OSCILLATOR SPACETIMES ARE RICCI SOLITONS

GIOVANNI CALVARUSO

Abstract. We consider the four-dimensional oscillator group, equipped with a well-known one-parameter family of left-invariant Lorentzian metrics, which includes the bi-invariant one [15]. In a suitable system of global coordinates, the Ricci soliton equation for these metrics translates into a system of partial differential equations. Solving such system, we prove that all these metrics are Ricci solitons. In particular, the bi-invariant metric on the oscillator group gives rise to infinitely many Ricci solitons (and so, also to Yamabe solitons).

Postprint version.

Nonlinear Analysis

Volume 140, July 2016, Pages 254-269

Link to the published version: http://dx.doi.org/10.1016/j.na.2016.03.008

1. Introduction

The *oscillator group* is a four-dimensional connected, simply connected Lie group, whose Lie algebra (known as the *oscillator algebra*) coincides with the one generated by the differential operators, acting on functions of one variable, associated to the harmonic oscillator problem. This group is given by $\mathbb{R} \times \mathbb{C} \times \mathbb{R}$, with the product

$$(x_1, z_1, y_1) \cdot (x_2, z_2, y_2) = (x_1 + x_2 + \frac{1}{2}\mathrm{Im}(\bar{z}_1 e^{iy_1} z_2), z_1 + e^{iy_1} z_2, y_1 + y_2).$$

After its introduction [22], the oscillator group has been extended to a one parameter family G_μ ($\mu > 0$), then generalized in any even dimension $2n \geq 4$, and proved several times to be an interesting object to study both in differential geometry and in mathematical physics. Among others, the following aspects of the geometry of the oscillator group(s) have been investigated: Yang-Baxter [2] and Einstein-Yang-Mills equations [14], compact quotients by lattices [3], parallel hypersurfaces [9], Ricci collineations and other curvature symmetries [11], homogeneous structures [15], electromagnetic waves [19], the Laplace-Beltrami operator [20].

The four-dimensional oscillator group is a well known homogeneous spacetime [13]. Its bi-invariant metric g_0 has been generalized to a one-parameter family g_a, $-1 < a < 1$,

2010 *Mathematics Subject Classification.* 53C50, 53B30, 35A01.

Key words and phrases. Oscillator group, Ricci solitons, Yamabe solitons.

The author partially supported by funds of the University of Salento and MIUR (PRIN).

Link: http://dx.doi.org/10.1016/j.na.2016.03.008.

of left-invariant Lorentzian metrics, of which g_0 is the only bi-invariant and symmetric example [15]. Equipped with these left-invariant Lorentzian metrics, the oscillator group is "one of the most celebrated examples of Lorentzian naturally reductive spaces" [1].

It is natural to investigate the curvature properties of these renowned examples of homogeneous spacetimes. For example, it is well known that the bi-invariant metric g_0 is symmetric and conformally flat. The aim of the present paper is to prove that (G_μ, g_a) is a Lorentzian Ricci soliton, for any $-1 < a < 1$ (and $\mu > 0$).

A *Ricci soliton* is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field X, such that

$$\mathcal{L}_X g + \varrho = \lambda g, \tag{1.1}$$

where $\mathcal{L}_X$ and ϱ respectively denote the Lie derivative in the direction of X and the Ricci tensor and λ is a real number. A Ricci soliton is said to be *shrinking*, *steady* or *expanding*, according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

Ricci solitons are the self-similar solutions of the *Ricci flow*. As such, they are essential in understanding its singularities. We may refer to the recent survey [12] for more information and further references on Ricci solitons. Introduced by Hamilton [16] on Riemannian manifolds, Ricci solitons have recently been studied by several authors in pseudo-Riemannian settings, and in particular on Lorentzian spaces. Some examples of the study of Lorentzian Ricci solitons may be found in [4]-[8],[10].

In a setting of local coordinates, the Ricci soliton equation (1.1) translates into a system of partial differential equations, which in general is not possible to deal with. For this reason, when one considers a pseudo-Riemannian homogeneous space (in particular, a Lie group equipped with a left-invariant pseudo-Riemannian metric), the first approach in studying the Ricci soliton equation (1.1) is algebraic. A *homogeneous Ricci soliton* is a homogeneous space $M = G/H$, together with a G-invariant metric g, for which equation (1.1) holds. An *invariant Ricci soliton* is a homogeneous one, such that equation (1.1) holds for an invariant vector field.

Algebraic Ricci solitons, introduced by Lauret [18] for Riemannian manifolds, have been successively extended to pseudo-Riemannian settings [21]. An *algebraic Ricci soliton* is a simply connected Lie group G, equipped with a left-invariant pseudo-Riemannian metric g, such that

$$Ric = c\,\mathrm{Id} + D,$$

where Ric denotes the Ricci operator, c is a real number, and $D \in \mathrm{Der}(g)$. An algebraic Ricci soliton on a solvable Lie group is called a *solvsoliton.*

Any algebraic Ricci soliton metric g is also a Ricci soliton [18],[21]. Moreover, it is relevant to observe that all known examples of homogeneous Riemannian Ricci soliton metrics on non-compact homogeneous manifolds are isometric to some solvsolitons ([17, Remark 1.5]).

By the above definitions, when we are concerned with the Ricci soliton metrics on a homogeneous space G, it is clear that invariant and algebraic Ricci solitons are subclasses of the class of homogeneous Ricci soliton, which in general do not exhaust the whole class.

Moreover, an invariant Ricci soliton need not be algebraic [21]. As we shall see, neither does the converse hold: an algebraic Ricci soliton need not be invariant.

Algebraic Ricci solitons on oscillator groups of every even dimension were investigated in [21], proving that g_0 is a steady algebraic Ricci soliton (nontrivial, since the metric is not Einstein). In this paper, we focus on the four-dimensional case, also because of its greater physical motivation. We completely solve the system of partial differential equations, which translates (1.1) in a suitable set of global coordinates on (G_μ, g_a). The main result is the following.

Theorem 1.1. *Every left-invariant metric g_a, $-1 < a < 1$ on the four-dimensional oscillator group G_μ is a Ricci soliton. More precisely,*

(a) *The bi-invariant metric g_0 is a Ricci soliton (expanding, steady and shrinking, as it satisfies equation* (1.1) *for any real value of λ);*
(b) *The left-invariant metric g_a, for any $a \neq 0$, is a Ricci soliton, which is expanding when $a > 0$ and shrinking when $a < 0$.*

A pseudo-Riemannian manifold (M, g) is said to be a *Yamabe soliton* if it admits a vector field Y, such that

$$\mathcal{L}_Y g = (\tau - \rho) g, \tag{1.2}$$

where τ denotes the scalar curvature and ρ is a real constant. Clearly, a Yamabe soliton is nontrivial when equation (1.2) holds with $\tau \neq \rho$, otherwise it just reduces to the equation for Killing vector fields. As we shall explain at the end of Section 3, the result listed in point (a) of Theorem 1.1 has the following consequence.

Corollary 1.2. *The bi-invariant metric g_0 on the four-dimensional oscillator group G_μ is a Yamabe soliton.*

The paper is organized in the following way. In Section 2 we shall report the description of a set of global coordinates (x_1, x_2, x_3, x_4) on the oscillator group and explicitly compute all the curvature information with respect to the corresponding basis $\left\{\frac{\partial}{\partial x_i}\right\}$ of coordinate vector fields. In Section 3 we shall introduce the system of ten PDE that express the Ricci soliton equation (1.1) in these global coordinates, and we shall solve it in the case of the bi-invariant metric g_0. The case of the remaining left-invariant Lorentzian metrics $g_a, a \neq 0$ is dealt with in Section 4.

2. The oscillator group

The four-dimensional *oscillator algebra* is the real Lie algebra $\mathfrak{g}_\mu$ with generators X, Y, P, Q, whose non-vanishing Lie brackets are

$$[X, Y] = P, \qquad [Q, X] = \mu Y, \qquad [Q, Y] = -\mu X, \tag{2.1}$$

where $\mu > 0$ is a real constant (with respect to the standard notations used for example in [9] and [21], here we use μ instead of λ, to avoid confusion with equation (1.1)). The corresponding connected simply connected Lie group is called the *(four-dimensional) oscillator group*, and we shall denote it by G_μ. In [9], generalizing the argument used in [22] for the case $\mu = 1$, equation (2.1) was proved to hold for matrices

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, the oscillator group corresponds to the four-dimensional subgroup of $\mathrm{GL}(4,\mathbb{R})$

$$G_\mu = \{M_\mu(x_1,x_2,x_3,x_4) \in \mathrm{GL}(4,\mathbb{R}) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\},$$

having as typical group element

$$M_\mu(x_i) = \exp(x_1 P)\exp(x_2 X)\exp(x_3 Y)\exp(x_4 Q),$$

that is,

$$M_\mu(x_i) = \begin{pmatrix} 1 & x_2\sin(\mu x_4) - x_3\cos(\mu x_4) & x_2\cos(\mu x_4) + x_3\sin(\mu x_4) & 2x_1 + x_2x_3 \\ 0 & \cos(\mu x_4) & -\sin(\mu x_4) & x_2 \\ 0 & \sin(\mu x_4) & \cos(\mu x_4) & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

More precisely, M_μ provides a diffeomorphism between G_μ and $\mathbb{R}^3 \times \mathbb{R}/\frac{2\pi}{\mu}\mathbb{Z}$.

