $|g|$ we deduce

$$\begin{aligned}&\int\limits_H e^{\alpha g^2} d\nu\\ &\leq 1 + 2\alpha \int\limits_{\mathbb{R}} \exp\left(-\frac{1}{2\varepsilon}\left[t + m_\nu(|g|) - \ln K - 2\beta(\varepsilon)\right]^2 - 2\beta(\varepsilon) + \alpha(m_\nu(|g|) + t)^2\right)\end{aligned}$$

$$\times (t + m_\nu(|g|))dt,$$

whence, choosing $\varepsilon < (2\alpha)^{-1}$, we can conclude. □

As in Theorem 4.5, using the relation $\nabla_R \varphi = R^2 \nabla \varphi$ in (5.13) we obtain the following alternative result.

Proposition 5.7. *Assume Hypotheses 5.1 hold true. If the family of inequalities (5.13) is satisfied, then for any $g \in \mathrm{Lip}(H)$ with $\mathrm{Lip}(g) \le 1$ and $t > 0$ it holds*

$$\nu(\{x \in H \mid g(x) \ge m_\nu(g) + t\}) \le \inf_{\varepsilon > 0} \exp\left(-\frac{1}{2\varepsilon}[t + m_\nu(g) - \ln K - 2\beta(\varepsilon)]^2 - 2\beta(\varepsilon)\right),$$

for some positive constant K. Consequently, for any $\alpha > 0$

$$\int_H e^{\alpha g^2} d\nu < +\infty.$$

We are going to prove that the supercontractivity property of $P(t)$ implies (5.13).

Theorem 5.8. *Assume that Hypotheses 5.1 hold true with $\zeta_R < 0$. If the semigroup $\{P(t)\}_{t \ge 0}$ is supercontractive then the family of inequalities (5.13) holds true.*

Proof. We start by proving the statement for functions $f \in \mathcal{F}C_b^1(H)$ such that $\|f\|_{L^2(H,\nu)} = 1$. So, let us fix such a function f and $1 < p < q < +\infty$. We claim that there exist a constant $K = K(p,q) \in (0,1)$ and a positive function $\eta_{p,q} : (0, +\infty) \to (0, +\infty)$ blowing up as $t \to 0^+$, independent of f, such that

$$\int_H (P(t)f^2) \ln(P(t)f^2) d\nu \le K(p,q) \int_H f^2 \ln f^2 d\nu + \eta_{p,q}(t), \qquad t > 0. \tag{5.18}$$

To this aim, let us recall that $P(t)$ is supercontractive, i.e., $\|P(t)\|_{\mathcal{L}(L^p(H,\nu);L^q(H,\nu))} \le c_{p,q}(t)$ for any $t > 0$. Moreover, being $\|P(t)\|_{\mathcal{L}(L^1(H,\nu))} \le 1$ for any $t > 0$, by the Riesz–Thorin interpolation theorem (see [35]), we deduce that

$$\|P(t)\|_{\mathcal{L}(L^{p_h}(H,\nu);L^{q_h}(H,\nu))} \le [c_{p,q}(t)]^{r_h}$$

for any h belonging to $[0, 1 - 1/p]$, $r_h = ph(p-1)^{-1}$, $p_h = (1-h)^{-1}$ and $q_h = (1 - r_h + r_h/q)^{-1}$. In particular,

$$\int_H \left(P(t)|f|^{2(1-h)}\right)^{q_h} d\nu \le [c_{p,q}(t)]^{r_h q_h}$$

and consequently, for any $h > 0$,

$$\frac{1}{h}\left[\int_H \left(P(t)|f|^{2(1-h)}\right)^{q_h} d\nu - 1\right] \leq \frac{1}{h}\left[(c_{p,q}(t))^{r_h q_h} - 1\right].$$

Thus, recalling that $\|f\|_{L^2(H,\nu)} = 1$, by the previous inequality, we get

$$\frac{1}{h}\left(\|P(t)|f|^{2(1-h)}\|^{q_h}_{L^{q_h}(H,\nu)} - \|P(t)|f|^2\|_{L^1(H,\nu)}\right) \leq \frac{1}{h}\left[(c_{p,q}(t))^{r_h q_h} - 1\right].$$

Observing that $r_0 = 0$, $q_0 = 1$ and that the functions $h \to \|P(t)|f|^{2(1-h)}\|^{q_h}_{L^{q_h}(H,\nu)}$ and $h \to [c_{p,q}(t)]^{r_h q_h}$ are differentiable at $h = 0$, by the previous inequality, we obtain

$$\frac{d}{dh}\left(\|P(t)|f|^{2(1-h)}\|^{q_h}_{L^{q_h}(H,\nu)}\right)_{|h=0} \leq \frac{d}{dh}(c_{p,q}(t))^{r_h q_h})_{|h=0}$$

that, thanks to the invariance of ν, implies

$$\begin{aligned}\frac{p(q-1)}{q(p-1)}\int_H (P(t)f^2)\ln(P(t)f^2)d\nu &\leq \int_H P(t)\left(f^2\ln f^2\right)d\nu + \frac{p}{p-1}\ln(c_{p,q}(t)) \\ &\leq \int_H f^2\ln f^2 d\nu + \frac{p}{p-1}\ln(c_{p,q}(t))\end{aligned}$$

