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Static and free vibration analysis of anisotropic doubly-curved shells with general boundary conditions

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Abstract. In the present work, a two-dimensional model based on a higher order Layer-Wise (LW) approach is presented for the static and dynamic analysis of doubly-curved anisotropic shell structures. The Equivalent Single Layer (ESL) methodology is also obtained as particular case of LW. Each lamina of the stacking sequence is modelled as an anisotropic continuum. The fundamental equations account for both surface and concentrated loads, as well as the effects of the Winkler-Pasternak foundation. Moreover, non-conventional boundary conditions are introduced, and the numerical solution is assessed from the Generalized Differential Quadrature (GDQ) method. The proposed formulation is validated with respect to refined three-dimensional simulations, pointing out its accuracy and computational efficiency.

Introduction

In many engineering applications, layered structures with complex shapes are very frequently adopted in many branches of engineering. In this context, novel design perspectives require more complicated models capable of providing accurate predictions in terms of structural response.

Among two-dimensional methodologies, the Layer-Wise (LW) formulation [1] seems to provide very accurate results with respect to three-dimensional solutions, accounting for the compatibility conditions at the interface between adjacent laminae. More specifically, the governing equations are solved directly within each lamina. On the other hand, when the Equivalent Single Layer (ESL) approach [2-3] is adopted, a reference surface is provided for the entire structure, and a higher order through-the-thickness expansion of the field variable is adopted taking into account a generalized formulation.

From literature, closed-form solutions can be derived only for a limited number of cases, such that numerical procedures like classical finite elements are more suitable to solve approximately more complicated cases. In this context, refined simulations can be very high computationally demanding, thus spectral collocation approaches like the Generalized Differential Quadrature (GDQ) are adopted [4] since they lead to very accurate results with a reduced number of Degrees of Freedom (DOFs).

In the present contribution, a generalized higher order two-dimensional formulation based on a LW approach is proposed to study the linear statics and dynamics of laminated shell structures featuring a double curvature, general lamination schemes, and enforced with unconventional external constrains [5]. Then, a unified higher order ESL theory is outlined as a particular case of the LW. A numerical solution of the fundamental equations is provided, taking into account the GDQ method. A doubly-curved shell structure is here selected as benchmark, characterized by a softcore lamination scheme, and unconventional boundary conditions. The results are compared to those ones obtained from a 3D Finite Element Method (FEM), pointing out the accuracy of the proposed formulation, and its computational efficiency. The present ESL and LW higher order formulations have been implemented in the DiQuMASPAB software [4], and all the material properties are obtained from its database.

Theoretical formulation

Let consider a laminated doubly-curved shell made of l laminae, described in a reference surface $\mathbf{r}(\alpha_1,\alpha_2)$ located in its middle thickness. In particular, a global coordinate system $O'\alpha_1,\alpha_2\zeta$ is introduced starting from the principal directions of $\mathbf{r}(\alpha_1,\alpha_2)$. Furthermore, a local coordinate system $O'\alpha_1,\alpha_2\zeta^{(k)}$ is assessed in each k-th layer of thickness h_k, with $k=1,...,l$. As a consequence, the position vector $\mathbf{R}^{(k)}(\alpha_1,\alpha_2,\zeta)$ of an arbitrary point of the shell can be described as [1]:

$$\mathbf{R}^{(k)}\left(\alpha_1,\alpha_2,\zeta^{(k)}\right)=\mathbf{r}\left(\alpha_1,\alpha_2\right)+\left(\frac{\zeta_{k+1}+\zeta_k}{2}+\frac{h_k}{2}z_k\right)\mathbf{n}\left(\alpha_1,\alpha_2\right) \tag{1}$$

where $(\alpha_1,\alpha_2)\in[\alpha_1^0,\alpha_1^1]\times[\alpha_2^0,\alpha_2^1]$ and $\zeta\in[\zeta_k,\zeta_{k+1}]$. In addition, ζ_k,ζ_{k+1} denote the locations of the intrados and the extrados of the lamina at issue in the global reference system, respectively, whereas the dimensionless local out-of-plane coordinate is defined as $z_k=2\zeta^{(k)}/h_k$. In other words, in Eq. (1) a midsurface is provided for each lamina, so that the global and the local out-of-plane coordinates ζ and $\zeta^{(k)}$ are related as:

$$d\zeta^{(k)}=d\zeta \tag{2}$$

On the other hand, the geometry of the structure can be described in the ESL framework in terms of the global thickness coordinate ζ, as follows [2]:

$$\mathbf{R}\left(\alpha_1,\alpha_2,\zeta\right)=\mathbf{r}\left(\alpha_1,\alpha_2\right)+\zeta\mathbf{n}\left(\alpha_1,\alpha_2\right) \tag{3}$$