Throughout the paper, we shall denote by $\partial_j := \partial/\partial_{x_j}$ the coordinate vector field corresponding to the x_j-coordinate. As a matrix in $\mathfrak{gl}(4,\mathbb{R})$, this corresponds to $\frac{\partial M_\mu}{\partial x_j}(x_1,x_2,x_3,x_4)$. With respect to coordinate vector fields ∂_i, a basis $\{e_1,e_2,e_3,e_4\}$ of left-invariant vector fields on G_μ is explicitly given by

$$\begin{aligned} e_1 &= \partial_1, \\ e_2 &= -x_3\cos(\mu x_4)\partial_1 + \cos(\mu x_4)\partial_2 + \sin(\mu x_4)\partial_3, \\ e_3 &= x_3\sin(\mu x_4)\partial_1 - \sin(\mu x_4)\partial_2 + \cos(\mu x_4)\partial_3, \\ e_4 &= \partial_4. \end{aligned} \tag{2.2}$$

For this basis, one has $(e_j)_I = (\partial_{x_j})_I$, where $I = M_\mu(0,0,0,2k\pi/\mu)$, for any integer k, is the identity matrix. By equation (2.2), a direct calculation yields that the only non-vanishing Lie brackets $[e_i,e_j]$ are given by

$$[e_2,e_3] = e_1, \qquad [e_2,e_4] = -\mu e_3, \qquad [e_3,e_4] = \mu e_2, \tag{2.3}$$

so that the above Lie algebra coincides with the oscillator Lie algebra $\mathfrak{g}_\mu$, via the identifications $X = e_2$, $Y = e_3$, $P = e_1$ and $Q = e_4$. Further details on this description may be found in [9], and in the original paper [22] for the classic case $\mu = 1$.

In [15], the oscillator group G_μ has been equipped with the one-parameter family of left-invariant Lorentzian metrics $g_a = \langle , \rangle$, described by having as the possibly nonvanishing products

$$\langle e_1, e_1\rangle = \langle e_4, e_4\rangle = a, \qquad \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1, \qquad \langle e_1, e_4\rangle = \langle e_4, e_1\rangle = 1, \tag{2.4}$$

for any real constant with $-1 < a < 1$. The case when $a = 0$ and $\mu = 1$ gives the bi-invariant metric on the classic oscillator group G_1 [15]. When $a \neq 0$, g_a is only left-invariant. As proved in [9], with respect to the coordinates (x_1, x_2, x_3, x_4) described above, the metric g_a is explicitly given by

$$g_a = adx_1^2 + 2ax_3dx_1dx_2 + (1 + ax_3^2)dx_2^2 + dx_3^2 + 2dx_1dx_4 + 2x_3dx_2dx_4 + adx_4^2. \tag{2.5}$$

It may be observed that the above explicit description (2.5) is the same for any value of μ, since this parameter is used in (2.2) for the description of the left-invariant basis.

The explicit description of these metrics makes possible to explicitly compute their Levi-Civita connection and curvature. With respect to the basis $\{\partial_i\}$ of coordinate vector fields, the Levi-Civita connection ∇ is completely determined by the following possibly non-vanishing components:

$$\begin{array}{lll} \nabla_{\partial_1}\partial_2 = -\frac{a}{2}\partial_3, & \nabla_{\partial_1}\partial_3 = -\frac{ax_3}{2}\partial_1 + \frac{a}{2}\partial_2, & \nabla_{\partial_2}\partial_2 = -ax_3\partial_3, \\ \nabla_{\partial_2}\partial_3 = \frac{1-ax_3^2}{2}\partial_1 + \frac{ax_3}{2}\partial_2, & \nabla_{\partial_2}\partial_4 = -\frac{1}{2}\partial_3, & \nabla_{\partial_3}\partial_4 = -\frac{x_3}{2}\partial_1 + \frac{1}{2}\partial_2. \end{array} \tag{2.6}$$

Remark 2.1. The above description of the Levi-Civita connection of (G, g_a) yields that if $a \neq a'$, then (G, g_a) is not homothetic to $(G, g_{a'})$ (in particular, they are not isometric).

In fact, for the Levi-Civita connections ∇ and ∇' of g_a and $g_{a'}$ respectively, we have $\nabla_{\partial_1}\partial_2 = -\frac{a}{2}\partial_3 \neq -\frac{a'}{2}\partial_3 = \nabla'_{\partial_1}\partial_2$.

We can then describe the Riemann-Christoffel curvature tensor R of (G_λ, g_a) with respect to $\{\partial_i\}$, computing $R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$ for all indices i, j, k. Denoting by R_{ij} the matrix describing $R(\partial_i, \partial_j)$ with respect to the basis of coordinate vector fields, we have

$$R_{12} = \begin{pmatrix} \frac{a^2x_3}{4} & \frac{a^2x_3^2+a}{4} & 0 & \frac{ax_3}{4} \\ -\frac{a^2}{4} & -\frac{a^2x_3}{4} & 0 & -\frac{a}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{13} = \begin{pmatrix} 0 & 0 & \frac{a}{4} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a^2}{4} & -\frac{a^2x_3}{4} & 0 & -\frac{a}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{14} = 0, \quad R_{23} = \begin{pmatrix} 0 & 0 & ax_3 & 0 \\ 0 & 0 & -\frac{3a}{4} & 0 \\ -\frac{a^2x_3}{4} & \frac{3a-a^2x_3^2}{4} & 0 & -\frac{ax_3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{24} = \begin{pmatrix} -\frac{ax_3}{4} & -\frac{ax_3^2+1}{4} & 0 & -\frac{x_3}{4} \\ \frac{a}{4} & \frac{ax_3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{4} & \frac{ax_3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, the Ricci tensor of (G_μ, g_a) is obtained as a contraction of the curvature tensor, by the equation $\varrho(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y)$. With respect to $\{\partial_i\}$, the Ricci tensor is then described by the matrix

$$\varrho = \begin{pmatrix} \frac{1}{2}a^2 & \frac{1}{2}a^2x_3 & 0 & \frac{1}{2}a \\ \frac{1}{2}a^2x_3 & \frac{1}{2}a(ax_3^2-1) & 0 & \frac{1}{2}ax_3 \\ 0 & 0 & -\frac{1}{2}a & 0 \\ \frac{1}{2}a & \frac{1}{2}ax_3 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.7}$$

and the Ricci operator Q, defined by $g(QX,Y) := \varrho(X,Y)$, is determined by the matrix

$$Q = \begin{pmatrix} \frac{1}{2}a & ax_3 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2}a & 0 & 0 \\ 0 & 0 & -\frac{1}{2}a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Comparison between equations (2.7) and (2.5) easily yields that these metrics are never Einstein (see also [21]). Moreover, the Ricci eigenvalues are 0, $\frac{1}{2}a$ and $-\frac{1}{2}a$ (twice), and so, the Ricci tensor is degenerate, for any value of a. Finally, the Weyl conformal tensor W is completely determined by the following possibly non-vanishing matrices W_{ij}, describing

$W(\partial_i, \partial_j)$ with respect to the coordinate vector fields $\{\partial_i\}$:
(2.8)

$$W_{12} = \begin{pmatrix} \frac{a^2 x_3}{6} & \frac{a(1+a x_3^2)}{6} & 0 & \frac{a x_3}{6} \\ -\frac{a^2}{6} & -\frac{a^2 x_3}{6} & 0 & -\frac{a}{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad W_{13} = \begin{pmatrix} 0 & 0 & \frac{a}{6} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a^2}{6} & -\frac{a^2 x_3}{6} & 0 & -\frac{a}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{14} = \begin{pmatrix} -\frac{a}{3} & -\frac{a x_3}{3} & 0 & -\frac{a^2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a^2}{3} & \frac{a^2 x_3}{3} & 0 & \frac{a}{3} \end{pmatrix}, \qquad W_{23} = \begin{pmatrix} 0 & 0 & \frac{a x_3}{2} & 0 \\ 0 & 0 & -\frac{a}{3} & 0 \\ -\frac{a^2 x_3}{6} & \frac{a(2-a x_3^2)}{6} & 0 & -\frac{a x_3}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{24} = \begin{pmatrix} -\frac{a x_3}{2} & -\frac{a x_3^2}{2} & 0 & -\frac{a^2 x_3}{2} \\ \frac{a}{6} & \frac{a x_3}{6} & 0 & \frac{a^2}{6} \\ 0 & 0 & 0 & 0 \\ \frac{a^2 x_3}{3} & \frac{a(2a x_3^2-1)}{6} & 0 & \frac{a x_3}{3} \end{pmatrix}, \quad W_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{6} & \frac{a x_3}{6} & 0 & \frac{a^2}{6} \\ 0 & 0 & -\frac{a}{6} & 0 \end{pmatrix}.$$

In particular, by (2.8), g_a is locally conformally flat if and only if $a = 0$. Starting from the above equations, it is also easy to check the well-known fact that $\nabla R = 0$ (that is, (G_μ, g_a) is locally symmetric) if and only if $a = 0$.

