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The Quality of Content Publishing in the Digital Era

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Abstract. We propose and analyse a game describing the interactions between readers and publishers, with the aim of understanding to what extent the strategic behaviour of the latter may influence the quality of content publishing in the World Wide Web. For games with identical publishers, we provide a wide characterization of the cases in which pure Nash equilibria are guaranteed to exist, which mainly depends on the number of publishers and, subordinately, on some of the parameters we use to model their writing abilities. Then, for any game possessing pure Nash equilibria, we show that the price of anarchy is at most 2, even in presence of heterogeneous publishers. Finally, we provide better and tight bounds for some special cases of games with identical publishers.

1 Introduction

The digital revolution has dramatically changed the way in which publishing is approached. The fact that digital contents can be created and made available worldwide at an almost negligible cost grants an opportunity to an uncountable number of potential publishers. At the same time, huge profits can be raised, for instance through the selling of advertising slots, once a sufficiently high popularity is reached. To show some numbers, according to the stats publicised by *WordPress*, which refer to blogs hosted on *WordPress.com* only, over 409 million people access more than 20 billion pages each month, during which about 70 million new posts are created [1]. Moreover, a report published by *ConvertKit* on the state of the blogging industry in 2017 [9], states that professional bloggers reach an average yearly profit of $\$138,046$. This makes online publishing a huge market, where strategic and economic behavior comes naturally into play.

Within the range of all possible informative topics, there are usually some which, at a certain time, happen to be very popular and attract the interest of lots of readers. Thus, in order to boost their popularity, publishers may be tempted to discuss "hot topics", or simply topics on which less competition with other publishers is expected, even when they might not have the necessary competence to deal with them, at the risk of releasing low quality contents. The objective of this work is to propose and analyse a game, that we call the *content publishing game*, describing the interactions between readers and publishers, with the aim of understanding to what extent the strategic behaviour of the latter may influence the quality of content publishing in the World Wide Web.

We represent the set of possible topics by the $[0, 1]$ interval. Although this may seem a rough approximation, it is a common choice in the literature (see, for instance, [5]). We assume that there is a set of players (the publishers) who want to publish a content. Each player is associated with a point in $[0, 1]$ (her *topic of expertise*) representing her competence. When a player produces a content on her topic of expertise, she reaches a top quality, which is normalized to 1 for every player. As soon as the player departs from her topic of expertise, the quality of the produced content starts decreasing according to the distance from the topic of expertise scaled by a factor $\alpha \in [0, 1]$, which we assume to be common to all players. In general, the set of readers may be generated according to a probability distribution in $[0, 1]$, so that every reader is associated with a point modelling her *topic of interest*. In this work, we shall restrict our attention to the uniform distribution only. The *satisfaction* of a reader, when accessing a content, is defined as the product between its quality and the distance of the content topic from her topic of interest. Each reader will select the content maximizing her satisfaction. This function aims at balancing the wish of a reader for contents that are both interesting to her and of good quality. On the other hand, publishers aim at attracting as many readers as possible.

Beyond Hotelling: a New Model of Competitive Facility Location. Content publishing games can also be interpreted as a natural and interesting generalization of the classic *Hotelling model* for competitive facility location. Hotelling games [15], in fact, are equivalent to content publishing games with $\alpha = 0$, i.e., games in which the content quality does not affect the satisfaction of the readers. These games suitably model scenarios in which sellers/players produce different types of goods and strategically focus on those attracting as many customers as possible: the interaction between sellers and customers generates a market in which sellers compete with each other to maximize their profit. In general, a seller may be more specialised in producing certain types of goods and less in other ones. Therefore, in highly competitive and globalized markets, the quality of the produced goods cannot be neglected when trying to model marketing strategic behaviour. To accommodate this issue, we introduce a *quality factor* for each type of produced goods that generally depends on the sellers' abilities.

We point out that the content publishing model also fits with some scenarios of party competition in politics, widely studied in [2]. Indeed, each player can be seen as a political candidate, and her topic of interest can model a political ideology, ranging from the far-left (associated to 0) to the far-right (associated to 1). Instead of readers, we have voters whose political ideology can range from the far-left to the far-right (as for candidates), and each voter chooses the candidate maximizing her satisfaction, according to her political ideology. Each player/candidate can strategically choose a political party whose ideology is slightly different from her personal one, with the aim of maximizing the number of people voting her, and this turns into a strategic behavior similar as that of the content publishing game.

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1.1 Our Contribution

Recall that, for $\alpha = 0$, the content publishing game boils down to the classic Hotelling model [15] for competitive facility location, whose characterization required a considerable research effort [14, 11, 12]. Hence, it is legitimate to expect the achievement of a complete characterization of the properties of the content publishing game to be quite a challenging task. In this work, as a first step towards the understanding of this game, we mainly focus on the case of identical publishers, all sharing the same topic of expertise p. Our results, summarized in the following, show that even this basic version exhibits a rich variety of complex situations.

Existence of Pure Nash Equilibria. We start by considering the problem of determining the existence, and eventually uniqueness, of pure Nash equilibria in content publishing games.

For identical publishers, we provide a wide characterization of the cases in which pure Nash equilibria are guaranteed to exist, which mainly depends on the number of publishers n and, subordinately, on both the scaling factor α and the topic of expertise p. Our analysis exploits the following fundamental property that holds if and only if publishers are identical: in any strategy profile, the set of readers who prefer the content released by a given publisher forms a (possibly empty) subinterval of $[0, 1]$ (Lemma 1). This property allows us to obtain a tight characterization of the utility that each publisher gets in a given strategy profile (Lemma 4). By leveraging on this characterization, we derive a set of necessary conditions that must be satisfied by a strategy profile in order to be a pure Nash equilibrium (Theorem 1). This theorem is then exploited to obtain both positive and negative existential results.

It turns out that, for $n = 2$, there exists a unique pure Nash equilibrium, with both players paired on a same topic depending on both α and p (Theorem 2), while, for $n = 3$, a pure Nash equilibrium never exists (Theorem 3). Both these results hold independently of the values of α and p. However, as the number of publishers increases, dependence from these two parameters comes into play. For $n = 4$, in fact, pure Nash equilibria are guaranteed to exist only if one of the following five cases holds: *(i)* $\alpha = 0$, *(ii)* $\alpha \in [4/5, 1]$ and $p \in [0, 5/4 - 1/\alpha]$, *(iii)* $\alpha \in [4/5, 1]$ and $p = 3 - 1/\alpha - \sqrt{145/16 - 6/\alpha}$, *(iv)* $\alpha = 1$ and $p \in [1/4, 1/2]$, *(v)* $p = 1/2$. In case of existence, the Nash equilibrium is unique (up to a permutation of the players) and we give an exact formula for its computation (Theorem 4). For $n \geq 5$, existence of pure Nash equilibria is always guaranteed in the two extremal cases of $\alpha \in \{0, 1\}$, with the set of equilibria forming an $(n - 5)$-dimensional polytope, independently of the value of p (Theorems 5 [12] and 6). On the negative side, we show that there exists an $\alpha^* \in (0, 1)$ such that no game with $n \geq 5$ players admits a pure Nash equilibrium when $0 < \alpha < \alpha^*$ (Theorem 7), and that, for any $\alpha \in (0, 1)$, there exists a positive integer $n^* \geq 2$ such that no game with $n > n^*$ players admits a pure Nash equilibrium (Theorem 8). This last result, in particular, shows that $\alpha = 0$ and $\alpha = 1$ are the only two cases for which the existence of pure Nash equilibria can be guaranteed in *large games* where the number of players goes to infinity.

We point out that the structure of equilibria in content publishing games exhibits some similarities with that of Hotelling games, but there are some substantial differences. In Hotelling games, pure Nash equilibria do not exist only for 3 players, and are completely described by simple linear inequalities. In content publishing games, instead, they may not exist even for $n \geq 4$ players; furthermore, except for the cases $\alpha = 1$ or $n = 2$, the set of pure Nash equilibria is generally determined by complex non-linear inequalities. Due to these difficulties, we resort to a qualitative study of the structure of equilibria to characterize some of their properties, such as existence and uniqueness.