Referring to the local geometric reference system $O'\alpha_1\alpha_2\zeta^{(k)}$, the three-dimensional displacement field vector $\mathbf{U}^{(k)}\left(\alpha_1,\alpha_2,\zeta^{(k)}\right)=\left[U_1^{(k)}\ U_2^{(k)}\ U_3^{(k)}\right]^T$ is described by means of generalized thickness functions $F_\tau^{\alpha_i(k)}$ for $i=1,...,3$ collected in the matrix $\mathbf{F}_\tau^{(k)}$ defined in each k-th layer for each $\tau=0,...,N+1$. Thus, $U_i^{(k)}$ is expressed in terms of the so-called generalized displacement field components $u_i^{(k\tau)}$, setting $i=1,2,3$:

$$\begin{bmatrix}U_1^{(k)}\\U_2^{(k)}\\U_3^{(k)}\end{bmatrix}=\sum_{\tau=0}^{N+1}\begin{bmatrix}F_\tau^{\alpha_1(k)} & 0 & 0\\0 & F_\tau^{\alpha_2(k)} & 0\\0 & 0 & F_\tau^{\alpha_3(k)}\end{bmatrix}\begin{bmatrix}u_1^{(k\tau)}\\u_2^{(k\tau)}\\u_3^{(k\tau)}\end{bmatrix}\leftrightarrow\mathbf{U}^{(k)}\left(\alpha_1,\alpha_2,\zeta^{(k)}\right)=\sum_{\tau=0}^{N+1}\mathbf{F}_\tau^{(k)}\mathbf{u}^{(k\tau)} \tag{4}$$

Based on the ESL approach, the relation $\mathbf{u}^{(k\tau)}=\mathbf{u}^{(\tau)}$ should be considered in Eq. (4). The constitutive equation considered in the problem is valid for generally anisotropic materials relating each component of the three-dimensional stress and strain vectors $\boldsymbol{\sigma}^{(k)}$ and $\boldsymbol{\varepsilon}^{(k)}$:

$$\boldsymbol{\sigma}^{(k)} = \overline{\mathbf{E}}^{(k)}\boldsymbol{\varepsilon}^{(k)} \leftrightarrow \begin{bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \tau_{12}^{(k)} \\ \tau_{13}^{(k)} \\ \tau_{23}^{(k)} \\ \sigma_3^{(k)} \end{bmatrix} = \begin{bmatrix} \overline{E}_{11}^{(k)} & \overline{E}_{12}^{(k)} & \overline{E}_{16}^{(k)} & \overline{E}_{14}^{(k)} & \overline{E}_{15}^{(k)} & \overline{E}_{13}^{(k)} \\ \overline{E}_{12}^{(k)} & \overline{E}_{22}^{(k)} & \overline{E}_{26}^{(k)} & \overline{E}_{24}^{(k)} & \overline{E}_{25}^{(k)} & \overline{E}_{23}^{(k)} \\ \overline{E}_{16}^{(k)} & \overline{E}_{26}^{(k)} & \overline{E}_{66}^{(k)} & \overline{E}_{46}^{(k)} & \overline{E}_{56}^{(k)} & \overline{E}_{36}^{(k)} \\ \overline{E}_{14}^{(k)} & \overline{E}_{24}^{(k)} & \overline{E}_{46}^{(k)} & \overline{E}_{44}^{(k)} & \overline{E}_{45}^{(k)} & \overline{E}_{34}^{(k)} \\ \overline{E}_{15}^{(k)} & \overline{E}_{25}^{(k)} & \overline{E}_{56}^{(k)} & \overline{E}_{45}^{(k)} & \overline{E}_{55}^{(k)} & \overline{E}_{35}^{(k)} \\ \overline{E}_{13}^{(k)} & \overline{E}_{23}^{(k)} & \overline{E}_{36}^{(k)} & \overline{E}_{34}^{(k)} & \overline{E}_{35}^{(k)} & \overline{E}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \varepsilon_1^{(k)} \\ \varepsilon_2^{(k)} \\ \gamma_{12}^{(k)} \\ \gamma_{13}^{(k)} \\ \gamma_{23}^{(k)} \\ \varepsilon_3^{(k)} \end{bmatrix} \quad \text{for} \quad k = 1,...,l \tag{5}$$