Remark 2.2. Using equation (2.2), we can easily determine the components u^i of a vector field X with respect to the basis of left-invariant vector fields $\{e_1, e_2, e_3, e_4\}$, in terms of its components X^i with respect to the basis of coordinate vector fields $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ (and conversely). Explicitly, if $X = X^i \partial_i = u^j e_j$, then by (2.2) we have

$$\begin{aligned} &(u^1, u^2, u^3, u^4) \\ &\quad = (X^1 + x_3 X^2, \cos(\mu x_4) X^2 + \sin(\mu x_4) X^3, \cos(\mu x_4) X^3 - \sin(\mu x_4) X^2, X^4). \end{aligned} \tag{2.9}$$

In particular, X is a left-invariant vector field if and only if the above Eq. (2.9) holds for some constants u^i, $i = 1, \dots, 4$.

3. The general system of equations and the solutions for g_0

With respect to the coordinate system (x_1, x_2, x_3, x_4), let $X = X^i \partial_i$ denote an arbitrary vector field on (G_μ, g_a), where $X^i = X^i(x_1, x_2, x_3, x_4)$, $i = 1, \dots, 4$ are arbitrary smooth functions. We now determine the Lie derivative $\mathcal{L}_X g_a$ of the metric g_a, as explicitly described in (2.5), with respect to X. To do so, we calculate $(\mathcal{L}_X g_a)(\partial_i, \partial_j)$, for all indices $i \leq j$. These components of $\mathcal{L}_X g_a$, together with (2.5) and (2.7), yield that the left-invariant metric g_a, together with the smooth vector field X, is a solution of the Ricci

soliton equation (1.1) if and only if the following system of 10 PDE is satisfied:
(3.1)

$$\begin{cases} 2\,a\partial_1 X^1 + 2\,ax_3\partial_1 X^2 + 2\,\partial_1 X^4 + \frac{1}{2}a^2 - a\lambda = 0, \\ ax_3\partial_1 X^1 + a\partial_2 X^1 + \partial_1 X^2 + ax_3^2\partial_1 X^2 + ax_3\partial_2 X^2 + aX^3 + x_3\partial_1 X^4 + \partial_2 X^4 \\ \quad +\frac{1}{2}a^2 x_3 - a\lambda x_3 = 0, \\ a\partial_3 X^1 + ax_3\partial_3 X^2 + \partial_1 X^3 + \partial_3 X^4 = 0, \\ \partial_1 X^1 + a\partial_4 X^1 + x_3\partial_1 X^2 + ax_3\partial_4 X^2 + a\partial_1 X^4 + \partial_4 X^4 + \frac{1}{2}a - \lambda = 0, \\ 2\,ax_3\partial_2 X^1 + 2\,\partial_2 X^2 + 2\,ax_3^2\partial_2 X^2 + 2\,ax_3 X_3 + 2\,x_3\partial_2 X^4 + \frac{1}{2}a^2x_3^2 - \frac{1}{2}a - \lambda - a\lambda x_3^2 = 0, \\ ax_3\partial_3 X^1 + \partial_3 X^2 + ax_3^2\partial_3 X^2 + \partial_2 X^3 + x_3\partial_3 X^4 = 0, \\ \partial_2 X^1 + ax_3\partial_4 X^1 + x_3\partial_2 X^2 + \partial_4 X^2 + ax_3^2\partial_4 X^2 + X^3 + a\partial_2 X^4 + x_3\partial_4 X^4 + \frac{1}{2}ax_3 \\ \quad -\lambda x_3 = 0, \\ 2\,\partial_3 X^3 - \frac{1}{2}a - \lambda = 0, \\ \partial_3 X^1 + x_3\partial_3 X^2 + \partial_4 X^3 + a\partial_3 X^4 = 0, \\ 2\,\partial_4 X^1 + 2\,x_3\partial_4 X^2 + 2\,a\partial_4 X^4 + \frac{1}{2} - a\lambda = 0. \end{cases}$$

We will completely solve the system of PDE (3.1), determining the Ricci solitons of the four-dimensional oscillator group.

Integrating the eight equation of (3.1), and the first equation of (3.1) with respect to X^4, we find

$$\text{(3.2)} \qquad \begin{cases} X^3 = (\frac{1}{4}a + \frac{1}{2}\lambda)x_3 + F_3(x_1, x_2, x_4), \\ X^4 = -aX^1 - ax_3X^2 + \left(\frac{1}{2}a\lambda - \frac{1}{4}a^2\right)x_1 + F_4\left(x_2, x_3, x_4\right), \end{cases}$$

for some smooth functions F_3, F_4. Replacing into the third equation of (3.1), it becomes

$$\text{(3.3)} \qquad \partial_1 F_3\left(x_1, x_2, x_4\right) - aX^2 + \partial_3 F_4\left(x_2, x_3, x_4\right) = 0.$$

It is now evident that the Eq. (3.3) (and so, the whole system (3.1)) will have different sets of solutions, depending on whether $a = 0$ or $a \neq 0$. In the remaining part of this section, we shall continue assuming $a = 0$, while in the next one we shall solve (3.1) for $a \neq 0$.

Taking into account $a=0$ and (3.2), system (3.1) now reduces to

$$
\left\{
\begin{array}{l}
\partial_1 X^2 + \partial_2 F_4\,(x_2,x_3,x_4) = 0,\\
\partial_1 F_3(x_1,x_2,x_4) + \partial_3 F_4\,(x_2,x_3,x_4) = 0,\\
\partial_1 X^1 + x_3\partial_1 X^2 + \partial_4 F_4\,(x_2,x_3,x_4) - \lambda = 0,\\
2\,\partial_2 X^2 + 2\,x_3\partial_2 F_4\,(x_2,x_3,x_4) - \lambda = 0,\\
\partial_3 X^2 + \partial_2 F_3(x_1,x_2,x_4) + x_3\partial_3 F_4\,(x_2,x_3,x_4) = 0,\\
\partial_2 X^1 + x_3\partial_2 X^2 + \partial_4 X^2 + F_3(x_1,x_2,x_4) + x_3\partial_4 F_4\,(x_2,x_3,x_4) - \frac{1}{2}\lambda x_3 = 0,\\
\partial_3 X^1 + x_3\partial_3 X^2 + \partial_4 F_3(x_1,x_2,x_4) = 0,\\
2\,\partial_4 X^1 + 2\,x_3\partial_4 X^2 + \frac{1}{2} = 0.
\end{array}
\right.
\tag{3.4}
$$

In the second equation of (3.4), function F_3 depends on (x_1,x_2,x_4), while F_4 depends on (x_2,x_3,x_4). Therefore, this equation implies that there exists some smooth function $K(x_2,x_4)$, such that

$$\partial_1 F_3(x_1,x_2,x_4) = -\partial_3 F_4\,(x_2,x_3,x_4) = K(x_2,x_4).$$

Integrating, we get

$$
\begin{aligned}
F_3\,(x_1,x_2,x_4) &= K\,(x_2,x_4)\,x_1 + G_3\,(x_2,x_4)\,,\\
F_4\,(x_2,x_3,x_4) &= -K\,(x_2,x_4)\,x_3 + G_4\,(x_2,x_4)\,,
\end{aligned}
$$

for some smooth functions G_3, G_4. Substituting the above into system (3.4), it becomes

$$
\left\{
\begin{array}{l}
\partial_1 X^2 - x_3\partial_2 K\,(x_2,x_4) + \partial_2 G_4\,(x_2,x_4) = 0,\\
\partial_1 X^1 + x_3\partial_1 X^2 - x_3\partial_4 K\,(x_2,x_4) + \partial_4 G_4\,(x_2,x_4) - \lambda = 0,\\
2\,\partial_2 X^2 - 2\,x_3^2\,\partial_2 K\,(x_2,x_4) + 2\,x_3\partial_2 G_4\,(x_2,x_4) - \lambda = 0,\\
\partial_3 X^2 + x_1\partial_2 K\,(x_2,x_4) + \frac{\partial}{\partial x_2} G_3\,(x_2,x_4) - x_3 K\,(x_2,x_4) = 0,\\
\partial_2 X^1 + x_3\partial_2 X^2 + \partial_4 X^2 + x_1 K\,(x_2,x_4) + G_3\,(x_2,x_4) - x_3^2\,\partial_4 K\,(x_2,x_4)\\
\quad + x_3\partial_4 G_4\,(x_2,x_4) - \frac{1}{2}\,\lambda\, x_3 = 0,\\
\partial_3 X^1 + x_3\partial_3 X^2 + x_1\partial_4 K\,(x_2,x_4) + \partial_4 G_3\,(x_2,x_4) = 0,\\
2\,\partial_1 X^1 + 2\,x_3\partial_4 X^2 + \frac{1}{2} = 0.
\end{array}
\right.
\tag{3.5}
$$

We now integrate the first two equations in (3.5) and we get

$$
\left\{
\begin{array}{l}
X^1 = \big((x_3\partial_4 - x_3^2\partial_2)K\,(x_2,x_4) + (x_3\partial_2 - \partial_4)G_4\,(x_2,x_4) + \lambda\big)\,x_1 + F_1\,(x_2,x_3,x_4)\,,\\
X^2 = \;(x_3\partial_2 K\,(x_2,x_4) - \partial_2 G_4\,(x_2,x_4))\,x_1 + F_2\,(x_2,x_3,x_4)\,,
\end{array}
\right.
\tag{3.6}
$$

for some smooth functions F_1, F_2. Replacing into the fourth equation of (3.5), it gives

$$2\;(\partial_2 K\,(x_2,x_4))\,x_1 + \partial_3 F_2\,(x_2,x_3,x_4) + \partial_2 G_3\,(x_2,x_4) - x_3 K\,(x_2,x_4) = 0.$$