whence the claim with $K(p,q) = \frac{q(p-1)}{p(q-1)}$ and $\eta_{p,q}(t) = K(p,q)\frac{p}{p-1}\ln(c_{p,q}(t))$ which, clearly, blows up as $t \to 0^+$. To complete the proof, it suffices to integrate (5.7) over H with respect to ν, using its invariance to deduce that

$$\int_H f^2\ln f^2 d\nu \leq C(t)\int_H \|\nabla_R f\|_R^2 d\nu + \int_H P(t)f^2\ln P(t)f^2 d\nu,$$

where $C(t)$ is defined in estimate (5.7). Finally, thanks to (5.18) we conclude that

$$\int_H f^2\ln f^2 d\nu \leq \frac{p(q-1)}{(q-p)}C(t)\int_H \|\nabla_R f\|_R^2 d\nu + \frac{qp}{q-p}\ln(c_{p,q}(t)). \tag{5.19}$$

Since $C(t)$ vanishes as $t \to 0^+$ and $c_{p,q}(t)$ blows up as $t \to 0^+$, estimate (5.13) is proved for functions $f \in \mathcal{F}C_b^1(H)$ with $\|f\|_{L^2(H,\nu)} = 1$. Condition $\|f\|_{L^2(H,\nu)} = 1$ can be removed by applying (5.19) to the function $f\|f\|^{-1}_{L^2(H,\nu)}$ to get

$$\int_H f^2\ln\left(\frac{f^2}{\|f\|^2_{L^2(H,\nu)}}\right)d\nu \leq \frac{p(q-1)}{(q-p)}C(t)\int_H \|\nabla_R f\|_R^2 d\nu + \frac{qp}{q-p}\ln(c_{p,q}(t))\|f\|^2_{L^2(H,\nu)}. \tag{5.20}$$

The general case can be obtained by approximation. Indeed, for any $f \in W^{1,2}_R(H,\nu)$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}C^1_b(H)$ such that $\|f_n - f\|_{W^{1,2}_R(H,\nu)}$ vanishes as $n \to +\infty$. The claim will follow writing (5.20) for f_n, letting $n \to \infty$ after observing that

$$\lim_{n\to+\infty} \int_H f_n^2 \ln f_n^2 d\nu = \int_H f^2 \ln f^2 d\nu. \tag{5.21}$$

The convergence result in (5.21) can be proved writing $\ln f_n^2 = \ln_+ f_n^2 - |\ln f_n^2|\chi_{\{f_n^2<1\}}$ for any $n \in \mathbb{N}$, where $\ln_+$ denotes the positive part of the natural logarithm. Thus the Fatou lemma and the dominated convergence theorem allow us to deduce (5.21). □

We are now in position to prove a complete characterization of the supercontractivity of $P(t)$ in terms of some families of logarithmic Sobolev inequalities and of some integrability conditions for the functions $x \mapsto e^{\lambda\|x\|_R^2}$ for any $\lambda > 0$. We underline that estimate (5.1) plays a key role in its proof.

Theorem 5.9. *Assume that Hypotheses 5.1 hold true with $\zeta_R < 0$ and $\nu(H_R) = 1$. The following statements are equivalent:*

(i) the semigroup $\{P(t)\}_{t\geq 0}$ is supercontractive;
(ii) the family of inequalities (5.13) holds true;
(iii) for any $\lambda > 0$ it holds

$$\int_{H_R} e^{\lambda\|x\|_R^2} \nu(dx) < +\infty. \tag{5.22}$$

Proof. By Theorem 5.8 we know that (i) implies (ii). Observe that (ii) implies (iii) follows immediately by Proposition 5.6 taking $g(x) = \|x\|_R$.

To prove that (iii) implies (i) it suffices to show that $\|P(t)f\|_{L^q(H,\nu)} \leq M_{p,q}(t)\|f\|_{L^p(H,\nu)}$ for any $f \in C_b(H)$, any $t > 0$ and some function $M_{p,q} : (0,+\infty) \to (0,+\infty)$. The general case when f belongs to $L^p(H,\nu)$ can be obtained by approximation and the dominated convergence theorem. Let us consider $f \in C_b(H)$, $1 < p < +\infty$ and $t > 0$. By using the Harnack inequality in (5.1), the invariance of ν and the fact that $\nu(H_R) = 1$ we deduce

$$\begin{aligned}
\int_H |f(y)|^p \nu(dy) &= \int_H P(t)|f|^p(y)\nu(dy) = \int_{H_R} P(t)|f|^p(y)\nu(dy) \\
&\geq |P(t)f(x)|^p \int_{H_R} \exp\left(-\frac{p(e^{2\zeta_R t}-1)}{2\zeta_R(p-1)t^2}\|x-y\|_R^2\right)\nu(dy) \\
&\geq |P(t)f(x)|^p \nu(B_R(0,r)) \exp\left(-\frac{p(e^{2\zeta_R t}-1)}{2\zeta_R(p-1)t^2}(r^2+\|x\|_R^2)\right),
\end{aligned}$$

where $B_R(0,r)$ is the ball in H_R with center 0 and radius $r>0$. Hence for any $x\in H_R$, choosing $\overline{r}=\overline{r}_p>0$ such that $\nu(B_R(0,\overline{r}))>2^{-p}$, we have

$$|P(t)f(x)|\le 2\exp\left(\frac{e^{2\zeta_R t}-1}{2\zeta_R(p-1)t^2}(\overline{r}^2+\|x\|_R^2)\right)\|f\|_{L^p(H,\nu)}. \tag{5.23}$$