Efficiency of Pure Nash Equilibria. We also focus on the problem of evaluating the impact of strategic behaviour on the overall quality of the published contents.

For each content publishing game admitting pure Nash equilibria, we provide suitable bounds on the price of anarchy [17]. The social function we adopt to measure the quality of a strategy profile is the sum of the qualities (or, equivalently, the average quality) of the contents released by the publishers. This allows us to quantify to what extent the publishers' strategic behavior impacts on the quality of the documents populating the World Wide Web.

For games with identical publishers, we show the following tight bounds. For $n = 2$ and $n = 4$, the price of anarchy is $3/2$ and $10/7$, respectively (Theorems 9 and 10). For $\alpha = 0$, as there is no degradation in the quality of any published content, the price of anarchy is trivially equal to 1 for any game. For $\alpha = 1$, we show that it is equal to $\sqrt{2}$ when the number of publishers tends to infinity (Theorems 11 (upper bound) and 12 (lower bound)). Due to Theorem 8, this is enough to characterize the price of anarchy of large games.

More generally, for any game admitting pure Nash equilibria, we show that the price of anarchy is at most 2 even in presence of heterogeneous players (Theorem 14). This result is obtained by determining a relaxed, but significant characterization of the properties fulfilled by any pure Nash equilibrium for a content publishing game with general players (Theorem 13).

Due to the lack of space, some proofs are omitted and left to the full version of this paper.

1.2 Related Work

The Hotelling model for competitive facility location, which coincides with the content publishing game when $\alpha = 0$, has been introduced in the seminal paper [15] and further investigated and extended in [7, 14, 11, 12, 16, 21]. It admits a unique pure Nash equilibrium for $n = 2, 4, 5$, no equilibria for $n = 3$, and infinitely many ones for $n \geq 6$, see [12]. A very similar model is that of Voroni games, which have been addressed in [3, 10, 18]. Models for competitive facility location have been also studied in the political context of party competition [2].

The content publishing game models the interactions between publishers and readers in a direct way. A different approach, which has been explored in the literature, assumes the presence of a mediator, such as a search engine or a recommendation system, between them. Several models about

Games with strategic publishers under the mediation of a search engine fall within the field of *search engine optimization* [13]. In these models, strategic modifications are performed by the authors of a web content with the aim of improving the position the content will occupy in the rankings generated by the search engine in response to a sequence of queries. In [4], the characterization of the price of anarchy of a game in which the search engine adopts the *probability ranking principle* [20] is presented, while in [6], it is shown that any learning dynamics converges to a pure Nash equilibrium.

Games with strategic publishers under the mediation of a recommendation system have been recently considered in [5, 8] under a mechanism design approach [19]. In [5], a recommendation system is designed which, from the one hand, minimizes a metric called the *intervention cost*, and, from the other hand, induces a game whose unique pure Nash equilibrium minimizes the social cost. In [8], instead, a recommendation system based on the Shapley value is designed so as to ensure fairness, stability, economic efficiency and fast implementation.

Content publishing games exhibit some similarities with the framework of *valid utility games*, introduced in [22]. The price of anarchy of these games is known to be upper bounded by 2. However, content publishing games are not valid utility games and the results achieved in [22], so as similar proof arguments, do not apply to re-obtain the upper bound on the price of anarchy we achieve in Theorem 14.

2 Model

A content publishing game $G = (n, (p_i)_{i\in[n]}, \alpha)$ is defined by a finite set $[n] = \{1, \ldots, n\}$ of n players, each having an expertise $p_i \in [0, 1]$ representing the topic she is specialized in, and a parameter $\alpha \in [0, 1]$. If player i publishes a content on topic $x \in [0, 1]$, she achieves a quality of $q_i(x) = 1 - \alpha|x - p_i|$. There are infinitely many users, one for each point in $[0, 1]$.[4] A user located at point x (i.e., interested on topic x), when accessing a content on topic t having quality q, obtains a satisfaction equal to $s_x(t, q) = q\cdot(1-|x-t|)$. From now on, we shall refer to the user located at point x as to user x. Given a strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, where each player $i \in [n]$ publishes a content on topic σ_i, and a user x, define $P_x(\boldsymbol{\sigma}) = \text{argmax}_{i\in[n]} \{s_x(\sigma_i, q_i(\sigma_i))\}$ as the set of players publishing a content maximizing the satisfaction of x. Each user x chooses the content maximizing her satisfaction (breaking ties uniformly at random), that is, the content published by any player belonging to $P_x(\boldsymbol{\sigma})$. With a little abuse of notation, we denote by $s_x(\boldsymbol{\sigma}) = s_x(\sigma_i, q_i(\sigma_i))$, with $i \in P_x(\boldsymbol{\sigma})$, the satisfaction of user x in $\boldsymbol{\sigma}$. For each $i \in [n]$, denote by $X_i(\boldsymbol{\sigma})$ the set of users who can potentially choose the content published by player i in $\boldsymbol{\sigma}$, that is, $X_i(\boldsymbol{\sigma}) = \{x \in [0, 1] : i \in P_x(\boldsymbol{\sigma})\}$ and let $\mu_i(\boldsymbol{\sigma}, x)$ be the probability that x chooses the content published by i, that is,

$$\mu_i(\boldsymbol{\sigma}, x) = \begin{cases} |P_x(\boldsymbol{\sigma})|^{-1} & \text{if } x \in X_i(\boldsymbol{\sigma}), \\ 0 & \text{otherwise.} \end{cases}$$

The utility of player i in $\boldsymbol{\sigma}$ is defined as the fraction of users accessing her content, i.e.,

$$u_i(\boldsymbol{\sigma}) = \int_0^1 \mu_i(\boldsymbol{\sigma}, x)dx$$

and each player aims at maximizing it.

A strategy profile $\boldsymbol{\sigma}$ is a pure Nash equilibrium if, for each $i \in [n]$ and for each $t \in [0, 1]$, we have $u_i(\boldsymbol{\sigma}) \geq u_i(\boldsymbol{\sigma}_{-i}, t)$. Denote by $\mathsf{NE}(G)$ the set of pure Nash equilibria of game G.

To measure the quality of a strategy profile, we consider the overall (or, equivalently, the average) quality of the published contents, defined as $Q(\boldsymbol{\sigma}) = \sum_{i\in[n]} q_i(\sigma_i)$. The price of anarchy of G is defined as $\mathsf{PoA}(G) = \max_{\boldsymbol{\sigma}\in\mathsf{NE}(G)} \{n/Q(\boldsymbol{\sigma})\}$, that is, as the worst-case ratio between the maximum possible overall quality, which is equal to n, and the overall quality of any pure Nash equilibrium.

3 Existence of Equilibria for Identical Players

In this section, we consider games with identical players, that is, games where all publishers have the same topic of expertise p, which yields the same quality function $q(x) = 1-\alpha|x-p|$, and we denote a content publishing game with identical players as $G = (n, p, \alpha)$. We shall assume, without loss of generality, that $p \in [0, 1/2]$ (indeed, the case $p \in [1/2, 1]$ can be treated in a symmetric way). Moreover, as players are identical, every strategy profile is defined up to an ordering of the players. Hence, we also assume that, in every strategy profile $\boldsymbol{\sigma}$, we have $\sigma_i \leq \sigma_j$ for any $i < j$.