The above equation must be satisfied for any value of x_1. Therefore, it yields $\partial_2 K(x_2,x_4)=0$, that is, $K(x_2,x_4)=H(x_4)$, where H is a smooth function, and then reduces to

$$\partial_3 F_2(x_2,x_3,x_4)+\partial_2 G_3(x_2,x_4)-x_3 K(x_2,x_4)=0,$$

which easily yields

$$F_2(x_2,x_3,x_4)=\tfrac{1}{2}\,x_3^2\,H(x_4)-x_3\partial_2 G_3(x_2,x_4)+F_5(x_2,x_4),$$

for a smooth function F_5. Replacing into (3.5), it reduces to

(3.7)
$$\begin{cases} 2\,\partial_2 G_2(x_2,x_4)-2\,x_1\partial^2_{22}G_4(x_2,x_4)-2\,x_3\partial^2_{22}G_3(x_2,x_4)+2\,x_3\partial_2 G_4(x_2,x_4)-\lambda=0,\\ -2\,x_1\partial^2_{24}G_4(x_2,x_4)+\partial_2 F_1(x_2,x_3,x_4)-x_3^2\,\partial^2_{22}G_3(x_2,x_4)+x_3\partial_2 G_2(x_2,x_4)\\ \quad -\tfrac{1}{2}\,x_3^2\,\partial_4 H(x_4)-x_3\partial^2_{24}G_3(x_2,x_4)+\partial_4 G_2(x_2,x_4)+x_1 H(x_4)+G_3(x_2,x_4)\\ \quad +x_3\partial_4 G_4(x_2,x_4)-\tfrac{1}{2}\,\lambda\,x_3=0,\\ (\partial_2 G_4(x_2,x_4)+2\partial_4 H(x_4))\,x_1+\partial_3 F_1(x_2,x_3,x_4)-x_3\partial_2 G_3(x_2,x_4)+x_3^2\,H(x_4)\\ \quad +\partial_4 G_3(x_2,x_4)=0,\\ x_1x_3\partial^2_{44}H(x_4)-2\,x_1\partial^2_{44}G_4(x_2,x_4)+2\,\partial_4 F_1(x_2,x_3,x_4)+x_3^3\,\frac{d}{dx_4}H(x_4)\\ \quad -2\,x_3^2\,\partial^2_{24}G_3(x_2,x_4)+2\,x_3\partial_4 G_2(x_2,x_4)+\tfrac{1}{2}=0. \end{cases}$$

The third equation in (3.7), holding for all values of x_1, implies that

$$\partial_2 G_4(x_2,x_4)+2\partial_4 H(x_4)=0.$$

Integrating, we then find

$$G_4(x_2,x_4)=-2\,x_2 H'(x_4)+H_4(x_4).$$

The third equation in (3.7) now reduces to

$$\partial_3 F_1(x_2,x_3,x_4)-x_3\partial_2 G_3(x_2,x_4)+x_3^2\,H(x_4)+\partial_4 G_3(x_2,x_4)=0$$

and integrating we get

$$F_1(x_2,x_3,x_4)=-\tfrac{1}{3}x_3^3\,H(x_4)+\tfrac{1}{2}\,x_3^2\,\partial_2 G_3(x_2,x_4)-x_3\partial_4 G_3(x_2,x_4)+G_1(x_2,x_4).$$

Substituting the above expressions of F_1 and G_4 into (3.7), it becomes

(3.8)
$$\begin{cases} 2\,\partial_2 G_2(x_2,x_4)-2\,x_3\partial^2_{22}G_3(x_2,x_4)-4\,x_3 H'(x_4)-\lambda=0,\\ 4\,x_1 H''(x_4)-\tfrac{1}{2}\,x_3^2\,\partial^2_{22}G_3(x_2,x_4)-2\,x_3\partial^2_{24}G_3(x_2,x_4)+\partial_2 G_1(x_2,x_4)\\ \quad +x_3\partial_2 G_2(x_2,x_4)-\tfrac{1}{2}\,x_3^2\,H'(x_4)+\partial_4 G_2(x_2,x_4)+x_1 H(x_4)+G_3(x_2,x_4)\\ \quad -2\,x_2x_3 H''(x_4)+x_3 H_4'(x_4)-\tfrac{1}{2}\,\lambda\,x_3=0,\\ 2\,x_1x_3 H''(x_4)+4\,x_1x_2 H'''(x_4)-2\,x_1 H_4''(x_4)+\tfrac{1}{3}\,x_3^3 H_1'(x_4)-x_3^2\partial^2_{24}G_3(x_2,x_4)\\ \quad -2\,x_3\partial^2_{44}G_3(x_2,x_4)+2\,\partial_4 G_1(x_2,x_4)+2\,x_3\partial_4 G_2(x_2,x_4)+\tfrac{1}{2}=0. \end{cases}$$

Since the first equation in (3.8) must hold for all values of x_3, it is equivalent to requiring that

$$\partial^2_{22} G_3\left(x_2,x_4\right)+2H'\left(x_4\right)=2\,\partial_2 G_2\left(x_2,x_4\right)-\lambda=0.$$

Thus, integrating we obtain

$$\begin{aligned} G_3\left(x_2,x_4\right)&=-x_2^2\,H'\left(x_4\right)+U_3\left(x_4\right)x_2+W_3\left(x_4\right),\\ G_2\left(x_2,x_4\right)&=\tfrac{1}{2}\,\lambda\,x_2+H_2\left(x_4\right). \end{aligned}$$

We then replace into (3.8) and it gives

(3.9)
$$\left\{\begin{aligned} &4\,x_1H''\left(x_4\right)+\tfrac{1}{2}\,x_3^2\,H'\left(x_4\right)+2\,x_2x_3H''\left(x_4\right)-2\,x_3U_3'\left(x_4\right)+\partial_2G_1\left(x_2,x_4\right)+H_2'\left(x_4\right)\\ &\quad+x_1H\left(x_4\right)-x_2^2\,H'\left(x_4\right)+U_3\left(x_4\right)x_2+W_3\left(x_4\right)+x_3H_4'\left(x_4\right)=0,\\ &2\,x_1x_3H''\left(x_4\right)+4\,x_1x_2H'''\left(x_4\right)-2\,x_1H_4''\left(x_4\right)+\tfrac{1}{3}\,x_3^3\,H'\left(x_4\right)+2\,x_2{x_3}^2H''\left(x_4\right)\\ &\quad-x_3^2\,U_3'\left(x_4\right)+2\,x_2^2\,x_3H'''\left(x_4\right)-2\,x_2x_3U_3''\left(x_4\right)-2\,x_2x_3W_3''\left(x_4\right)\\ &\quad+2\,\partial_4G_1\left(x_2,x_4\right)+2\,x_3H_2'\left(x_4\right)+\tfrac{1}{2}=0. \end{aligned}\right.$$

By the same argument already used several times, collecting the terms with x_1 in the second equation of (3.9), we find that necessarily

$$2x_3H''(x_4)+4x_2H'''(x_4)-2H_4''(x_4)=0,$$

and the above equation must hold for all values of x_2 and x_3. Henceforth, it yields $H''(x_4)=H_4''(x_4)=0$ and integrating we get

$$H(x_4)=C_1x_4+P_1,\qquad H_4(x_4)=a_4x_4+b_4,$$

for some real constants C_1,P_1,a_4,b_4. We replace into system (3.9) and it reduces to

$$\text{(3.10)}\qquad\left\{\begin{aligned} &\partial_2G_1\left(x_2,x_4\right)-2\,x_3U_3'\left(x_4\right)+\tfrac{1}{2}\,C_1\,{x_3}^2+H_2'\left(x_4\right)+C_1\,x_1x_4+P_1x_1\\ &\quad-C_1\,{x_2}^2+x_2U_3\left(x_4\right)+W_3\left(x_4\right)+a_4x_3=0,\\ &\tfrac{1}{3}\,C_1\,x_3^3-U_3'\left(x_4\right)x_3^2-2\left(x_2U_3''\left(x_4\right)-2\,W_3''\left(x_4\right)+2H_2'\left(x_4\right)\right)x_3\\ &\quad+2\,\partial_4G_1\left(x_2,x_4\right)+\tfrac{1}{2}=0. \end{aligned}\right.$$

We wrote the last equation in (3.10) as a polynomial in x_3. Since this equation must hold for any value of x_3, it yields

$$C_1=0,\quad U_3'\left(x_4\right)=0,\quad x_2U_3''\left(x_4\right)-2\,W_3''\left(x_4\right)+2H_2'\left(x_4\right)=0,\quad 2\,\partial_4G_1\left(x_2,x_4\right)+\tfrac{1}{2}=0,$$

which, integrating, gives

$$C_1=0,\quad U_3\left(x_4\right)=a_3,\quad H_2\left(x_4\right)=W_3'\left(x_4\right)+b_2,\quad G_1\left(x_2,x_4\right)=-\tfrac{1}{4}x_4+H_1(x_2)=0.$$

Replacing the above expressions into (3.10), it reduces to

$$\text{(3.11)}\qquad {H_1}'\left(x_2\right)+W_3''\left(x_4\right)+P_1x_1+a_3\,x_2+W_3\left(x_4\right)+a_4x_3=0.$$