Now, let $1<p<q<+\infty$. By (5.23) we have

$$\begin{aligned}\|P(t)f\|^q_{L^q(H,\nu)}&=\int_H |P(t)f(x)|^q\nu(dx)=\int_{H_R}|P(t)f(x)|^q\nu(dx)\\&\le 2^q\left(\int_{H_R}\exp\left(\frac{q(e^{2\zeta_R t}-1)}{2\zeta_R(p-1)t^2}(\overline{r}^2+\|x\|_R^2)\right)\nu(dx)\right)\|f\|^q_{L^p(H,\nu)}.\end{aligned}$$

Thanks to (5.22),

$$(M_{p,q}(t))^q:=2^q\int_{H_R}\exp\left(\frac{q(e^{2\zeta_R t}-1)}{2\zeta_R(p-1)t^2}(\overline{r}^2+\|x\|_R^2)\right)\nu(dx)<+\infty,$$

and, consequently, for any $f\in C_b(H)$, it holds $\|P(t)f\|_{L^q(H,\nu)}\le M_{p,q}(t)\|f\|_{L^p(H,\nu)}$, whence the claim. □

Remark 5.10.

(i) As already pointed out in the introduction, even if it should be quite natural to exploit [5, Theorem 3.1] and the approximation in (5.6) to prove directly the equivalence (i) ⇔ (ii) in Theorem 5.9, this approach does not work. Indeed, for every $n\in\mathbb{N}$ and $\delta>0$ the semigroup $\{P_{\delta,n}(t)\}_{t\ge0}$ admits a unique invariant measure $\nu_{\delta,n}$, but in our general setting, the relation between $\nu_{\delta,n}$ and ν is not explicit and then it is not possible to relate the supercontractivity property of $\{P(t)\}_{t\ge0}$ to that of $\{P_{\delta,n}(t)\}_{t\ge0}$ for any $n\in\mathbb{N}$ and $\delta>0$ and viceversa.

(ii) Note that in the case $H=\mathbb{R}^n$, in [5, Lemma 3.8] the authors provide sufficient conditions on the drift of (7.9) to get property (iii) of Theorem 5.9 and consequently, to deduce the supercontractivity of $\{P(t)\}_{t\ge0}$. However, these conditions are related to the existence of a Lyapunov function for the second order Kolmogorov operator associated to (7.9) and in the infinite dimensional setting, the theory of Lyapunov functions is not well-developed.

(iii) In many cases, the assumption $\nu(H_R)=1$ is checked assuming that the mild solution of the stochastic partial differential equation (7.9) takes values in H_R. In these situations, Theorem 5.9 can be applied with $R=\mathrm{Id}$ and replacing H_R with H. However, we did not choose this approach to highlight the fact that the hypothesis $\nu(H_R)=1$ is just used in Theorem 5.9 to prove that (iii) implies (i). We

believe that it is possible to avoid the condition $\nu(H_R) = 1$ if one were able to prove a Harnack inequality of the type (5.1) with $\|\cdot\|_R$ replaced by $\|\cdot\|_H$. This type of Harnack inequality has already been proven in the linear case in [38], via a technique that is not applicable in our case. The main problem considering stochastic partial differential equation of the type (1.3) is proving a pointwise version of the estimate in [14, Proposition 6.4.1].

6. Applications to ultraboundness

Here, we collect some consequences of the results in the previous sections. We start by finding conditions that ensure summability properties stronger than the supercontractivity. In particular, the following result allows us to characterize the ultraboundedness of the semigroup $P(t)$, i.e. its boundedness as an operator from $L^p(H,\nu)$ into $L^\infty(H,\nu)$ for any $p > 1$ and any $t > 0$.

Proposition 6.1. *Assume that Hypotheses 5.1 hold true with $\zeta_R < 0$, $\nu(H_R) = 1$ and consider the following assertions:*

(i) *the semigroup $P(t)$ is ultrabounded, i.e., for any $p > 1$ and $t > 0$:*

$$\|P(t)\|_{\mathcal{L}(L^p(H,\nu);L^\infty(H,\nu))} < +\infty;$$

(ii) *for any $\delta > 0$ and $\lambda > 0$, there exists $c_{\delta,\lambda} > 0$ such that $\|P(t)\varphi_\lambda\|_{L^\infty(H,\nu)} \le c_{\delta,\lambda}$ for any $t \ge \delta$ where $\varphi_\lambda(x) := e^{\lambda\|x\|_R^2}$.*

Then (i) $\Rightarrow$ (ii). Further, if $P(t)$ is supercontractive then (ii) $\Rightarrow$ (i).