Given a user x and two topics y and z, with $y \leq z$, define $\delta_x(y, z) = s_x(z, q(z)) - s_x(y, q(y))$. We say that z *dominates* y (resp. y *dominates* z) if $\delta_x(y, z) > 0$ (resp. $\delta_x(y, z) < 0$) for any $x \in [0, 1]$. Moreover, we say that a user x^* is a *separator* for y and z, with $y < z$, if $\delta_x(y, z) < 0$ for any user $x < x^*$ (users in $[0, x^*)$ strictly prefer a content on topic y), $\delta_x(y, z) > 0$ for any $x > x^*$ (users in $(x^*, 1]$ strictly prefer a content on topic z), and $\delta_{x^*}(y, z) = 0$ (user x^* is indifferent between the two topics). Let $sep(y, z)$ be the function which, given two topics y and z with $y < z$, returns the separator for y and z whenever it exists. For the sake of simplicity and readability, we define $sep(y, z) = 0$ (resp. $sep(y, z) = 1$) whenever z dominates y (resp. y dominates z). This technicality does not alter the definition of the game, as the utility of a player adopting a dominated strategy remains zero (the contribution of a single point in $[0, 1]$ to the utility of a player is clearly equal to zero), but guarantees that function sep is defined for any two topics y and z, with $y < z$, as shown in the following lemma, where an explicit formula defining function $sep(y, z)$ is also provided.

Lemma 1. *Fix a game $G = (n, p, \alpha)$ and two topics y and z, with $y < z$. The following claims hold, with the interpretation that $1/\alpha = \infty$ when $\alpha = 0$.*

(a) *Assume $p < y$. Then, $sep(y, z) = \min\{1, y+z-p+1-1/\alpha\}$ if $y \geq p-1+1/\alpha$, and $sep(y, z) = \frac{\alpha(y^2+z^2+y-z)-(y+z)(\alpha p+1)}{\alpha(z+y-2p)-2}$ if $y \in (p, p-1+1/\alpha)$.*

(b) *Assume $z < p$. Then, $sep(y, z) = \max\{0, y+z-p-1+1/\alpha\}$ if $z \leq p+1-1/\alpha$, and $sep(y, z) = \frac{\alpha(z^2+y^2+y-z)-(y+z)(\alpha p-1)}{\alpha(z+y-2p)+2}$ if $z \in (p+1-1/\alpha, p)$.*

(c) *Assume $y \leq p \leq z$. Then, $sep(y, z) = \frac{\alpha(y^2-z^2-y(p-1)+z(p+1)-2p)+y+z}{\alpha(y-z)+2}$.*

Sketch of the Proof. Within this proof, in order to simplify our analysis, we assume that the definition of function $\delta_x(y, z)$ is extended to the whole set of real numbers.

(a): By the assumption $p < y < z$, it follows that $\delta_x(y, z) < 0$ for each $x \leq y$. To characterize the preference of a user $x > y$, we show the following inequality:

$$\frac{\partial}{\partial x}\delta_x(y, z) > 0, \quad \forall x \geq y. \tag{1}$$

To show (1), we distinguish between two cases:

- if $x \leq z$, we have that $\frac{\partial}{\partial x}\delta_x(y, z) = -\alpha(z + y) + 2\alpha p + 2 \geq 2 - z - y > 0$;
- if $x > z$, we have that $\frac{\partial}{\partial x}\delta_x(y, z) = \alpha(z - y) > 0$.

Thus (1) is true. Because of (1), we have that function $\delta_x(y, z)$ is increasing in $[y, \infty)$ and, since $\delta_y(y, z) < 0$ and $\lim_{z\to\infty} \delta_z(y, z) > 0$, it follows that function $\delta_x(y, z)$ admits exactly one zero $x^* \in [y, \infty)$. Thus, as $\delta_x(y, z) < 0$ for each $x < x^*$ and $\delta_x(y, z) > 0$ for each $x > x^*$, we have that $sep(x, y) = \min\{x^*, 1\}$. The formula of part **(a)** comes by computing x^* in the two cases of $x^* \in [y, z]$ and $x^* \in [z, \infty)$, respectively, and noting that the first case happens

[4] This is equivalent to assuming that there is a finite set of users sampled uniformly at random in the interval $[0, 1]$.

when $y \leq p-1+1/\alpha$ and the second one happens when $y \geq p-1+1/\alpha$ (and in this last case $x^* \leq 1$).
(b): The proof uses similar arguments to case **(a)**.
(c): First assume that $p-y \leq z-p$. By the assumption $y \leq p \leq z$ and $p-y \leq z-p$, it follows that $\delta_x(y,z) < 0$ for each $x < p$. At the same time, denoted $\epsilon_1 := p-y$ and $\epsilon_2 := z-p$, we have $\delta_1(y,z) = -\alpha z(z-p)+\alpha y(p-y)+z-y = -\alpha z\epsilon_2 + \alpha y\epsilon_1 + \epsilon_1 + \epsilon_2 \geq 0$, as $\alpha z \leq 1$. Thus, there exists $x^* \in [p,1]$ such that $\delta_{x^*}(y,z) = 0$ and $\delta_x(y,z) > 0$ for each $x > x^*$, i.e., $x^* \in [p,1]$ is a separator for y and z. The formula of part **(c)** is obtained by computing the unique zero of function $\delta_x(y,z)$ in the cases in which $x^* \in [p,z]$ and $x^* \in [z,1]$, respectively. Observe that $(z-y)(p-y)+y+z-2p = (\epsilon_1+\epsilon_2)\epsilon_1 - \epsilon_1 + \epsilon_2 > 0$ as $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, $\epsilon_1 \leq \epsilon_2$ and $\epsilon_1+\epsilon_2 > 0$, so that $\epsilon_1 = 0 \implies \epsilon_2 > \epsilon_1$. Thus, the former case happens when $\alpha \leq \frac{z-y}{(z-y)(p-y)+y+z-2p}$ and the latter case happens when $\alpha \geq \frac{z-y}{(z-y)(p-y)+y+z-2p}$. As we have $\frac{z-y}{(z-y)(p-y)+y+z-2p} = \frac{\epsilon_1+\epsilon_2}{(\epsilon_1+\epsilon_2)\epsilon_1-\epsilon_1+\epsilon_2} \geq 1$, the case of $x^* \in [p,z]$ which requires $\alpha \leq 1$ always happens. The case of $p-y \geq z-p$ can be shown by using similar arguments. □

Observe that, from the above lemma, it follows that y can dominate z only when $p < y$ and $p \geq p-1+1/\alpha$, while z can dominate y only when $z < p$ and $z \leq p+1-1/\alpha$.

Next two lemmas give properties of function sep that we shall widely exploit in the following.

Lemma 2. *Fix a game $G = (n,p,\alpha)$ and a topic y. Then,*

$$\lim_{\epsilon \to 0^+} sep(y-\epsilon, y) = \lim_{\epsilon \to 0^+} sep(y, y+\epsilon)$$
$$= \begin{cases} y & \text{if } y \in [p+1-1/\alpha, p-1+1/\alpha], \\ 2y-p+1-1/\alpha & \text{if } y \geq p-1+1/\alpha, \\ 2y-p-1+1/\alpha & \text{if } y \leq p+1-1/\alpha. \end{cases}$$

By the previous lemma, given $y \in [0,1]$, we set $sep(y,y) := \lim_{\epsilon \to 0^+} sep(y-\epsilon, y)$. Let $\mathcal{T} = \{(y,z) : 0 \leq y < z \leq 1 \text{ and } sep(y,z) \in (0,1)\}$.

Lemma 3. *Function sep is continuous and non-decreasing in both of its arguments. Furthermore, restricted to $\mathcal{T}$, it is increasing in both of its arguments.*

Fix a strategy profile $\boldsymbol{\sigma}$. Given an index $i \in [n]$, let $l(i)$ be the index of the first player at the left of i adopting a different strategy, with $l(i) = 0$ if $\sigma_1 = \sigma_i$, and $r(i)$ be the index of the first player at the right of i adopting a different strategy, with $r(i) = n+1$ if $\sigma_n = \sigma_i$. We define $first$ as the maximum index $i \in [n-1]$ such that σ_i is not dominated by $\sigma_{r(i)}$ and $last$ as the minimum index $i \in [n] \setminus \{1\}$ such that σ_i is not dominated by $\sigma_{l(i)}$. Clearly, $first \leq last$, as $last < first$ implies that either σ_{first} dominates σ_{last} and σ_{last} dominates σ_{first}: a contradiction. For the sake of conciseness, we set $first-1 := 0$, $last+1 := n+1$, and $sep(\sigma_0, x) := 0$, $sep(x, \sigma_{n+1}) := 1$ for each $x \in [0,1]$. Next lemma characterizes the set of users choosing the content published by every player in $\boldsymbol{\sigma}$.