From (3.11) we get at once $P_1=a_4=0$ and the equation becomes

$${H_1}'\left(x_2\right)+W_3''\left(x_4\right)\,+a_3\,x_2+W_3\left(x_4\right),$$

which, since H_1 and W_3 only depend on x_2 and x_4 respectively, yields

$$H_1\,(x_2) = -\tfrac{1}{2}a_3x_2^2 + Kx_2 + b_2, \quad W_3\,(x_4) = a_3\sin(x_4) + b_3\cos(x_4) - K,$$

for some real constant K. Now, all equations in (3.1) are satisfied. Replacing the functions we found by integration into X^i, we explicitly get

(3.12)
$$\begin{cases} X^1 = \lambda x_1 + \frac{1}{2}\,a_3{x_3}^2 - a_3x_3\cos\,(x_4) + b_3x_3\sin\,(x_4) - \frac{1}{4}\,x_4 - \frac{1}{2}\,a_3{x_2}^2 + K\,x_2 + b_2, \\ X^2 = -a_3x_3 + \frac{1}{2}\,\lambda\,x_2 + a_3\cos\,(x_4) - b_3\sin\,(x_4) + b_2, \\ X^3 = a_3\,x_2 + a_3\sin\,(x_4) + b_3\cos\,(x_4) - \frac{1}{2}K\lambda\,x_3, \\ X^4 = b_4. \end{cases}$$

As a check, if we compute $\mathcal{L}_X g_0$, where $X = X^i\partial_i$ with X^i described by (3.12), we find that $\mathcal{L}_X g_0$ is completely determined by the following possibly non-vanishing components $(\mathcal{L}_X g_0)_{ij} = \mathcal{L}_X g_0(\partial_i, \partial_j), i \leq j$:

$$(\mathcal{L}_X g_0)_{14} = (\mathcal{L}_X g_0)_{22} = (\mathcal{L}_X g_0)_{33} = \lambda, \quad (\mathcal{L}_X g_0)_{24} = \lambda x_3, \quad (\mathcal{L}_X g_0)_{44} = -\tfrac{1}{2},$$

which, by (2.5) and (2.7), ensures that the Ricci soliton equation (1.1) is satisfied. Writing $X = u^i e_i$ as a linear combination of left-invariant vector fields $\{e_i\}$ and using (2.9), we conclude that X cannot be left-invariant, as $X^1 + x_3X^2$ cannot be a real constant for any choice of $\lambda, K, b_2, a_3, b_3$.

Finally, we check that the above Ricci soliton is not a gradient one, that is, there does not exist a smooth function $f(x_1, x_2, x_3, x_4)$, such that $X = \text{grad}(f) = \sum_{i,j} g^{ij}\frac{\partial f}{\partial x_i}\partial x_j$, except in the steady case. In fact, suppose that such a function exists. Then, by (3.12), we have

$$\begin{cases} x_3^2\partial_1 f - x_3\partial_2 f + \partial_4 f = \lambda x_1 + \frac{1}{2}\,a_3{x_3}^2 - a_3x_3\cos\,(x_4) + b_3x_3\sin\,(x_4) - \frac{1}{4}\,x_4 \\ -\frac{1}{2}\,a_3{x_2}^2 + K\,x_2 + b_2, \\ -x_3\partial_1 f + \partial_2 f = -a_3x_3 + \frac{1}{2}\,\lambda\,x_2 + a_3\cos\,(x_4) - b_3\sin\,(x_4) + b_2, \\ \partial_3 f = a_3\,x_2 + a_3\sin\,(x_4) + b_3\cos\,(x_4) + \frac{1}{2}\lambda\,x_3 - K, \\ \partial_1 f = b_4. \end{cases} \tag{3.13}$$

Integrating the fourth equation in (3.13), we obtain

$$f(x_1, x_2, x_3, x_4) \;=\; b_4\,x_1 + Q\,(x_2, x_3, x_4)$$

for some smooth function Q. Replacing into the third equation in (3.13), it gives

$$\partial_3 Q\,(x_2, x_3, x_4) = \tfrac{1}{2}\,\lambda\,x_3 + a_3\,x_2 + a_3\sin\,(x_4) - b_3\cos\,(x_4) - K$$

and integrating we obtain

$$Q\,(x_2, x_3, x_4) = \frac{1}{4}\,\lambda\,{x_3}^2 + (a_3\,x_2 + a_3\sin\,(x_4) + b_3\cos\,(x_4)\; - K)\,x_3 + W\,(x_2, x_4)\,,$$

where W is a smooth function. The second equation in (3.13) then reduces to

$$\partial_2 W\left(x_2, x_4\right) = (b_4 - 2a_3)x_3 + \frac{1}{2}\lambda\, x_2 + a_3 \cos\left(x_4\right) - b_3 \sin\left(x_4\right) + b_2,$$

which must hold for all values of x_3. Hence, $b_4 = 2a_3$ and integrating the above we get

$$W\left(x_2, x_4\right) = \frac{1}{4}\lambda\, x_2^2 + \big(a_3 \cos\left(x_4\right) - b_3 \sin\left(x_4\right) + b_2\big)x_2 + S\left(x_4\right),$$

where S is a smooth function. Finally, replacing into the first equation in (3.13) and writing it as a polynomial in x_3, we find

$$\begin{aligned} &a_3 \tfrac{1}{2}\, x_3^2 + \big(-b_2 + \cos\left(x_4\right) a_3 - \sin\left(x_4\right) b_3 - \tfrac{1}{2}\,\lambda\, x_2\big)\, x_3 - \lambda x_1 \\ &- a_3 x_2 \sin\left(x_4\right) - b_3 x_2 \cos\left(x_4\right) + \tfrac{d}{dx_4} S\left(x_4\right) - b_2 + \tfrac{1}{4}\, x_4 + \tfrac{1}{2}\, a_3\, {x_2}^2 - K\, x_2 = 0 \end{aligned} \tag{3.14}$$

for all values of x_3. Therefore, $a_3 = 0$ and (3.14) reduces to

$$\big(-b_2 - \sin\left(x_4\right) b_3 - \tfrac{1}{2}\,\lambda\, x_2\big)\, x_3 - \lambda x_1 - b_3 x_2 \cos\left(x_4\right) + \tfrac{d}{dx_4} S\left(x_4\right) - b_2 + \tfrac{1}{4}\, x_4 - K\, x_2 = 0.$$

The coefficient of x_3 in the above equation must vanish for all values of x_2 and x_4 and so, $b_2 = b_3 = 0$ and $\lambda = 0$, that is, the Ricci soliton is necessarily steady. The above equation now reduces to

$$S'\left(x_4\right) + \frac{1}{4}\, x_4 - K\, x_2 = 0,$$

which must hold for all values of x_2 and so, it yields $K = 0$ and $S\left(x_4\right) = -\frac{1}{8}\, x_4^2 + R$, where R is a real constant. All equations in (3.13) are now satisfied. Therefore, this Ricci soliton is gradient only when $\lambda = 0$. Replacing into $f(x_1, x_2, x_3, x_4)$ we then explicitly have $X = \mathrm{grad}(f)$, where

$$X^1 = -\frac{1}{4}\, x_4, \quad X^2 = X^3 = X^4 = 0 \quad \text{and} \quad f(x_1, x_2, x_3, x_4) = -\frac{1}{8} x_4^2 + R.$$

Thus, we have the following result, which proves part (a) of Theorem 1.1.

Theorem 3.1. *The bi-invariant metric g_0 is a Ricci soliton, which satisfies equation* (1.1) *for any real value of λ, where $X = X^i \partial_i$ is a smooth vector field, whose components X^i with respect to $\{\partial_i\}$ are described by* (3.12). *This vector field X is never left-invariant, and the Ricci soliton is gradient only in the steady case.*

The bi-invariant metric g_0 gives rise to an algebraic Ricci soliton only when $\lambda = 0$ [21]. On the other hand, we proved that g_0 satisfies equation (1.1) for any value of λ. As a consequence, g_0 also gives rise to a Yamabe soliton. In fact, for any distinct real constants λ_1, λ_2, let us consider two smooth vector fields X_{λ_1}, X_{λ_2}, with components of the form (3.12) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. Since X_{λ_1}, X_{λ_2} satisfy the Ricci soliton equation (1.1), vector field $Y = X_{\lambda_1} - X_{\lambda_2}$ then satisfies

$$\mathcal{L}_Y g_0 = \mathcal{L}_{X_{\lambda_1}} g_0 - \mathcal{L}_{X_{\lambda_2}} g_0 = (\lambda_1 - \lambda_2) g_0.$$

Since the scalar curvature of g_0 vanishes and $\lambda_1 - \lambda_2 \neq 0$, Y is a nontrivial solution of the Yamabe soliton equation (1.2). This proves Corollary 1.2.