Proof. Observe that if $P(t)$ is ultrabounded, then it is supercontractive as well, since ν is a probability measure. According to Theorem 5.9 the functions φ_λ belong to $L^p(H,\nu)$ for any $\lambda > 0$ and $p \ge 1$. As a consequence, $P(t)\varphi_\lambda$ is well defined ν-a.e. in H. To prove that (i) implies (ii), it is enough to observe that

$$\|P(t)\varphi_\lambda\|_{L^\infty(H,\nu)} \le \|P(t)\|_{\mathcal{L}(L^2(H,\nu),L^\infty(H,\nu))}\|\varphi_\lambda\|_{L^2(H,\nu)}.$$

The map $t \mapsto \|P(t)\|_{\mathcal{L}(L^2(H,\nu),L^\infty(H,\nu))}$ blows up at $t = 0$ and, by (2.15), it is bounded in every interval of the form $[\varepsilon, +\infty)$, for $\varepsilon > 0$. So we get that $\|P(t)\varphi_\lambda\|_\infty \le c_{\delta,\lambda}$ for any $t \ge \delta$ and some positive constant $c_{\delta,\lambda}$.

Conversely, if $P(t)$ is supercontractive, in order to prove that (ii) implies (i) we take $f \in L^p(H,\nu)$ and for any $\delta > 0$, let M_δ be such that $2e^{\zeta_R t} \ge 2 + \zeta_R M_\delta t^2$ for any $t \ge \delta$. Applying both the semigroup law and inequality (5.23), we can then deduce that:

$$|P(t)f(x)| = |P(t/2)P(t/2)f(x)| \le P(t/2)|P(t/2)f(x)|$$

$$\begin{aligned}
&\leq P(t/2)\left(2\exp\left(\frac{2(e^{\zeta_R t}-1)}{\zeta_R(p-1)t^2}(\overline{r}^2+\|x\|_R^2)\right)\|f\|_{L^p(H,\nu)}\right)\\
&\leq 2\exp\left(\frac{2(e^{\zeta_R t}-1)}{\zeta_R(p-1)t^2}\overline{r}^2\right)\|f\|_{L^p(H,\nu)}\left(P(t/2)\left(\exp\left(\frac{2(e^{\zeta_R t}-1)}{\zeta_R(p-1)t^2}(\|\cdot\|_R^2)\right)\right)\right)(x)\\
&\leq 2c_{\delta,(p-1)^{-1}M_\delta}\exp\left(\frac{2(e^{\zeta_R t}-1)}{\zeta_R(p-1)t^2}\overline{r}^2\right)\|f\|_{L^p(H,\nu)},
\end{aligned}$$

for any $t \geq \delta$ where $\overline{r}$ is chosen as in the proof of Theorem 5.9. The arbitrariness of δ allows us to conclude. □

6.1. Sufficient conditions for ultraboundness

We assume the following additional hypotheses on the perturbation F.

Hypotheses 6.2. Under Hypotheses 5.1, assume that A_R is a negative operator and that it holds $F_{|_E}(H_R \cap E) \subseteq H_R$. Suppose further that there exist $a > 0$ and a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ belonging to $C([0,+\infty))$ such that $1/\phi \in L^1([b,+\infty))$, for every $b > 0$, $1/\phi \notin L^1([0,+\infty))$, and

$$\lim_{s\to+\infty} s^{-1}\phi(s) = +\infty, \tag{6.1}$$

$$\langle F_E(x) - F_E(y), x-y\rangle_R \leq a - \phi(\|x-y\|_R^2), \qquad x,y \in H_R \cap E. \tag{6.2}$$

Proposition 6.3. *Assume Hypotheses 6.2 hold true. For every $x, y \in H_R$ and $t > 0$, it holds*

$$\|X(t,x) - X(t,y)\|_R^2 \leq 2\phi^{-1}(2a) + \psi^{-1}(t/4), \qquad \mathbb{P}\text{-a.s.} \tag{6.3}$$

where $\psi(s) := \int_s^{+\infty} 1/\phi(r)dr$.

Proof. It is not restrictive to assume that both $\{X(t,x)\}_{t\geq 0}$ and $\{X(t,y)\}_{t\geq 0}$ are strict solutions of (1.1) (with initial datum x and y, respectively). Otherwise we proceed as in [8, Proposition 3.6] or [14, Proposition 6.2.2] approximating $\{X(t,x)\}_{t\geq 0}$ and $\{X(t,y)\}_{t\geq 0}$ by means of sequences of more regular processes using the Yosida approximation of A_R. Estimates similar to (2.11) and (3.2) yield that $X(t,x) - X(t,y) \in E \cap H_R$ for any $t > 0$, $x, y \in H_R \cap E$. Now, let us fix $x, y \in E \cap H_R$, multiplying the equation solved by $X(t,x) - X(t,y)$ by the difference $X(t,x) - X(t,y)$ and using Hypotheses 6.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|X(t,x) - X(t,y)\|_R^2 \leq a - \phi\left(\|X(t,x) - X(t,y)\|_R^2\right).$$

Arguing as in [19, pp. 1010–1011] and using the density of $E \cap H_R$ in H_R, we conclude the proof. □

Proposition 6.4. *Assume that Hypotheses 6.2 hold true, $\nu(H_R) = 1$ and that $P(t)$ is supercontractive. Then $P(t)$ is ultrabounded.*