Lemma 4. *Fix a strategy profile $\boldsymbol{\sigma}$ and a player $i \in [n]$. We have $X_i(\boldsymbol{\sigma}) = \emptyset$ if $i \notin \{first,\dots,last\}$ and $X_i(\boldsymbol{\sigma}) = [sep(\sigma_{l(i)},\sigma_i), sep(\sigma_i,\sigma_{r(i)})]$ if $i \in \{first,\dots,last\}$.*

Proof. First, we show that $X_i(\boldsymbol{\sigma}) = \emptyset$ for each $i \notin \{first,\dots,last\}$. Towards this end, fix a player $i \notin \{first,\dots,last\}$. Assume $i < first$ which implies $first > 1$. By the definition of $first$, $\sigma_{r(i)}$ dominates σ_i which implies $X_i(\boldsymbol{\sigma}) = \emptyset$. The case of $i > last$ can be treated with a symmetric argument.

Now fix a player $i \in \{first,\dots,last\}$. By Lemma 3, function sep is non-decreasing, so it follows that interval $[sep(\sigma_{l(i)},\sigma_i), sep(\sigma_i,\sigma_{r(i)})]$ is well defined. Thus, as we have $sep(\sigma_{first-1},\sigma_{first}) := 0$ and $sep(\sigma_{last},\sigma_{last+1}) := 1$, we get that the multi-set of intervals $\{X_{first}(\boldsymbol{\sigma}),\dots,X_{last}(\boldsymbol{\sigma})\}$ realizes a partition of the $[0,1]$ interval. By the definition of separator, we have $X_i(\boldsymbol{\sigma}) \subseteq [sep(\sigma_{l(i)},\sigma_i), sep(\sigma_i,\sigma_{r(i)})]$. However, since $\{X_{first}(\boldsymbol{\sigma}),\dots,X_{last}(\boldsymbol{\sigma})\}$ realizes a partition of the $[0,1]$ interval, it must necessarily be $X_i(\boldsymbol{\sigma}) = [sep(\sigma_{l(i)},\sigma_i), sep(\sigma_i,\sigma_{r(i)})]$. In fact, if there exists a player $i \in \{first,\dots,last\}$ such that $X_i(\boldsymbol{\sigma}) \subset [sep(\sigma_{l(i)},\sigma_i), sep(\sigma_i,\sigma_{r(i)})]$, then there must be a player $j \in \{first,\dots,last\}$ such that $X_j(\boldsymbol{\sigma}) \supset [sep(\sigma_{l(j)},\sigma_j), sep(\sigma_j,\sigma_{r(j)})]$, which rises a contradiction. □

For an interval $I \subseteq [0,1]$ (either open or closed) of left extreme a and right extreme b, denote by $|I| = b-a$ the length of I. Given a strategy profile $\boldsymbol{\sigma}$ and a player $i \in \{first,\dots,last\}$, set $\widetilde{X}_i(\boldsymbol{\sigma}) := [sep(\sigma_{l(i)},\sigma_i), sep(\sigma_i,\sigma_i)]$ the closed interval whose left extreme coincides with the left extreme of $X_i(\boldsymbol{\sigma})$ and its right extreme is equal to $\lim_{\epsilon \to 0^+} sep(\sigma_i-\epsilon,\sigma_i)$. The following theorem gives necessary conditions that each pure Nash equilibrium must satisfy (the proof is principally based on the previous lemmas, and due to the lack of space, is omitted).

Theorem 1 (Equilibrium conditions for identical players)**.** *Fix a game $G = (n,p,\alpha)$ and let $\boldsymbol{\sigma}$ be a pure Nash equilibrium for G. The following properties hold:*

(i) *$|X_i(\boldsymbol{\sigma})| > 0$ for each $i \in [n]$, i.e., $first = 1$ and $last = n$,*
(ii) *$\sigma_1 = \sigma_2$ and $\sigma_{n-1} = \sigma_n$,*
(iii) *if $\sigma_i = \sigma_{i+1}$, then $|\widetilde{X}_i(\boldsymbol{\sigma})| = |X_i(\boldsymbol{\sigma}) \setminus \widetilde{X}_i(\boldsymbol{\sigma})| = \frac{1}{2}|X_i(\boldsymbol{\sigma})|$,*
(iv) *there are at most two players selecting the same strategy,*
(v) *if $\sigma_i = \sigma_{i+1}$ and $\sigma_j = \sigma_{j+1}$ for some $j > i+1$, then $|X_i(\boldsymbol{\sigma})| = |X_j(\boldsymbol{\sigma})|$,*
(vi) *$\sigma_1 > \max\left\{0, \frac{p}{2}+\frac{1}{2}-\frac{1}{2\alpha}\right\}$ and $\sigma_n < \min\left\{1, \frac{p}{2}+\frac{1}{2\alpha}\right\}$.*

Furthermore, $\boldsymbol{\sigma}$ is a pure Nash equilibrium if and only if

(vii) *$u_i(\boldsymbol{\sigma}) = |[sep(\sigma_{i-1},\sigma_i), sep(\sigma_i,\sigma_{i+1})]|$ and $|[sep(\sigma_{i-1},\sigma_i), sep(\sigma_i,\sigma_{i+1})]| \geq |[sep(\sigma_j,x), sep(x,\sigma_{j+1})]|$ for any $i \in [n]$, $j \in [n] \cup \{0\} \setminus \{i, i-1\}$, $x \in [\sigma_j,\sigma_{j+1}]$, and $|[sep(\sigma_{i-1},\sigma_i), sep(\sigma_i,\sigma_{i+1})]| \geq |[sep(\sigma_{i-1},x), sep(x,\sigma_{i+1})]|$ for any $x \in [\sigma_{i-1},\sigma_{i+1}]$.*

3.1 The Case $n \leq 4$

In this subsection, we address games with at most four players. For two players, there exists a unique pure Nash equilibrium.

Theorem 2 (Equilibria for two players)**.** *For any game $G = (2,p,\alpha)$, there exists a unique pure Nash equilibrium $\boldsymbol{\sigma}$ such that:*

$$\sigma_1 = \sigma_2 = \begin{cases} \frac{p}{2}-\frac{1}{4}+\frac{1}{2\alpha} & \text{if } 0 \leq p \leq \frac{3}{2}-\frac{1}{\alpha} \text{ and } \frac{2}{3} \leq \alpha \leq 1, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Proof. Consider a game $G = (2,p,\alpha)$. If $\boldsymbol{\sigma}$ is a pure Nash equilibrium for G, by Theorem 1, we have that $\sigma_1 = \sigma_2$, and $1/2 = |[0,1]|/2 = |X_1(\boldsymbol{\sigma})|/2 = |\widetilde{X}_1(\boldsymbol{\sigma})| = sep(\sigma_1,\sigma_1)$. Thus, we have that $\boldsymbol{\sigma} = (y,y)$ for some $y \in [0,1]$ such that $1/2 = sep(y,y)$. Conversely, if a strategy profile $\boldsymbol{\sigma}$ of G verifies $\boldsymbol{\sigma} = (y,y)$ and $sep(y,y) = 1/2$, we get $1/2 = 1 -$

$sep(y, y) = |[sep(\sigma_1, \sigma_2), 1]|$ and $|[0, sep(\sigma_1, \sigma_2)]| = sep(y, y) = 1/2$, so that $|[sep(\sigma_1, \sigma_2), 1]| = |[0, sep(\sigma_1, \sigma_2)]|$. Observe that, since $|[sep(\sigma_1, x), 1]| \leq |[sep(\sigma_1, \sigma_2), 1]| = |[0, sep(\sigma_1, \sigma_2)]|$ for any $x \in [\sigma_1, 1]$, and $|[sep(\sigma_1, \sigma_2), 1]| = |[0, sep(\sigma_1, \sigma_2)]| \geq |[0, sep(x, \sigma_2)]|$ for any $x \in [0, \sigma_1]$, we have that the sufficient condition of claim **(vii)** from Theorem 1 holds, and so $\boldsymbol{\sigma}$ is a pure Nash equilibrium.