4. The solutions for g_a, $a \neq 0$

We now restart from (3.1) and determine solutions to the Ricci soliton equation under the assumption that $a \neq 0$. As we already observed, equations (3.2) are valid for any admissible value of a. Since $a \neq 0$, equation (3.3) now gives at once

$$X^2(x_1,x_2,x_3,x_4) = \tfrac{1}{a}\left(\partial_1 F_3(x_1,x_2,x_4) + \partial_3 F_4(x_2,x_3,x_4)\right).$$

Hence, multiplying by a when needed, system (3.1) now becomes

(4.1)
$$\begin{cases} 2\partial^2{}_{11}F_3(x_1,x_2,x_4) + a^3x_3 + 2\,a^2F_3(x_1,x_2,x_4) + 2\,a\partial_2F_4(x_2,x_3,x_4) = 0, \\ 4\,a^3\partial_1X^1 - 4\,a\partial_1X^1 - 4\,x_3\partial^2_{11}F_3(x_1,x_2,x_4) + 4\,a^2x_3\,\partial^2_{11}F_3(x_1,x_2,x_4) \\ \quad - 4\,a\partial_4F_4(x_2,x_3,x_4) + a^4 - 2\,a^2 - 2\,a^3\lambda + 4a\,\lambda = 0, \\ 4\,\partial^2_{12}F_3(x_1,x_2,x_4) + 4\,\partial^2_{23}F_4(x_2,x_3,x_4) + 2\,a^3{x_3}^2 + 4\,a^2x_3F_3(x_1,x_2,x_4) \\ \quad +4a\,x_3\partial_2F_4(x_2,x_3,x_4) - a^2 - 2\,\lambda\,a = 0, \\ \partial^2_{33}F_4(x_2,x_3,x_4) + a\partial_2F_3(x_1,x_2,x_4) - ax_3\,\partial_1F_3(x_1,x_2,x_4) = 0, \\ 4\,a^3\partial_2X^1 - 4\,a\partial_2X^1 - 4\,x_3\partial^2_{12}F_3(x_1,x_2,x_4) - 4\,x_3\partial^2_{23}F_4(x_2,x_3,x_4) - 4\,\partial^2_{14}F_3(x_1,x_2,x_4) \\ \quad - 4\,\partial^2_{34}F_4(x_2,x_3,x_4) - 3\,a^2x_3 + 2\,\lambda\,ax_3 - 4\,aF_3(x_1,x_2,x_4) + 4\,a^2x_3\partial^2_{12}F_3(x_1,x_2,x_4) \\ \quad + 4\,a^2x_3\partial^2_{23}F_4(x_2,x_3,x_4) - 4\,a^2\partial_2F_4(x_2,x_3,x_4) - 4a\,x_3\partial_4F_4(x_2,x_3,x_4) = 0, \\ a^3\partial_3X^1 - a\partial_3X^1 - x_3\partial^2_{33}F_4(x_2,x_3,x_4) - a\partial_4F_3(x_1,x_2,x_4) + a^2\partial_1F_3(x_1,x_2,x_4) \\ \quad + a^2x_3\partial^2_{33}F_4(x_2,x_3,x_4) = 0, \\ 4\,a^3\partial_4X^1 - 4\,a\partial_4X^1 - 4\,x_3\partial^2_{14}F_3(x_1,x_2,x_4) - 4\,x_3\partial^2_{34}F_4(x_2,x_3,x_4) - 4\,a^2\partial_4F_4(x_2,x_3,x_4) \\ \quad + 4\,a^2x_3\,\partial^2_{14}F_3(x_1,x_2,x_4) + 4\,a^2x_3\,\partial^2_{34}F_4(x_2,x_3,x_4) - a + 2\,\lambda\,a^2 = 0. \end{cases}$$

The argument we follow is then similar to the one used in the previous Section for the case $a = 0$, that is, we integrate the equations in (4.1) one by one and each time we replace the corresponding solutions into the system. For this reason, we shall skip a few details.

Integrating the first equation of (4.1) (in which we observe that F_3 depends on (x_1,x_2,x_4) while F_4 depends on (x_2,x_3,x_4)), by a standard calculation we find

$$\begin{aligned} F_3(x_1,x_2,x_4) &= \sin(ax_1)\,G_3(x_2,x_4) + \cos(ax_1)\,H_3(x_2,x_4) - \tfrac{1}{a}\partial_2G_4(x_2,x_4), \\ F_4(x_2,x_3,x_4) &= G_4(x_2,x_4) + H_4(x_3,x_4) - \tfrac{1}{2}\,a^2x_2x_3, \end{aligned}$$

for some smooth functions G_3, G_4, H_3, H_4. We now substitute the above expressions to F_3 and F_4 into (4.1) and it becomes

(4.2)

$$\begin{cases} \partial_1 X^1 - a^2\partial_1 X^1 - a x_3\, \sin(a x_1)\, G_3(x_2,x_4) - a x_3\, \cos(a x_1)\, H_3(x_2,x_4) \\ \; + a^3 x_3 \sin(a x_1)\, G_3(x_2,x_4) + a^3 x_3 \cos(a x_1)\, H_3(x_2,x_4) \\ \; - \frac14 a^3 + \frac12 \lambda\, a^2 + \partial_4 G_4(x_2,x_4) + \partial_4 H_4(x_3,x_4) + \frac12 a - \lambda = 0, \\ 2\, \cos(a x_1)\, \partial_2 G_3(x_2,x_4) - 2\, \sin(a x_1)\, \partial_2 H_3(x_2,x_4) - \frac32 a + 2\, a x_3\, \sin(a x_1)\, G_3(x_2,x_4) \\ \; + 2\, a x_3\, \cos(a x_1)\, H_3(x_2,x_4) - \lambda = 0, \\ \partial^2_{33} H_4(x_3,x_4) + a \sin(a x_1)\, \partial_2 G_3(x_2,x_4) + a\cos(a x_1)\, \partial_2 H_3(x_2,x_4) - \partial^2_{22} G_4(x_2,x_4) \\ \; - a^2 x_3 \cos(a x_1)\, G_3(x_2,x_4) + a^2 x_3 \sin(a x_1)\, H_3(x_2,x_4) = 0, \\ 4\, a^3 \partial_2 X^1 - 4\, a\partial_2 X^1 - a^2 x_3 + 2\,\lambda\, a x_3 + 4\, a^3 x_3\, \cos(a x_1) + \partial_2 G_3(x_2,x_4) \\ \; - 4\, a^3 x_3 \sin(x_1 a)\, \partial_2 H_3(x_2,x_4) - 4\, a \sin(a x_1)\, G_3(x_2,x_4) - 4\, a\cos(a x_1)\, H_3(x_2,x_4) \\ \; + 4\, \partial_2 G_4(x_2,x_4) - 4\, a x_3\, \cos(x_1 a)\, \partial_2 G_3(x_2,x_4) + 4\, a x_3\, \sin(x_1 a)\, \partial_2 H_3(x_2,x_4) \\ \; + 4\, a \sin(a x_1)\, \partial_4 H_3(x_2,x_4) - 4\, a\cos(a x_1)\, \partial_4 G_3(x_2,x_4) - 4\, \partial^2_{34} H_4(x_3,x_4) \\ \; - 4\, a^2 \partial_2 G_4(x_2,x_4) - 4\, a x_3\, \partial_4 G_4(x_2,x_4) - 4\, a x_3\, \partial_4 H_4(x_3,x_4) = 0, \\ a^3 \partial_3 X^1 - a\partial_3 X^1 - x_3\, \partial^2_{33} H_4(x_3,x_4) - a \sin(a x_1)\, \partial_4 G_3(x_2,x_4) - a\cos(a x_1)\, \partial_4 H_3(x_2,x_4) \\ \; + \partial^2_{24} G_4(x_2,x_4) + a^3 \cos(a x_1)\, G_3(x_2,x_4) - a^3 \sin(a x_1)\, H_3(x_2,x_4) + a^2 x_3\, \partial^2_{33} H_4(x_3,x_4) = 0, \\ 4\, a^3 \partial_4 X^1 - 4\, a\partial_4 X^1 - 4\, a x_3 \cos(a x_1)\, \partial_4 G_3(x_2,x_4) + 4\, x_3 \sin(a x_1)\, \partial_4 H_3(x_2,x_4) \\ \; - 4\, x_3 \partial^2_{34} H_4(x_3,x_4) + 4\, a^3 x_3 \cos(x_1 a)\, \partial_4 G_3(x_2,x_4) - 4\, a^3 x_3 \sin(a x_1)\, \partial_4 H_3(x_2,x_4) \\ +4\, a^2 x_3 \partial^2_{34} H_4(x_3,x_4) - 4\, a^2 \partial_4 G_4(x_2,x_4) - 4\, a^2 \partial_4 H_4(x_3,x_4) - a + 2\,\lambda\, a^2 = 0. \end{cases}$$

The second equation in the above system (4.2) is a linear combination of functions $\cos(a x_1)$, $\sin(a x_1)$ and x_1^0, with coefficients independent of variable x_1, that is,

$$2\big(\partial_2 G_3(x_2,x_4) + a x_3 H_3(x_2,x_4)\big)\cos(a x_1) - 2\big(\partial_2 H_3(x_2,x_4) - a x_3\, G_3(x_2,x_4)\big)\sin(a x_1) \\ - \tfrac32 a - \lambda = 0.$$

Since this equation must hold for any value of x_1 (and x_3), it then easily implies $\lambda = -\frac32 a$ and $G_3(x_2,x_4) = H_3(x_2,x_4) = 0$. Therefore, system (4.2) reduces to