Proof. By Proposition 6.1 it is sufficient to prove that for any $\delta, \lambda > 0$ there exists $c_{\delta,\lambda} > 0$ such that $\|P(t)\varphi_\lambda\|_{L^\infty(H,\nu)} \leq c_{\delta,\lambda}$ for any $t \geq \delta$ where $\varphi_\lambda(x) := e^{\lambda\|x\|_R^2}$. For every $\beta > 0$, we consider the function

$$\varphi_{\lambda,\beta}(x) = \frac{e^{\lambda\|x\|_R^2}}{1 + \beta e^{\lambda\|x\|_R^2}}, \qquad x \in H_R.$$

By the invariance of ν and the Jensen inequality we have, for $p \in [1, \infty)$,

$$\|P(s)\varphi_{\lambda,\beta}\|_{L^p(H,\nu)}^p = \int_H (P(s)\varphi_{\lambda,\beta})^p d\nu \leq \int_H P(s)(\varphi_{\lambda,\beta})^p d\nu = \int_H (\varphi_{\lambda,\beta})^p d\nu.$$

Thus, letting $\beta \to 0^+$ by the monotone convergence theorem and Theorem 5.9 we obtain

$$\|P(s)\varphi_\lambda\|_{L^p(H,\nu)}^p \leq \int_H \varphi_{p\lambda} d\nu < +\infty$$

so $P(s)\varphi_\lambda \in L^p(H,\nu)$ for every $s \geq 0$, $\lambda > 0$ and $p > 1$. By formula (5.1), for every $s \geq 0$, $\beta, \lambda > 0$ and $x, y \in H$ such that $x - y \in H_R$ we get

$$(P(s)\varphi_{\lambda,\beta}(x))^2 \leq P(s)\varphi_{\lambda,\beta}^2(y) e^{(e^{2\zeta_R t} - 1)(\zeta_R t^2)^{-1}\|x-y\|_R^2}.$$

Letting $\beta \to 0$ by the monotone convergence theorem we obtain

$$(P(s)\varphi_\lambda(x))^2 \leq P(s)\varphi_{2\lambda}(y) e^{(e^{2\zeta_R t} - 1)(\zeta_R t^2)^{-1}\|x-y\|_R^2}. \tag{6.4}$$

By (6.3), (6.4) and the Jensen inequality, for every $s \geq 0$, $\lambda > 0$ and $x, y \in H_R$ we obtain

$$(P(t)\varphi_\lambda(x))^2 \leq P(t/2)\left[P(t/2)\varphi_\lambda\right]^2(x) \leq P(t)\varphi_{2\lambda}(y) e^{4(e^{\zeta_R t} - 1)(\zeta_R t^2)^{-1}(2\phi^{-1}(2a) + \psi^{-1}(t/8))}.$$

By Hypotheses 6.2 we have that $\psi^{-1}(t) \to 0$ as t approaches $+\infty$. Hence

$$\lim_{t \to +\infty} e^{4(e^{\zeta_R t} - 1)(\zeta_R t^2)^{-1}(2\phi^{-1}(2a) + \psi^{-1}(t/8))} = 1$$

so there exists $\delta, c_\delta > 0$ such that for every $t > \delta$

$$(P(t)\varphi_\lambda(x))^2 \leq c_\delta P(t)\varphi_{2\lambda}(y). \tag{6.5}$$

Integrating both sides of (6.5) with respect to the measure ν and using the invariance of ν and Theorem 5.9 we get

$$(P(t)\varphi_\lambda(x))^2 \le c_\delta \|\varphi_\lambda\|^2_{L^2(H,\nu)}, \quad t > \delta.$$

Thus, applying Proposition 6.1 we get the claim. □

7. Examples

Here we will provide examples of equations like (1.1) where our results can be applied.

7.1. Infinite dimensional polynomial

Let $E = H = L^2([0,1]^d, \lambda)$ with $d \in \mathbb{N}$ and λ is the Lebesgue measure on $[0,1]^d$. Let A be the realization of the Laplace operator in H with Dirichlet boundary condition. Let $F : H \to H$ be the function defined by

$$[F(f)](\xi) := \zeta_F f(\xi) + \int_0^1 \int_0^1 \int_0^1 K(\xi_1, \xi_2, \xi_3, \xi) f(\xi_1) f(\xi_2) f(\xi_3) d\xi_1 d\xi_2 d\xi_3,$$

where $f \in H$, $\xi \in [0,1]^d$, $\zeta_F \in \mathbb{R}$ and $K \in L^2([0,1]^{4d}, \lambda)$. For a fixed $\beta > 1/2$, we consider the following stochastic partial differential equation

$$\begin{cases} dX(t) = \left[-(-A)^\beta X(t) + F(X(t))\right] dt + (-A)^{-1/2} dW(t), \quad t > 0; \\ X(0) = x \in L^2([0,1]^d, \lambda). \end{cases}$$

In this case $R = (-A)^{-1/2}$ and $H_R = W_0^{1,2}([0,1]^d, \lambda)$, i.e. the set of functions belonging to $W^{1,2}([0,1]^d, \lambda)$ and having null trace on the boundary of $[0,1]^d$. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence of eigenvalues of A in H. Recall that, for every $d \in \mathbb{N}$ we have

$$\lambda_k \approx -k^{2/d}, \qquad k \in \mathbb{N}. \tag{7.1}$$

Note that, thanks to (7.1), Hypothesis 2.1(iv) holds true for every $\beta > d/2 - 1$. Moreover assuming that $(\partial K/\partial \xi) \in L^2([0,1]^{4d}, \lambda)$ and that K and $(\partial K/\partial \xi)$ have symmetric versions (see [4, Formula (5.10)]) then all the assumptions in Theorems 4.1 and 4.3 are satisfied. We refer to [4, Examples 5.3 and 5.4] for detailed calculations.