We conclude that, if the number of players is two, a strategy profile $\boldsymbol{\sigma}$ is a pure Nash equilibrium if and only if $\boldsymbol{\sigma} = (y, y)$ for some $y \in [0, 1]$ and $sep(y, y) = 1/2$. First of all, observe that $y \geq p$. Otherwise, if $y < p$, player 2 gets a utility that is higher than $1-p$ by deviating to strategy p. As $u_2(\sigma_1, \sigma_2) = 1/2 < u_2(\sigma_1, p)$, $\boldsymbol{\sigma}$ cannot be a pure Nash equilibrium. Furthermore, by exploiting the definition of $sep(y, y)$, we have that either **(a)** $1/2 = sep(y, y) = 2y-p+1-1/\alpha \geq y \geq p$, or **(b)** $1/2 = sep(y, y) = y \leq p-1+1/\alpha$. **(a)** holds if and only if inequalities $2y - p + 1 - \frac{1}{\alpha} = \frac{1}{2}$ and $0 \leq p \leq y \leq \frac{1}{2}$ are satisfied, which requires

$$y = \frac{p}{2} - \frac{1}{4} + \frac{1}{2\alpha}, \quad 0 \leq p \leq \frac{3}{2} - \frac{1}{\alpha}, \quad \frac{2}{3} \leq \alpha \leq 1. \tag{2}$$

We also have that **(b)** holds if and only if inequalities $y \leq p - 1 + \frac{1}{\alpha}$ and $0 \leq p \leq y = \frac{1}{2}$ are satisfied, which requires either:

$$y = \frac{1}{2}, \quad 0 \leq \alpha \leq \frac{2}{3}, \quad 0 \leq p \leq \frac{1}{2}, \text{ or} \tag{3}$$

$$y = \frac{1}{2}, \quad \frac{2}{3} \leq a \leq 1, \quad \frac{3}{2} - \frac{1}{\alpha} \leq p \leq \frac{1}{2}. \tag{4}$$

By considering the union of (2), (3) and (4), the claim follows. □

For three players, no game can ever admit a pure Nash equilibrium.

Theorem 3 (Equilibria for three players)**.** *For any game $G = (3, p, \alpha)$, there is no pure Nash equilibrium.*

Proof. Because of Theorem 1, claim **(ii)**, we have that, for any pure Nash equilibrium $\boldsymbol{\sigma}$, it must be $\sigma_1 = \sigma_2 = \sigma_3$. But this contradicts Theorem 1, claim **(iv)**. Thus, no strategy profile $\boldsymbol{\sigma}$ can be a pure Nash equilibrium. □

For four players, we give some necessary conditions on the existence of pure Nash equilibria depending on the mutual relationships between α and p. Only two situations may happen: either there are no equilibria, or there is a unique one (up to a permutation of the players).

Theorem 4 (Equilibria for four players)**.** *For any game $G = (4, p, \alpha)$, pure Nash equilibria exist only if one of the following cases holds:* **(i)** $\frac{4}{5} \leq \alpha \leq 1$ *and* $0 \leq p \leq \frac{5}{4} - \frac{1}{\alpha}$, **(ii)** $\frac{4}{5} \leq \alpha \leq 1$ *and* $p = -\sqrt{\frac{145}{16} - \frac{6}{\alpha}} + 3 - \frac{1}{\alpha}$, **(iii)** $\alpha = 1$ *and* $\frac{1}{4} \leq p \leq \frac{1}{2}$, **(iv)** $p = \frac{1}{2}$, **(v)** $\alpha = 0$. *Cases* **(i)**, **(iii)** *and* **(v)** *are also sufficient ones. Furthermore, pure Nash equilibria do not exist when $\alpha \in (0, 4/5)$ independently of the value of p. In all cases of existence, there is a unique pure Nash equilibrium $\boldsymbol{\sigma}$ defined as follows:*

(a) $\sigma_1 = \sigma_2 = \frac{p}{2} - \frac{3}{8} + \frac{1}{2\alpha}$ *and* $\sigma_3 = \sigma_4 = \frac{p}{2} - \frac{1}{8} + \frac{1}{2\alpha}$ *if* **(i)** *holds;*
(b) $\sigma_1 = \sigma_2 = \frac{1}{4}$ *and* $\sigma_3 = \sigma_4 = \frac{11}{8} - \sqrt{\frac{145}{64} - \frac{3}{2\alpha}}$ *if* **(ii)** *holds;*
(c) $\sigma_1 = \sigma_2 = \frac{p}{2} + \frac{1}{8}$ *and* $\sigma_3 = \sigma_4 = \frac{p}{2} + \frac{3}{8}$ *if* **(iii)** *holds;*
(d) $\sigma_1 = \sigma_2 = \frac{7}{8} - \frac{1}{2\alpha}$ *and* $\sigma_3 = \sigma_4 = \frac{1}{8} + \frac{1}{2\alpha}$ *if* **(iv)** *holds and* $\frac{4}{5} \leq \alpha < 1$*;*
(e) $\sigma_1 = \sigma_2 = \frac{1}{4}$ *and* $\sigma_3 = \sigma_4 = \frac{3}{4}$ *if* **(v)** *holds.*

Corollary 1 (Equilibria for four players and $\alpha = 1$)**.** *Any game $G = (4, p, 1)$ admits a unique pure Nash equilibrium.*

Proof. The claim follows by combining cases **(i)** and **(iii)** from Theorem 4. □

3.2 The Case $\alpha \in \{0, 1\}$ and $n \geq 5$

In this subsection, we consider the two extremal cases of $\alpha \in \{0, 1\}$ for which a similar characterization is possible. For $\alpha = 0$, the content publishing game boils down to the classical Hotelling game for which the following theorem is known.

Theorem 5 (Equilibria for $\alpha = 0$ (Fournier and Scarsini [12]))**.** *Given a game $G = (n, p, 0)$, with $n \geq 5$, the set of pure Nash equilibria of G forms an $(n-5)$-dimensional polytope of $\mathbb{R}^n$.*

For the other case of $\alpha = 1$, pure Nash equilibria are always guaranteed to exist as stated by the following theorem.

Theorem 6 (Equilibria for $\alpha = 1$)**.** *Given a game $G = (n, p, 1)$ with $n \geq 5$, the following facts hold:*

(i) *a strategy profile $\boldsymbol{\sigma}$ is a pure Nash equilibrium if and only if:*

$$\begin{cases} \sigma_{i+1} - \sigma_{i-1} \geq \sigma_j - \sigma_{j-1}, \\ \qquad \forall j \in [n-1] \setminus \{1, 2, i, i+1\}, \forall i \in [n-1] \setminus \{1\}, \\ 2\sigma_1 - p = \sigma_3 - \sigma_1 = \sigma_n - \sigma_{n-2} = 1 - 2\sigma_n + p, \\ \frac{p}{2} \leq \sigma_1 = \sigma_2 \leq \sigma_3 \leq \ldots \leq \sigma_{n-1} = \sigma_n \leq \frac{p+1}{2}; \end{cases} \tag{5}$$

(ii) *the set of pure Nash equilibria of G forms an $(n-5)$-dimensional polytope of $\mathbb{R}^n$.*

3.3 The Case $\alpha \in (0, 1)$ and $n \geq 5$

In this subsection, we consider the case of $\alpha \in (0, 1)$ and $n \geq 5$ for which we provide two negative results.