(4.3)

$$\begin{cases} \partial_1 X^1 - a^2\partial_1 X^1 - a^3 + \partial_4 G_4(x_2,x_4) + \partial_4 H_4(x_3,x_4) + 2\, a = 0, \\ \partial^2_{22} G_4(x_2,x_4) - \partial^2_{33} H_4(x_3,x_4) = 0, \\ a^3 \partial_2 X^1 - a\partial_2 X^1 - a^2 x_3 - \partial^2_{34} H_4(x_3,x_4) - a^2 \partial_2 G_4(x_2,x_4) - a x_3\, \partial_4 G_4(x_2,x_4) \\ \; - a x_3\, \partial_4 H_4(x_3,x_4) + \partial_2 G_4(x_2,x_4) = 0, \\ a^3 \partial_3 X^1 - a\partial_3 X^1 - x_3 \partial^2_{33} H_4(x_3,x_4) + \partial^2_{24} G_4(x_2,x_4) + a^2 x_3\, \partial^2_{33} H_4(x_3,x_4) = 0, \\ 4\, a^3 \partial_4 X^1 - 4\, a\partial_4 X^1 - 4\, x_3 \partial^2_{34} H_4(x_3,x_4) + 4\, a^2 x_3 \partial^2_{34} H_4(x_3,x_4) \\ \; - 4\, a^2 \partial_4 G_4(x_2,x_4) - 4\, a^2 \partial_4 H_4(x_3,x_4) - a - 3\, a^3 = 0. \end{cases}$$

Integrating the first equation of (4.3), we find

$$X_1 \;=\; \tfrac{1}{a^2-1} x_1 \left(\partial_4 G_4\,(x_2,x_4) + \partial_4 H_4\,(x_3,x_4) + 2\,a - a^3\right) + F_1\,(x_2,x_3,x_4)$$

and substituting the above into system (4.3), it becomes

(4.4)

$$\begin{cases} \partial^2_{22} G_4\,(x_2,x_4) - \partial^2_{33} H_4\,(x_3,x_4) = 0, \\ a^3\partial_2 F_1\,(x_2,x_3,x_4) - a^2\partial_2 G_4\,(x_2,x_4) - a^2 x_3 - a\partial_2 F_1\,(x_2,x_3,x_4) + a x_1\,\partial^2_{24} G_4\,(x_2,x_4) \\ \;- a x_3\,\partial_4 G_4\,(x_2,x_4) - a x_3\,\partial_4 H_4\,(x_3,x_4) + \partial_2 G_4\,(x_2,x_4) - \partial^2_{34} H_4\,(x_3,x_4) = 0, \\ a^3\partial_3 F_1\,(x_2,x_3,x_4) + a^2 x_3\,\partial^2_{33} H_4\,(x_3,x_4) + a x_1\,\partial^2_{34} H_4\,(x_3,x_4) - a\partial_3 F_1\,(x_2,x_3,x_4) \\ \;- x_3\partial^2_{33} H_4\,(x_3,x_4) + \partial^2_{24} G_4\,(x_2,x_4) = 0, \\ 4a^3\partial_4 F_1\,(x_2,x_3,x_4) - 3\,a^3 - 4\,a^2\partial_4 G_4\,(x_2,x_4) + 4\,a^2 x_3\partial^2_{34} H_4\,(x_3,x_4) \\ \;- 4\,a^2\partial_4 H_4\,(x_3,x_4) - a + 4a x_1\,\partial^2_{44} H_4\,(x_3,x_4) + 4a x_1\,\partial^2_{44} G_4\,(x_2,x_4) \\ \;- 4\,a\partial_4 F_1\,(x_2,x_3,x_4) - 4\,x_3\partial^2_{34} H_4\,(x_3,x_4) = 0. \end{cases}$$

By the second equation of system (4.4) (which must hold for any value of x_1), we have $\partial^2_{24} G_4\,(x_2,x_4) = 0$ and integrating we obtain

$$G_4\,(x_2,x_4) \;=\; U_4\,(x_2) + V_4\,(x_4)\,.$$

Replacing into the first equation of (4.4), it gives

$$\partial^2_{33} H_4\,(x_3,x_4) - U_4''\,(x_2) = 0,$$

where H_4 only depends on (x_3,x_4) and U_4 only on x_2. Therefore, there exists some real constant H_1, such that

$$\partial^2_{33} H_4\,(x_3,x_4) = U_4''\,(x_2) = H_1.$$

Integrating, we get

$$H_4\,(x_3,x_4) \;=\; \tfrac{1}{2} H_1\,{x_3}^2 + x_3\,P_4\,(x_4) + Q_4\,(x_4)\,, \quad U_4\,(x_2) \;=\; \tfrac{1}{2} H_1\,{x_2}^2 + a_4\,x_2 + b_4,$$

for some smooth functions P_4, Q_4 and real constants a_4, b_4. Replacing the above expressions of G_4, H_4 and U_4, system (4.4) becomes

$$\text{(4.5)}\quad \begin{cases} -a^3\partial_2 F_1\,(x_2,x_3,x_4) + a^2 x_3 + a^2 H_1\,x_2 + a_4 a^2 + a\partial_2 F_1\,(x_2,x_3,x_4) + a x_3 V_4'\,(x_4) \\ \;+ a {x_3}^2 P_4'\,(x_4) + a x_3 Q_4'\,(x_4) + P_4'\,(x_4) - H_1\,x_2 - a_4 = 0, \\ a^3\partial_3 F_1\,(x_2,x_3,x_4) + a^2 H_1 x_3 - a\partial_3 F_1\,(x_2,x_3,x_4) + a x_1 P_4'\,(x_4) - H_1\,x_3 = 0, \\ 4a^3\partial_4 F_1\,(x_2,x_3,x_4) - 3\,a^3 - 4\,a^2 Q_4'\,(x_4) - 4\,a^2 V_4'\,(x_4) + 4\,a x_1 x_3\,P_4''\,(x_4) \\ \;+ 4\,a x_1 Q_4''\,(x_4) - a + 4\,a x_1 V_4''\,(x_4) - 4\,a\partial_4 F_1\,(x_2,x_3,x_4) - 4 x_3\,P_4'\,(x_4) = 0. \end{cases}$$

In the second equation of (4.5), the only term involving x_1 is $a x_1 P_4'\,(x_4)$, and $a \neq 0$. Henceforth, $P_4'\,(x_4) = 0$, that is, $P_4\,(x_4) = c_4$, for some real constant c_4. The second equation in (4.5) then reduces to

$$a(a^2-1)\partial_3 F_1\,(x_2,x_3,x_4) + (a^2-1)H_1 x_3 = 0,$$

which, by integration, gives

$$F_1(x_2,x_3,x_4) \;=\; -\tfrac{1}{2a}H_1\,{x_3}^2 + G_1(x_2,x_4)\,,$$

so that (4.5) now reduces to

$$(4.6)\quad \begin{cases} a\partial_2 G_1(x_2,x_4) - a^3\partial_2 G_1(x_2,x_4) + a^2 H_1\,x_2 + a^2 x_3 + a^2 a_4 \\ \;+ a x_3 V_4'(x_4) + a x_3 Q_4'(x_4) - H_1\,x_2 - a_4 = 0, \\ 2\,\partial_4 G_1(x_2,x_4) - 2\,a^2\partial_4 G_1(x_2,x_4) + \tfrac{1}{2} + 2\,aQ_4'(x_4) + 2\,aV_4'(x_4) - 2\,x_1 Q_4''(x_4) \\ \;+ \tfrac{3}{2}\,a^2 - 2\,x_1 V_4''(x_4) = 0. \end{cases}$$

The first equation in (4.6) is determined by a polynomial in the variable x_3, where the coefficients of x_3^1 and x_3^0 must vanish, that is,

$$\begin{aligned} &a^2 + aV_4'(x_4) + aQ_4'(x_4) = 0, \\ &a\partial_2 G_1(x_2,x_4) - a^3\partial_2 G_1(x_2,x_4) + a^2 H_1\,x_2 + a^2 a_4 - H_1\,x_2 - a_4 = 0. \end{aligned}$$

Taking into account that $a(a^2-1) \neq 0$ and integrating, we then get

$$Q_4(x_4) \;=\; -V_4(x_4) - a x_4 + r_4, \quad G_1(x_2,x_4) \;=\; \tfrac{1}{2a}H_1\,{x_2}^2 + \tfrac{a_4}{a}\,x_2 + P_1(x_4)\,,$$

for some smooth function P_1 and a real constant r_4. Replacing the above into system (4.6), it reduces to the only equation

$$\tfrac{1}{2}(1-a^2) + 2(1-a^2)P_1'(x_4) = 0,$$

which, since $a^2-1 \neq 0$, yields

$$P_1(x_4) \;=\; -\tfrac{1}{4}\,x_4 + s_4,$$

for some real constant s_4. All equations in (3.1) are now satisfied. We replace the functions we found above into X^i and we find

$$(4.7)\qquad \begin{cases} X^1 = \tfrac{1}{4a}\left(-4a^2\,x_1 + 2\,H_1\,{x_2}^2 + 4\,a_4\,x_2 - 2\,H_1\,{x_3}^2 - a x_4 + 4\,a s_4\right), \\ X^2 = \tfrac{1}{2a}\left(2\,H_1\,x_3 - a^2 x_2 + 2\,c_4\right), \\ X^3 = -\tfrac{1}{2a}\left(2\,H_1\,x_2 + a^2 x_3 + 2\,a_4\right), \\ X^4 = -\tfrac{3}{4}a x_4 - a s_4 + b_4 + r_4. \end{cases}$$

Computing $\mathcal{L}_X g_0$, where $X = X^i\partial_i$ with X_i given by (4.7), we find that $\mathcal{L}_X g_0$ is completely determined by the following possibly non-vanishing components $(\mathcal{L}_X g_0)_{ij} = \mathcal{L}_X g_0(\partial_i,\partial_j), i \leq j$:

$$\begin{aligned} &(\mathcal{L}_X g_0)_{11} = -2a^2, && (\mathcal{L}_X g_0)_{12} = -2a^2 x_3, && (\mathcal{L}_X g_0)_{14} = -2a, \\ &(\mathcal{L}_X g_0)_{22} = -2a^2 x_3^2 - a, && (\mathcal{L}_X g_0)_{24} = -2a x_3, && (\mathcal{L}_X g_0)_{33} = -a, \\ &(\mathcal{L}_X g_0)_{44} = -\tfrac{3}{2}a^2 - \tfrac{1}{2}. \end{aligned}$$

Therefore, by (2.5) and (2.7), the Ricci soliton equation (1.1) is satisfied.