7.2. Radial perturbation

Let $f : [0, +\infty) \to [0, +\infty)$ be an increasing and differentiable function. We consider $H = L^2([0,1])$ and the function $U : H \to \mathbb{R}$ defined by

$$U(x) = -f(\|x\|^2_H), \qquad x \in H, \tag{7.2}$$

and the transition semigroup $P(t)$ associated to

$$\begin{cases} dX(t) = [AX(t) + \nabla U(X(t))]dt + dW(t), & t > 0, \\ X(0) = x \in H, \end{cases} \tag{7.3}$$

where A is the realization of the second order derivative with Dirichlet boundary condition in H. It is well known that $P(t)$ has a unique invariant measure given by

$$\nu(dx) = Ce^U \mu(dx), \qquad \mu \sim \mathcal{N}(0, 2^{-1}(-A)^{-1}),$$

where $C = (\int_H e^U \mu(dx))^{-1}$ is the normalization constant. We now want to find sufficient conditions that guarantee the supercontractivity of $P(t)$. We prove that if

$$\lim_{s \to +\infty} f(s)s^{-1} = +\infty, \tag{7.4}$$

then $P(t)$ is supercontractive, i.e., $\int_H e^{\lambda \|x\|^2} \nu(dx) < +\infty$ for any $\lambda > 0$. To this aim, let $\alpha_0 > 0$ be the optimal exponent given by Fernique Theorem such that for every $\delta < \alpha_0$ it holds

$$\int_H e^{\delta \|x\|_H^2} \mu(dx) < +\infty. \tag{7.5}$$

For every $\lambda > 0$ we have

$$\int_H e^{\lambda \|x\|^2} \nu(dx) = \int_H e^{\lambda \|x\|^2} e^U \mu(dx) = \int_H e^{\lambda \|x\|_H^2 - f(\|x\|_H^2)} \mu(dx) \tag{7.6}$$

Fix $\delta < \alpha_0$ and choose $r_0 > 0$ such that (by (7.4)) for every $x \in H$ with $\|x\|^2 > r_0$ it holds

$$f(\|x\|_H^2) > (\lambda - \delta)\|x\|^2. \tag{7.7}$$

Let B the ball of H with radius $\sqrt{r_0}$. By (7.5), (7.6) and (7.7) for every $\lambda > 0$ there exists $\delta < \alpha_0$ such that

$$\int_H e^{\lambda \|x\|^2} e^U \mu(dx) \leq e^{\lambda r_0} + \int_{H \setminus B} e^{\delta \|x\|^2} \mu(dx) < +\infty. \tag{7.8}$$

Hence by (7.8) and Theorem 5.9, $P(t)$ is supercontractive.

7.3. A reaction–diffusion system

We consider the following stochastic reaction-diffusion equation:

$$\begin{cases} dX(t) = \left[\frac{\partial^2}{\partial \xi^2} X(t) - b(X(t))\right] dt + dW(t), & t > 0; \\ X(t)(0) = X(t)(1), & t > 0; \\ X(0) = x \in L^2([0,1], \lambda), \end{cases} \tag{7.9}$$

where λ is the Lebesgue measure on $[0,1]$ and $b : \mathbb{R} \to \mathbb{R}$ is a smooth enough function satisfying the following conditions:

Hypotheses 7.1. The function $b : \mathbb{R} \to \mathbb{R}$ belongs to $C^3(\mathbb{R})$ and

(i) there exists an integer $m \geq 0$ such that for every $j = 0, 1, 2, 3$ it holds

$$\sup_{z \in \mathbb{R}} \frac{\left|\mathcal{D}_z^j b(z)\right|}{1 + |z|^{\max\{0, 2m+1-j\}}} < +\infty,$$

where $\mathcal{D}_z^0 b$ denotes the function b;

(ii) there exists $a > 0$ such that for every $z, h \in \mathbb{R}$ it holds $(b(z+h) - b(z))h \leq -ah^2$.

A simple example of function b that verifies Hypotheses 7.1 is

$$b(z) := C_{2m+1} z^{2m+1} + \sum_{k=0}^{2m} C_k z^k, \qquad z \in \mathbb{R}, \tag{7.10}$$

where $m \in \mathbb{N}$, $C_0, \ldots, C_{2m} \in \mathbb{R}$ and $C_{2m+1} > 0$. The stochastic equation (7.9) can be rewritten in an abstract way as (1.1) where A is the realization in $L^2([0,1], \lambda)$ of the second order derivative with Dirichlet boundary conditions, $R = \mathrm{Id}_{L^2([0,1],\lambda)}$ and $F : C([0,1]) \to C([0,1])$ is the Nemytskii operator defined as

$$F(x)(\xi) := -b(x(\xi)), \qquad \xi \in [0,1],\ x \in C([0,1]).$$

Note that Hypotheses 2.1 (with $E = C([0,1])$) are satisfied for (7.9), see [14, Chapter 6]) and since $R = \mathrm{Id}_{L^2([0,1],\lambda)}$ also Hypotheses 3.1 and 5.1 hold true. Hence all the results in this paper can be applied to the transition semigroup $P(t)$ associated to (7.9). Finally we show that under an additional assumption on b the invariant measure ν of $P(t)$ verifies the condition in Theorem 5.9(iii).