Theorem 7 (Equilibria for positive small values of α)**.** *There exists a sufficiently small α^*, with $0 < \alpha^* \leq 1/2$, such that, for any game $G = (n, p, \alpha)$ with $n \geq 5$ and $0 < \alpha < \alpha^*$, there is no pure Nash equilibrium.*

Proof. Assume by contradiction that, for any α^* such that $0 < \alpha^* \leq 1/2$, there exists a game $G = (n, p, \alpha)$ with $n \geq 5$ and $0 < \alpha < \alpha^*$ admitting a pure Nash equilibrium $\boldsymbol{\sigma}$. Assume that $\alpha^* \leq 1/2$. Because of Lemma 2 and since $\alpha^* \leq 1/2$, we have that $\lim_{\epsilon \to 0^+} sep(\sigma_i - \epsilon, \sigma_i) = \sigma_i$ for any player $i \in [n]$.

Now, by symmetry, we assume without loss of generality that $\sigma_{n-2} \geq p$, but with $p \in [0, 1]$. First of all, $p \neq 1$, otherwise we would have that $\sigma_{n-2} = \sigma_{n-1} = \sigma_n = 1$, that is, more than two players select the same strategy, thus contradicting Theorem 1. Let $x := \sigma_{n-2}$ and $z := \sigma_{n-1}$. Observe that, by Theorem 1, we get $z = \sigma_{n-1} = \sigma_n > \sigma_{n-2} = x$. Now we show that, if α^* is sufficiently small, there exists a topic y with $x < y < z$ such that $u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, y) > u_{n-1}(\boldsymbol{\sigma})$, contradicting that $\boldsymbol{\sigma}$ is a pure Nash equilibrium. By Lemma 4, we have that $u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, y) = sep(y, z) - sep(x, y)$. By using the characterization of $sep(y, z)$ and $sep(x, y)$ given in Lemma 1, we get that

$$\begin{aligned} \frac{\partial}{\partial y} u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, y) &= \frac{\partial}{\partial y}(sep(y, z) - sep(x, y)) \\ &= \frac{\alpha}{2}(-x + z - 2) \\ &+ \alpha^2 \frac{P(x, y, z, p, \alpha)}{(\alpha x + \alpha y - 2\alpha p - 2)^2(\alpha z + \alpha y - 2\alpha p - 2)^2}, \end{aligned} \tag{6}$$

where $P(x, y, z, p, \alpha)$ is a polynomial in its variables. Since $\alpha^* \leq 1/2$, $\alpha z + \alpha y - 2\alpha p - 2 = \alpha(z-p) + \alpha(y-p) - 2 \leq 1/2 + 1/2 - 2 = -1$ and, analogously $\alpha y + \alpha x - 2\alpha p - 2 \leq -1$, we have that

$$\alpha^2 \frac{P(x, y, z, p, \alpha)}{(\alpha x + \alpha y - 2\alpha p - 2)^2(\alpha z + \alpha y - 2\alpha p - 2)^2} \leq \alpha^2 P(x, y, z, p, \alpha) \leq \alpha^2 M \quad (7)$$

for some $M > 0$. Observe that such a value M exists since, as $P(\alpha, p, x, y, z)$ is a continuous function defined on a compact set, it admits a maximum. Observe that, by setting $\alpha^* := 1/(2M)$ and using (6) and (7), we get $\frac{\partial}{\partial y} u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, y) \leq \frac{\alpha}{2}(-x + z - 2) + \alpha^2 M \leq \frac{\alpha}{2}(-x + z - 2) + \alpha^2 M = \alpha\left(\frac{1}{2}(-x + z - 2) + \alpha M\right) \leq \alpha\left(-\frac{1}{2} + \alpha M\right) < \alpha\left(-\frac{1}{2} + \alpha^* M\right) < 0$. Hence, function $u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, y)$ is decreasing in $y \in (x, z)$, and we get $u_{n-1}(\boldsymbol{\sigma}) = sep(z, z) - sep(y, z) = \lim_{\epsilon \to 0^+} u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, z - \epsilon) < u_{n-1}(\boldsymbol{\sigma}_{-(n-1)}, y)$, for any $y \in (x, z)$. Thus, if player $n-1$ selects a strategy y with $x < y < z$, she improves her utility and that contradicts the hypothesis that $\boldsymbol{\sigma}$ is a pure Nash equilibrium. □

Theorem 8 (Equilibria for $\alpha \in (0, 1)$ and large values of n)**.** *For any $\alpha \in (0, 1)$, there exists an integer $n^* \geq 2$ such that, each game $G = (n, p, \alpha)$ with $n > n^*$ does not admit any pure Nash equilibrium.*

Proof. To show the theorem, we need some preliminary lemmas.

Lemma 5. *Given $\alpha \in [0, 1]$ and $\Delta > 0$, there exists a sufficiently large integer $n^* > 0$ such that, for any $p \in [0, 1/2]$, any pure Nash equilibrium $\boldsymbol{\sigma}$ of game $G = (n, p, \alpha)$ with $n > n^*$, and any interval $[a, b] \subseteq [p + 1 - 1/\alpha, p - 1 + 1/\alpha] \cap [0, 1]$ with $b - a \geq \Delta$, there exists a player $i \in [n]$ such that $\sigma_i \in [a, b]$.*

Lemma 6. *Given $\alpha \in (0, 1)$, $p \in [0, 1/2]$, and $w \in (p, \min\{1, p - 1 + 1/\alpha\})$, there exists $\Delta \in (0, \min\{1 - p, -1 + 1/\alpha\})$ such that $|[sep(x, w), sep(w, z)]| > |[sep(x, y), sep(y, z)]|$ for any x, y, z such that $p < x < w < y \leq z < p + \Delta < \min\{1, p - 1 + 1/\alpha\}$.*

Now, we proceed with the proof of the theorem. Let $\alpha \in (0, 1)$, and assume, by way of contradiction, that, for any $n^* > 0$, there exists $p \in [0, 1/2]$ such that, for some $n > n^*$, there exists a pure Nash equilibrium $\boldsymbol{\sigma}$ of game $G = (n, p, \alpha)$. Let $w \in (p, \min\{1, p - 1 + 1/\alpha\})$ be an arbitrary real number, and let $\Delta > 0$ be the real number defined as in Lemma 6, with respect to w. Observe that w is well-defined since, as $\alpha \in (0, 1)$, interval $(p, \min\{1, p - 1 + 1/\alpha\})$ is non-empty.

By Lemma 5, we have that there exists $n^* > 0$ such that, for any $n > n^*$, any $p \in [0, 1/2]$, and any pure Nash equilibrium $\boldsymbol{\sigma}$ of game $G = (n, p, \alpha)$, there exists a player i such that $p < \sigma_{i-1} < \sigma_i \leq \sigma_{i+1} \leq p + \Delta < \min\{1, p - 1 + 1/\alpha\}$. Indeed, by applying Lemma 5, one can show that there exist two sufficiently large integers n_1^*, n_2^* such that, for any pure Nash equilibrium of $G = (n, p, \alpha)$ with arbitrary $n > n_1^*$ and $p \in [0, 1/2]$, there exists at least a player choosing a strategy of $\boldsymbol{\sigma}$ belonging to interval (p, w), and, for any pure Nash equilibrium of $G = (n, p, \alpha)$ with arbitrary $n > n_2^*$ and $p \in [0, 1/2]$, there are at least two players choosing a strategy of $\boldsymbol{\sigma}$ belonging to interval $(w, p + \Delta)$. Thus, by setting $n^* := \{n_1^*, n_2^*\}$, we have that, for any $p \in [0, 1/2]$, any $n > n^*$, any pure Nash equilibrium $\boldsymbol{\sigma}$ of $G = (n, p, \alpha)$, by choosing i as the smallest index of a player selecting a strategy of $\boldsymbol{\sigma}$ belonging to interval $(w, \min\{1, p - 1 + 1/\alpha\})$, we get $p < \sigma_{i-1} < w < \sigma_i \leq \sigma_{i+1} \leq p + \Delta < \min\{1, p - 1 + 1/\alpha\}$.