It easily follows from equations (2.9) (4.7) that X is never left-invariant. In fact, writing $X = u^i e_i$ as a linear combination of left-invariant vector fields $\{e_i\}$, we see that $u_4 = X^4 = -\frac{3}{4}ax_4 - as_4 + b_4 + r_4$ cannot be constant, since $a \neq 0$.

We now prove that this Ricci soliton is never a gradient one. In fact, suppose that there exists a smooth function $f(x_1, x_2, x_3, x_4)$, such that $X = \mathrm{grad}(f)$. Then, (4.7) yields

$$\left\{\begin{array}{l} \frac{1}{a^2-1}\left(a\,\partial_1 f + (a^2-1)x_3^2\,\partial_1 f - (a^2-1)x_3\,\partial_2 f - \partial_4 f\right) \\ \quad = \frac{1}{4a}\left(-4a^2\,x_1 + 2\,H_1\,x_2{}^2 + 4\,a_4\,x_2 - 2\,H_1\,x_3{}^2 - ax_4 + 4\,as_4\right), \\ -x_3\partial_1 f + \partial_2 f = \frac{1}{2a}\left(2\,H_1\,x_3 - a^2 x_2 + 2\,c_4\right), \\ \partial_3 f = -\frac{1}{2a}\left(2\,H_1\,x_2 + a^2 x_3 + 2\,a_4\right), \\ -\frac{1}{a^2-1}\left(\partial_1 f - a\,\partial_4 f\right) = -\frac{3}{4}ax_4 - as_4 + b_4 + r_4. \end{array}\right. \tag{4.8}$$

Integrating the third equation in (4.8), we find

$$f(x_1, x_2, x_3, x_4) = -\tfrac{1}{4}ax_3{}^2 - \tfrac{1}{a}\left(H_1 x_2 + a_4\right)x_3 + Q(x_1, x_2, x_4),$$

for some smooth function Q. We replace into the second equation in (4.8) and obtain

$$-x_3\partial_1 Q(x_1, x_2, x_4) - \tfrac{1}{a}H_1 x_3 + \partial_2 Q(x_1, x_2, x_4) = \tfrac{1}{2a}\left(2\,H_1\,x_3 - a^2 x_2 + 2\,c_4\right). \tag{4.9}$$

Since (4.9) must hold for all values of x_3, in particular it implies $\partial_1 Q(x_1, x_2, x_4) = -\frac{2}{a}H_1$, which, integrated, gives

$$Q(x_1, x_2, x_4) = -\frac{2}{a}H_1 x_1 + W(x_2, x_4).$$

Replacing into (4.9), it now reduces to $\partial_2 W(x_2, x_4) = \frac{1}{2a}\left(-a^2 x_2 + 2\,c_4\right)$, which by integration yields

$$W(x_2, x_4) = -\frac{1}{4}ax_2^2 + \frac{1}{a}c_4 x_2 + S(x_4),$$

for some smooth function S. Finally, replacing into the first equation of (4.8), we have

$$\begin{array}{l} -\frac{1}{a(a^2-1)}\left(2a\,H_1 + (a^2-1)H_1\,x_3^2 + (a^2-1)\left(c_4 - \frac{1}{2}a^2 x_2\right)x_3 + aS'(x_4)\right) \\ \quad = \frac{1}{4a}\left(-4a^2\,x_1 + 2\,H_1\,x_2{}^2 + 4\,a_4\,x_2 - 2\,H_1\,x_3{}^2 - ax_4 + 4\,as_4\right). \end{array}$$

The above equation is polynomial in x_2 and x_3, and the coefficient of $x_2 x_3$ is $\frac{a}{2} \neq 0$. Therefore, the above equation cannot hold for all values of x_2 and x_3 and so, the Ricci soliton cannot be gradient. The above results, which prove part (b) of Theorem 1.1, are summarized as follows.

Theorem 4.1. *The (non-isometric) left-invariant metrics g_a, for any value of $a \in]-1, 1[$, $a \neq 0$, are Ricci solitons, which satisfy equation (1.1), where $X = X^i \partial_i$ is a smooth vector field, whose components X^i with respect to $\{\partial_i\}$ are described by (4.7), and $\lambda = -\frac{3}{2}a$. In particular, this Ricci soliton is either expanding or shrinking, depending on whether $a > 0$ or $a < 0$. This vector field X is never left-invariant, and the Ricci soliton is not gradient.*

References

[1] W. Batat, M. Castrillon-Lopez and E. Rosado, *Four-dimensional naturally reductive pseudo-Riemannian spaces*, Diff. Geom. Appl. **41** (2015), 48–64.

[2] M. Boucetta and A. Medina, *Solutions of the Yang-Baxter equations on quadratic Lie groups: the case of oscillator groups*, J. Geom. Phys. **61** (2011), 2309-2320.

[3] S. Bromberg and A. Medina, *Geometry of oscillator groups and locally symmetric manifolds*, Geom. Dedicata **106** (2004), 97–111.

[4] M. Brozos-Vazquez, G. Calvaruso, E. Garcia-Rio and S. Gavino-Fernandez, *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math. **188** (2012), 385–403.

[5] M. Brozos-Vazquez, E. Garcia-Rio and S. Gavino-Fernandez, *Locally conformally flat Lorentzian gradient Ricci solitons*, J. Geom. Anal. **23** (2013), 1196–1212.

[6] M. Brozos-Vazquez, E. Garcia-Rio, P. Gilkey and S. Gavino-Fernandez, *The structure of the Ricci tensor on locally homogeneous Lorentzian gradient Ricci solitons*, ArXiv: 1403.4400.

[7] G. Calvaruso and A. Fino, *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces*, Canad. J. Math. **64** (2012), no. 4, 778–804.

[8] G. Calvaruso and A. Fino, *Four-dimensional pseudo-Riemannian homogeneous Ricci solitons*, Int. J. Geom. Methods Mod. Phys. **12** (2015), 1550056 (21 pages).

[9] G. Calvaruso and J. Van der Veken, *Totally geodesic and parallel hypersurfaces of four-dimensional oscillator groups*, Results Math. 64 (2013), 135–153.

[10] G. Calvaruso and A. Zaeim, *A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous* 4*-spaces*, J. Geom. Phys. **80** (2014), 15–25.

[11] G. Calvaruso and A. Zaeim, *On the symmetries of the Lorentzian oscillator group*, submitted.

[12] H.-D. Cao, *Recent progress on Ricci solitons*, Adv. Lect. Math. (ALM), **11** (2009), 1-38.

[13] R. Duran Diaz, P.M. Gadea and J.A. Oubiña, *The oscillator group as a homogeneous spacetime*, Libertas Math. **19** (1999), 9–18.

[14] R. Duran Diaz, P.M. Gadea and J.A. Oubiña, *Reductive decompositions and Einstein-Yang-Mills equations associated to the oscillator group*, J. Math. Phys. **40** (1999), 3490–3498.

[15] P.M. Gadea and J.A. Oubiña, *Homogeneous Lorentzian structures on the oscillator groups*, Arch. Math. **73** (1999), 311–320.

[16] R.S. Hamilton, *The Ricci flow on surfaces*, Contemporary Mathematics **71** (1988) 237–261.

[17] M. Jablonski, *Homogeneous Ricci solitons*, J. Reine Angew. Math., **699** (2015), 159–182.

[18] J. Lauret, *Ricci solitons solvmanifolds*, J. Reine Angew. Math. **650** (2011), 1–21.

[19] A.V. Levitchev, *Chronogeometry of an electromagnetic wave given by a bi-invariant metric on the oscillator group*, Siberian Math. J. **27** (1986), 237–245.

[20] D. Müller and F. Ricci, *On the Laplace-Beltrami operator on the oscillator group*, J. Reine Angew. Math. **390** (1988), 193–207.

[21] K. Onda, *Examples of algebraic Ricci solitons in the pseudo-Riemannian case*, Acta Math. Hung. **144** (2014), 247–265.

[22] R.F. Streater, *The representations of the oscillator group*, Commun. Math. Phys. **4** (1967), 217–236.

Giovanni Calvaruso: Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Prov. Lecce-Arnesano, 73100 Lecce, Italy.

Email address: giovanni.calvaruso@unisalento.it