Let $B : \mathbb{R} \to \mathbb{R}$ be a primitive of b and consider $U : L^2([0,1], \lambda) \to \mathbb{R}$ given by

$$U(x) = \begin{cases} \int_0^1 B(x(\xi)) d\xi, & x \in C([0,1]); \\ 0, & x \notin C([0,1]). \end{cases}$$

Let $\mu \sim N(0, 2^{-1}(-A)^{-1})$. By [18, Proposition 5.2], the function U belongs to $W^{1,p}(L^2([0,1],\lambda),\mu)$, for any $p \geq 1$, and

$$\nabla U(x)(\xi) = b(x(\xi)) = -F(x)(\xi), \qquad x \in C([0,1]),\ \xi \in [0,1].$$

Since U is a potential of $-F$, then the invariant measure ν of $P(t)$ is the Gibbs measure given by

$$d\nu = e^{-U} d\mu.$$

Hence if we assume that there exist $c, \varepsilon > 0$ such that for any $\xi \in \mathbb{R}$ it holds that

$$B(\xi) \geq c|\xi|^{2+\varepsilon}, \tag{7.11}$$

then, be the Jensen inequality, for any $\lambda > 0$ we have

$$\begin{aligned}
&\int_{L^2([0,1],\lambda)} e^{\lambda||x||^2_{L^2([0,1],\lambda)}} \nu(dx) \\
&\quad = \int_{L^2([0,1],\lambda)} \exp\left(||x||^2_{L^2([0,1],\lambda)} - \int_0^1 B(x(\xi))d\xi \right) \mu(dx) \\
&\quad \leq \int_{L^2([0,1],\lambda)} \exp\left(||x||^2_{L^2([0,1],\lambda)} - c\int_0^1 |x(\xi)|^{2+\varepsilon} d\xi \right) \mu(dx) \\
&\quad \leq \int_{L^2([0,1],\lambda)} \exp\left(||x||^2_{L^2([0,1],\lambda)} - c\left(\int_0^1 |x(\xi)|^2 d\xi \right)^{(2+\varepsilon)/2} \right) \mu(dx) \\
&\quad = \int_{L^2([0,1],\lambda)} \exp\left(||x||^2_{L^2([0,1],\lambda)} - c||x||^{2+\varepsilon}_{L^2([0,1],\lambda)} \right) \mu(dx) < +\infty.
\end{aligned}$$

Thus ν verifies the conditions in Theorem 5.9(iii) and $P(t)$ is supercontractive. Observe that condition (7.11) is satisfied if $m \geq 1$ (see Hypothesis 7.1(i)). Moreover we stress that if, for example,

$$b(z) = Cz^{2m+1}$$

for some $m \in \mathbb{N}$ and $C > 0$, then the function F verifies Hypotheses 6.2 with $\phi(s) \approx s^{2m+2}$. Hence the assumptions of Proposition 6.4 hold true and the semigroup $P(t)$ is ultrabounded.

Declaration of competing interest

The authors do not have financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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Appendix A. A Lasry–Lions type approximations

In this appendix, we recall some properties of the Lasry-Lions type approximants along H_R (see [12,13]). The construction of such approximants is a modification of the original idea presented in [29]. For $\varepsilon > 0$ and $f \in \mathrm{BUC}(H)$ (the Banach space of real valued, bounded and uniformly continuous function, endowed with the usual norm), we consider the functions defined as follows:

$$f_\varepsilon(x) := \sup_{h\in H_R}\left\{\inf_{k\in H_R}\left\{f(x+k-h)+\frac{1}{2\varepsilon}\|k\|_R^2\right\}-\frac{1}{\varepsilon}\|h\|_R^2\right\}, \qquad x\in H. \tag{A.1}$$

In the following proposition, we summarize some basic properties of the approximants defined in (A.1) that will be useful in the paper. Some of the results we will present are already contained in [12] and [13]. However, we provide a proof of them to make the paper self-contained.