Since $p < \sigma_{i-1} < w < \sigma_i \leq \sigma_{i+1} \leq p + \Delta < \min\{1, p - 1 + 1/\alpha\}$, and by exploiting the definition of Δ given in Lemma 6, we have that $|[sep(\sigma_{i-1}, w), sep(w, \sigma_{i+1})]| > |[sep(\sigma_{i-1}, \sigma_i), sep(\sigma_i, \sigma_{i+1})]|$. However, this inequality does not satisfy the necessary condition given in claim **(vii)** of Theorem 1, and this contradicts the assumption that $\boldsymbol{\sigma}$ is a pure Nash equilibrium. Thus, we necessarily have that, for any $n > n^*$ and $p \in [1/2]$, game $G = (n, p, \alpha)$ does not admit any pure Nash equilibrium, and this shows the claim. □

4 The Price of Anarchy for Identical Players

In this section, we focus on the price of anarchy for games with identical players.

4.1 The Case $n \leq 4$

We start by considering games with few players. For two players, we have the following result.

Theorem 9 (PoA for two players)**.** *Given a game $G = (2, p, \alpha)$, we have that $\mathsf{PoA}(G) \leq 3/2$. Furthermore, there exists a game $G = (2, p, \alpha)$ such that $\mathsf{PoA}(G) = 3/2$.*

Proof. From Theorem 2, we have an explicit formula to compute the (unique) pure Nash equilibrium for the case of two identical players, and then we can compute the price of anarchy over all the possible values of α and p. Thus, by taking the maximum over $\alpha \in [0, 1]$ and $p \in [0, 1/2]$, we get $3/2$ which is attained for $\alpha = 2/3$ and $p = 0$, and the claim follows. □

For three players, we do not consider the price of anarchy since pure Nash equilibria never exist (Theorem 3). For four players, the price of anarchy is characterized by the following theorem.

Theorem 10 (PoA for four players)**.** *Given a game $G = (4, p, \alpha)$, we have that $\mathsf{PoA}(G) \leq 10/7$. Furthermore, there exists a game $G = (4, p, \alpha)$ such that $\mathsf{PoA}(G) = 10/7$.*

Proof. When the necessary conditions for the existence of a pure Nash equilibrium given in Theorem 4 are satisfied, we have an explicit formula to determine the (unique) candidate equilibrium. This gives an upper bound on the price of anarchy which can be computed by taking the maximum over all values of α and p satisfying the conditions. This value is $10/7$ and is attained for $\alpha = 4/5$ and $p = 0$ (case **(i)**). As for this case the conditions are also sufficient, it follows that there exists a game realizing a price of anarchy of $10/7$. □

4.2 The Case of Large Games

Here, we consider games in which the number of players goes to infinity. Because of Theorem 8, existence of pure Nash equilibria is guaranteed only in the two extremal cases of $\alpha \in \{0, 1\}$. Indeed, the case of $\alpha = 0$ is a trivial one as the quality of the content published by any players is always equal to one, so that $Q(\boldsymbol{\sigma}) = n$ for each strategy profile $\boldsymbol{\sigma}$ and the price of anarchy is 1 for any game. Hence, we only focus on the case of $\alpha = 1$. In Theorem 11, we show that, for any arbitrarily small $\epsilon > 0$, if the number of players is sufficiently large, then the price of anarchy is at most $\sqrt{2} + \epsilon$. Instead, in Theorem 12, we show that, for any arbitrarily small $\epsilon > 0$ and any $n^* > 0$, there always exists a game with more than n^* players having a price of anarchy of at least $\sqrt{2} - \epsilon$.

Theorem 11 (PoA for $\alpha = 1$ and $n \to \infty$ (upper bound)). *Given $p \in [0, 1/2]$, we have that $\limsup_{n\to\infty} \mathsf{PoA}((n, p, 1)) \leq \sqrt{2}$, i.e., for any $\epsilon > 0$, there exists $n^* > 0$ such that $\mathsf{PoA}((n, p, 1)) \leq \sqrt{2}+\epsilon$ for any $n > n^*$.*

Proof. Fix $p \in [0, 1/2]$. The following lemma characterizes the structure of a pure Nash equilibrium $\boldsymbol{\sigma}$ minimizing the overall quality $Q(\boldsymbol{\sigma})$[5], i.e., such that $n/Q(\boldsymbol{\sigma}) = \mathsf{PoA}((n, p, 1))$.

Lemma 7. *Given $n \geq 5$, let $\boldsymbol{\sigma}$ be a pure Nash equilibrium of game $G = (n, p, 1)$ minimizing the overall quality $Q(\boldsymbol{\sigma})$. Set $\Delta := \sigma_{n-1} - \sigma_{n-2}$. Then, the following conditions hold:*

(i) *there exists at most one strategy $\sigma_h \geq p$ and at most one strategy $\sigma_{h-1} \leq p$ with $h \in [n-2]\setminus\{1, 2, 3\}$, such that $0 < \sigma_h - \sigma_{h-1} < \Delta$;*
(ii) *there exist two strategies $\sigma_u \leq \sigma_v$, with $u, v \in [n-1] \setminus \{1\}$, such that, there are exactly two players selecting strategy σ_i if $i \leq u$ or if $i \geq v$, and exactly one player otherwise;*
(iii) *$\sigma_i - \sigma_{l(i)} = \Delta$ for any index $i \in [n-1] \setminus \{1, 2\}$, except for at most two indices.*

Now, by exploiting the structure of the pure Nash equilibria defined in Lemma 7, we can compute $\limsup_{n\to\infty} \mathsf{PoA}(n, p, 1)$.

Lemma 8. *Fix $p \in [0, 1/2]$. For any $n \geq 5$, let $\boldsymbol{\sigma}_n$ be a pure Nash equilibrium minimizing $Q(\boldsymbol{\sigma}_n)$ in game $G = (n, p, 1)$. We have that:*

$$\limsup_{n\to\infty} \mathsf{PoA}(n, p, 1) = \limsup_{n\to\infty} \frac{n}{Q(\boldsymbol{\sigma}_n)}$$
$$\leq \max_{a,b:\,\frac{p}{2}\leq a\leq b\leq \frac{p+1}{2}} \frac{\int_{\left[\frac{p}{2},\frac{p+1}{2}\right]\setminus[a,b]} 2dx + \int_{[a,b]} 1dx}{\int_{\left[\frac{p}{2},\frac{p+1}{2}\right]\setminus[a,b]} 2q(x)dx + \int_{[a,b]} q(x)dx}$$
$$= \max_{a,b:\,\frac{p}{2}\leq a\leq b\leq \frac{p+1}{2}} F(a, b, p), \tag{8}$$

where $F(a, b, p) := \frac{2\left(a-\frac{p}{2}\right)+(b-a)+2\left(\frac{p+1}{2}-b\right)}{\int_{\left[\frac{p}{2},\frac{p+1}{2}\right]\setminus[a,b]} 2(1-|x-p|)dx+\int_a^b(1-|x-p|)dx}$.

The formula obtained in (8) can be interpreted as the price of anarchy of a continuous variant of the content publishing game in which there are infinitely many players, the contribution of each player to the overall quality is infinitesimally small, and each strategy profile is a distribution on interval $[0, 1]$. According to formula (8), the pure Nash equilibrium minimizing the overall quality within this setting of continuous games, is a distribution with a density function μ defined as $\mu(x) = 2$ if $x \in [p/2, a] \cup [b,(p + 1)/2]$, $\mu(x) = 1$ if $x \in (a, b))$, and $\mu(x) = 0$ otherwise.

Now, let $F(a, b, p)$ as in Lemma 8. By (8), and by standard calculations, we get $\limsup_{n\to\infty} \mathsf{PoA}(n, p, 1) \leq \max_{a,b:\,\frac{p}{2}\leq a\leq b\leq \frac{p+1}{2}} F(a, b, p) = \max_{0\leq b\leq \frac{1}{2}} \frac{4-4b}{2b^2-4b+3} = \sqrt{2}$, thus showing the claim. □

Theorem 12 (PoA for $\alpha = 1$ and $n \to \infty$ (lower bound)). *We have that $\limsup_{n\to\infty} \mathsf{PoA}(n, 0, 1) \geq \sqrt{2}$, i.e., for any $\epsilon > 0$ and for any $n^* > 0$, there exists a game $G = (n, 0, 1)$ with $n > n^*$, such that $\mathsf{PoA}(G) \geq \sqrt{2} - \epsilon$.*

Proof. For any integer $t \geq 1$, let $\Delta(t) := \frac{1}{2(t+1)}$, and let $\boldsymbol{\sigma}_{n(t)}$ be the strategy profile of a game $G = (n(t), 0, 1)$ such that the set of strategies played by some player is $A_t := \{\sigma_{n(t),i} : i \in [n(t)]\} = \left\{\frac{\Delta(t)}{2} + h\cdot\Delta(t) : h \in [t] \cup \{0\}\right\}$, and such that, given $x \in A_t$, we have that $|P_x(\boldsymbol{\sigma}_{n(t)})| = 2$ if $x \geq 1 - \frac{1}{\sqrt{2}}$ or $x = \frac{\Delta(t)}{2}$, and $|P_x(\boldsymbol{\sigma}_{n(t)})| = 1$ otherwise. Observe that the number $n(t)$ of players is univocally determined by the definition of strategy profile $\boldsymbol{\sigma}_{n(t)}$. By Theorem 6, we get that strategy profile $\boldsymbol{\sigma}_{n(t)}$ is a pure Nash equilibrium for any $t \geq 1$. Given $t \geq 1$, let $h^*(t)$ be the smallest integer h such that $\frac{\Delta(t)}{2} + h \cdot \Delta(t) \geq 1 - \frac{1}{\sqrt{2}}$. By exploiting the definition of $\boldsymbol{\sigma}_{n(t)}$, and by using standard arguments of mathematical analysis, we get $\limsup_{n\to\infty} \mathsf{PoA}(n, 0, 1) \geq \lim_{t\to\infty} \frac{n(t)}{Q(\boldsymbol{\sigma}_{n(t)})} \cdot \frac{\Delta(t)}{\Delta(t)}$ $= \frac{4-4b}{2b^2-4b+3}\Big|_{b=1-\frac{1}{\sqrt{2}}} = \sqrt{2}$, thus showing the claim. □

5 The Price of Anarchy for Heterogeneous Players

In this section, we bound the price of anarchy of any content publishing game admitting pure Nash equilibria. Although we resort to the relaxed equilibrium conditions given in the following theorem, this is enough to obtain a general upper bound of 2.

Theorem 13 (Equilibrium conditions). *For any game $G = (n, (p_i)_{i\in[n]}, \alpha)$ admitting a pure Nash equilibrium $\boldsymbol{\sigma}$, the following properties hold:* **(i)** $0 < u_i(\boldsymbol{\sigma}) < 1$ *for any* $i \in [n]$, **(ii)** $\max\left\{0, \frac{p_i}{2} + \frac{1}{2} - \frac{1}{2\alpha}\right\} \leq \sigma_i \leq \min\left\{1, \frac{p_i}{2} + \frac{1}{2\alpha}\right\}$ *for any* $i \in [n]$.

Theorem 14 (Price of anarchy). *For any game $G = (n, (p_i)_{i\in[n]}, \alpha)$, we have that $\mathsf{PoA}(G) \leq \min\left\{\frac{1}{1-\alpha}, 2\right\} \leq 2$, with the interpretation that $1/0 := \infty$.*

Proof. Let $\boldsymbol{\sigma}$ be a pure Nash equilibrium of game G. Given $i \in [n]$, by Theorem 13, we have that, if $\sigma_i \geq p_i$, the following inequalities hold: $q_i(\sigma_i) = 1 - \alpha(\sigma_i - p_i) \geq 1 - \alpha \min\left\{1, \frac{p_i}{2} + \frac{1}{2\alpha}\right\} + \alpha p_i \geq 1 - \min\left\{\alpha, \frac{\alpha p_i}{2} + \frac{1}{2}\right\} + \alpha p_i = -\min\left\{-1 + \alpha - \alpha p_i, -\frac{\alpha p_i}{2} - \frac{1}{2}\right\} = \max\left\{1 - \alpha + \alpha p_i, \frac{\alpha p_i}{2} + \frac{1}{2}\right\} \geq \max\left\{1 - \alpha, \frac{1}{2}\right\}$, where the first inequality comes from Theorem 13. Analogously, if $\sigma_i < p_i$, we get $q_i(\sigma_i) = 1 - \alpha(p_i - \sigma_i) \geq 1 + \alpha \max\left\{0, \frac{p_i}{2} + \frac{1}{2} - \frac{1}{2\alpha}\right\} - \alpha p_i \geq 1 + \max\left\{0, \frac{\alpha p_i}{2} + \frac{\alpha}{2} - \frac{1}{2}\right\} - \alpha p_i \geq \max\left\{1 - \alpha p_i, 1 - \frac{\alpha p_i}{2} + \frac{\alpha}{2} - \frac{1}{2}\right\} \geq \max\left\{1 - \alpha, 1 - \frac{\alpha}{2} + \frac{\alpha}{2} - \frac{1}{2}\right\} \geq \max\left\{1 - \alpha, \frac{1}{2}\right\}$, where the first inequality comes from Theorem 13. Thus, we have that $q_i(\boldsymbol{\sigma}) \geq 1/2$ for any $i \in [n]$, and then the price of anarchy is $\mathsf{PoA}(G) = \frac{n}{\sum_{i=1}^n q_i(\boldsymbol{\sigma})} \leq \frac{n}{n\cdot\max\left\{1-\alpha,\frac{1}{2}\right\}} = \frac{1}{\max\left\{1-\alpha,\frac{1}{2}\right\}} = \min\left\{\frac{1}{1-\alpha}, 2\right\} \leq 2$, with the interpretation that $1/0 := \infty$, and this fact concludes the proof. □

6 Conclusions and Open Problems

We introduced the content publishing game, modelling interactions between readers and publishers, to evaluate to what extent the strategic behavior of the latter impacts on the quality of content publishing in the World Wide Web. The most technical and challenging part of our work has revealed to be the characterization of the set of pure Nash equilibria as a function of the parameters of the game and, although we provided a considerable amount of results, a complete picture is still missing, even for the case of identical publishers. Settling this question constitutes a challenging open problem which would likely provide further fundamental insights, also useful for the exact quantification of the price of anarchy as a function of n, α and p.

[5] Observe that, by Theorem 6, the set of pure Nash equilibria of game $G = (n, p, 1)$ is a compact set of $\mathbb{R}^n$, and the overall quality is a continuous function. Thus, by the Weierstrass Theorem, there exists a pure Nash equilibrium minimizing the overall quality.

Furthermore, a better characterization of the set of pure Nash equilibria could also imply better bounds for the price of anarchy of the content publishing game as function of the input parameters. By exploiting our results, we have an upper bound of 2 for heterogeneous players, and relatively to identical players, we have tight bounds of $3/2$, $10/7$, and $\sqrt{2}$, for $n = 2$, $n = 4$, and $(n, \alpha) \to (\infty, 1)$, respectively. As the highest lower bound is $3/2$ (attained by two identical players), it would be interesting to close the gap between $3/2$ and 2 for the price of anarchy in the general case of heterogeneous players. Finally, we conjecture that the price of anarchy for identical players decreases as the number of players increases.

Other interesting research directions would be to analyse distributions of users other than the uniform one, so as to model the presence of hot topics catalysing the interest of a huge portion of readers; to consider large games with a huge number of publishers whose topics of expertise are drawn according to some probability distribution; to relax the assumption of having a continuous space of topics; to consider publishers with different writing abilities; to address users interested in accessing multiple documents; to quantify the price of stability for the cases in which pure Nash equilibria are not unique; to focus on other social functions, such as the sum of the publishers' utilities, the sum of the readers' satisfactions, the minimum quality of a released document, and so forth.

Under a more general viewpoint, we believe that our work introduces an intriguing model of strategic interactions which may be applied in several practical settings such as, for instance, marketing strategies in business activities and shifting alliances in politics.

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