Proposition 7.2. *Assume Hypothesis 2.1(ii) and let $f \in \mathrm{Lip}_{H_R}(H)$. Let $\{f_\varepsilon\}_{\varepsilon\geq 0}$ be the family of functions introduced in (A.1). For every $\varepsilon > 0$ and $x \in H$ it holds*

$$\|f_\varepsilon\|_\infty \leq \|f\|_\infty; \tag{A.2}$$

$$0 \leq f(x) - f_\varepsilon(x) \leq 4\varepsilon(\mathrm{Lip}_R(f))^2; \tag{A.3}$$

$$\|\nabla_R f_\varepsilon\|_\infty \leq 4\sqrt{2}\mathrm{Lip}_R(f). \tag{A.4}$$

Proof. We start by proving (A.2).

$$\begin{aligned} f_\varepsilon(x) &= \sup_{h\in H_R}\left\{\inf_{k\in H_R}\left\{f(x+h-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\}-\frac{1}{\varepsilon}\|h\|_R^2\right\} \\ &\leq \sup_{h\in H_R}\left\{f(x)+\frac{1}{2\varepsilon}\|h\|_R^2-\frac{1}{\varepsilon}\|h\|_R^2\right\} \leq f(x) \leq \|f\|_\infty. \end{aligned} \tag{A.5}$$

In a similar way

$$f_\varepsilon(x) = \sup_{h\in H_R}\left\{\inf_{k\in H_R}\left\{f(x+h-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\}-\frac{1}{\varepsilon}\|h\|_R^2\right\}$$
$$\geq \inf_{k\in H_R}\left\{f(x-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\} \geq -\|f\|_\infty. \tag{A.6}$$

By (A.5) and (A.6) we get (A.2).

Now we prove (A.3). By (A.1), for every $\eta > 0$ there exists $k_\eta \in H_R$ such that

$$0 \leq f(x) - f_\varepsilon(x) \leq f(x) - \inf_{k\in H_R}\left\{f(x-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\}$$
$$\leq f(x) - f(x-k_\eta) - \frac{1}{2\varepsilon}\|k_\eta\|_R^2 + \eta$$
$$\leq \mathrm{Lip}_R(f)\|k_\eta\|_R - \frac{1}{2\varepsilon}\|k_\eta\|_R^2 + \eta. \tag{A.7}$$

By (A.7) we get the estimate

$$\|k_\eta\|_R^2 \leq 2\varepsilon \mathrm{Lip}_R(f)\|k_\eta\|_R + 2\varepsilon\eta.$$

Moreover, the Young inequality yields

$$\|k_\eta\|_R^2 \leq 4\varepsilon^2(\mathrm{Lip}_R(f))^2 + 4\varepsilon\eta. \tag{A.8}$$

Combining (A.7) and (A.8) we obtain

$$0 \leq f(x) - f_\varepsilon(x) \leq \mathrm{Lip}_R(f)(4\varepsilon^2(\mathrm{Lip}_R(f))^2 + 4\varepsilon\eta)^{1/2} + 2\varepsilon(\mathrm{Lip}_R(f))^2 + 3\eta.$$

Since the above estimate holds for every $\eta > 0$, by choosing η arbitrarily small, we get (A.3).

Let us now prove (A.4). The differentiability along H_R of f_ε is classical and can be found in [12]. By (A.1) for every $\sigma > 0$ there exists $h_\sigma \in H_R$ such that

$$f_\varepsilon(x) \leq \inf_{k\in H_R}\left\{f(x+h_\sigma-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\} - \frac{1}{\varepsilon}\|h_\sigma\|_R^2 + \sigma.$$

A straightforward calculation gives

$$\frac{1}{\varepsilon}\|h_\sigma\|_R^2 \leq f(x) - f_\varepsilon(x) + \sigma + \frac{1}{2\varepsilon}\|h_\sigma\|_R^2.$$

Thus from (A.3) we obtain

$$\|h_\sigma\|_R^2 \leq 8\varepsilon^2(\mathrm{Lip}_R(f))^2 + 2\varepsilon\sigma. \tag{A.9}$$

By (A.9) we get

$$
\begin{aligned}
f_\varepsilon(x+h)-f_\varepsilon(x) &\le \inf_{k\in H_R}\left\{f(x+h+h_\sigma-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\}-\frac{1}{\varepsilon}\|h_\sigma\|_R^2+\sigma\\
&\quad-\inf_{k\in H_R}\left\{f(x+h+h_\sigma-k)+\frac{1}{2\varepsilon}\|k\|_R^2\right\}+\frac{1}{\varepsilon}\|h+h_\sigma\|_R^2\\
&=\frac{1}{\varepsilon}\|h+h_\sigma\|_R^2-\frac{1}{\varepsilon}\|h_\sigma\|_R^2+\sigma=\frac{1}{\varepsilon}\|h\|_R^2+\frac{2}{\varepsilon}\langle h,h_\sigma\rangle_R+\sigma\\
&\le\frac{1}{\varepsilon}\|h\|_R^2+\frac{2}{\varepsilon}\|h\|_R(8\varepsilon^2(\mathrm{Lip}_R(f))^2+2\varepsilon\sigma)^{1/2}+\sigma.
\end{aligned}
$$

Since the above inequality holds for every $\sigma>0$, taking the infimum we get

$$
f_\varepsilon(x+h)-f_\varepsilon(x)\le\frac{1}{\varepsilon}\|h\|_R^2+4\sqrt{2}\|h\|_R\mathrm{Lip}_R(f). \tag{A.10}
$$

Arguing similarly, we deduce the same inequality for $f_\varepsilon(x)-f_\varepsilon(x+h)$. In this way estimate (A.10) holds true for $|f_\varepsilon(x+h)-f_\varepsilon(x)|$. Thus, dividing by $\|h\|_R$ and letting $\|h\|_R\to 0$ we complete the proof. □

Data availability

No data was used for the research described in the article.

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