The elastic stiffness matrix $\overline{\mathbf{E}}^{(k)} = \mathbf{T}^{(k)}\mathbf{E}^{(k)}\left(\mathbf{T}^{(k)}\right)^T$ of Eq. (5), referred to the geometric reference system $O'\alpha_1\alpha_2\zeta^{(k)}$, is derived from the rotation transformation $\mathbf{T}^{(k)}\left(\vartheta_k\right)$ of matrix $\mathbf{E}^{(k)}$, with coefficients $E_{ij}^{(k)}$ for $i, j = 1,...,6$, associated to the reference system of the material [4]. The fundamental equations and the related boundary conditions are derived from the Hamiltonian Principle, accounting for the virtual variation of the elastic strain energy, kinetic energy and external work. Referring to an arbitrary τ-th kinematic expansion order, one gets:

$$\sum_{\eta=0}^{N+1}\mathbf{L}^{(k\tau\eta)}\mathbf{u}^{(k\eta)} - \sum_{\eta=0}^{N+1}\mathbf{M}^{(k\tau\eta)}\ddot{\mathbf{u}}^{(k\eta)} + \mathbf{q}^{(k\tau)} = \mathbf{0} \qquad \text{for} \quad \tau = 0,...,N+1, \quad k = 1,...,l \tag{6}$$

where $\mathbf{L}^{(k\tau\eta)}$ and $\mathbf{M}^{(k\tau\eta)}$ denote the fundamental and mass matrix, respectively. The symbol $\mathbf{q}^{(k\tau)} = \left[q_1^{(k\tau)}\ q_2^{(k\tau)}\ q_3^{(k\tau)}\right]^T$ accounts for the vector of generalized external loads, whose components are determined according to a static equivalence principle.

The higher order two-dimensional problem of Eq. (6) is solved with the GDQ method, starting from a discretization of the physical domain in $I_N \times I_M$ discrete points according to the Chebyshev-Gauss-Lobatto (CGL) distribution [4]. Referring to an arbitrary univariate function $f = f\left(x\right)$, the GDQ technique provides the following expression for the n-th order derivative evaluated at an arbitrary point x_i for $i = 1,...,I_Q$:

$$f^{(n)}\left(x_i\right) = \left.\frac{\partial^n f\left(x\right)}{\partial x^n}\right|_{x=x_i} \cong \sum_{j=1}^{I_Q}\varsigma_{ij}^{(n)} f\left(x_j\right) \qquad i = 1, 2,..., I_Q \tag{7}$$

where the weighting coefficients $\varsigma_{ij}^{(n)}$ are computed with a recursive procedure.

Applications and results

We now present some results from the statics and dynamics of a doubly-curved laminated panel with a softcore, made of generally anisotropic materials. A revolution hyperbolic hyperboloid is considered [4], whose reference surface can be described with principal coordinates α_1, α_2 according to the following relation:

$$\mathbf{r}\left(\alpha_1, \alpha_2\right) = a\cosh\alpha_1\cos\alpha_2\,\mathbf{e}_1 - a\cosh\alpha_1\sin\alpha_2\,\mathbf{e}_2 + c\sinh\alpha_1\,\mathbf{e}_3 \tag{8}$$

with $a = 2.00\,\text{m}$ and $c = 1.50\,\text{m}$. The lamination scheme consists of four different layers with general orientation $(30/70/70/45)$. The two external sheets of the structure are made of graphite-epoxy $\left(\rho^{(k)} = 1450\ \text{kg/m}^3\right)$, here modeled as orthotropic material with elastic moduli $E_1^{(k)} = 13.79$ GPa, $E_2^{(k)} = E_3^{(k)} = 8.96$ GPa, shear moduli $G_{12}^{(k)} = G_{13}^{(k)} = 7.10$ GPa, $G_{23}^{(k)} = 6.21$ GPa and Poisson's ratios $\nu_{12}^{(k)} = \nu_{13}^{(k)} = 0.30$, $\nu_{23}^{(k)} = 0.49$. On the other hand, the central core is made of a triclinic material $\left(\rho^{(k)} = 7750\ \text{kg/m}^3\right)$, characterized by the following anisotropic stiffness matrix $\mathbf{E}^{(k)}$ [2]:

$$\mathbf{E}^{(k)} = \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} & E_{16}^{(k)} & E_{14}^{(k)} & E_{15}^{(k)} & E_{13}^{(k)} \\ E_{12}^{(k)} & E_{22}^{(k)} & E_{26}^{(k)} & E_{24}^{(k)} & E_{25}^{(k)} & E_{23}^{(k)} \\ E_{16}^{(k)} & E_{26}^{(k)} & E_{66}^{(k)} & E_{46}^{(k)} & E_{56}^{(k)} & E_{36}^{(k)} \\ E_{14}^{(k)} & E_{24}^{(k)} & E_{46}^{(k)} & E_{44}^{(k)} & E_{45}^{(k)} & E_{34}^{(k)} \\ E_{15}^{(k)} & E_{25}^{(k)} & E_{56}^{(k)} & E_{45}^{(k)} & E_{55}^{(k)} & E_{35}^{(k)} \\ E_{13}^{(k)} & E_{23}^{(k)} & E_{36}^{(k)} & E_{34}^{(k)} & E_{35}^{(k)} & E_{33}^{(k)} \end{bmatrix} = \begin{bmatrix} 98.84 & 53.92 & 0.03 & 1.05 & -0.1 & 50.78 \\ 53.92 & 99.19 & 0.03 & 0.55 & -0.18 & 50.87 \\ 0.03 & 0.03 & 22.55 & -0.04 & 0.25 & 0.02 \\ 1.05 & 0.55 & -0.04 & 21.1 & 0.07 & 1.03 \\ -0.1 & -0.18 & 0.25 & 0.07 & 21.14 & -0.18 \\ 50.78 & 50.87 & 0.02 & 1.03 & -0.18 & 87.23 \end{bmatrix} \text{GPa} \tag{9}$$

More specifically, the core is made of a lamina with triclinic-soft material, whose stiffness constants are equal to $1/1000$ of those reported in Eq. (9), whereas the third layer follows exactly the triclinic material of Eq. (9). Unconventional boundary conditions have been enforced accounting for a Double-Weibull distribution of linear springs. For more details on the topic, the interested reader is referred to [5].

In Table 1 the first ten mode frequencies, calculated with different higher order ESL and LW theories, are compared to those ones resulting from a 3D FEM simulation with 20-node brick elements.

Table 1. *Free vibration analysis of a revolution hyperbolic hyperboloid laminated with generally anisotropic materials employing higher order theories with both the ESL and LW approaches.*

Mode $f\,[\text{Hz}]$	$\left(\text{B}^{\text{K}}_{\text{SSS}}\text{CFF}\right)$										
	3D FEM	**FSDT**	**TSDT**	**ED3**	**EDZ3**	**ED4**	**EDZ4**	**LD1**	**LD2**	**LD3**	**LD4**
DOFs	327402	5046	10092	10092	12615	12615	15138	20184	30276	40368	50460
1	11.53	19.21	18.38	18.37	17.26	14.79	14.76	12.17	11.42	11.37	11.39
2	14.96	28.02	26.70	26.69	24.60	20.24	20.16	15.30	15.06	15.09	15.24
3	21.42	32.11	31.05	31.07	29.77	26.62	26.57	22.40	21.17	21.13	21.13
4	28.53	43.04	41.27	41.31	39.21	35.50	35.45	29.76	28.74	28.75	28.90
5	31.71	52.75	50.13	50.16	46.65	38.95	38.81	32.65	31.50	31.42	31.49
6	33.12	54.80	52.27	52.35	48.62	40.82	40.73	33.80	32.96	32.95	33.18
7	40.15	64.36	61.41	61.38	57.92	51.33	51.18	41.83	40.08	40.08	40.15
8	43.33	66.00	63.11	63.20	59.75	52.09	51.95	44.51	43.01	42.96	42.98
9	44.38	69.84	66.35	66.44	62.07	53.68	53.56	45.51	43.93	43.98	43.99
10	47.04	80.76	75.26	75.23	67.94	57.89	57.77	48.89	46.85	46.86	46.93

Geometric Inputs: Revolution Hyperbolic Hyperboloid, $a = 2.00$ m, $c = 1.50$ m
$\alpha_1^0 = -1$, $\alpha_1^1 = 1$, $\alpha_2^0 = 0$ and $\alpha_2^1 = \pi/2$, $h_1 = h_4 = 0.01$ m, $h_2 = 0.10$ m, $h_3 = 0.03$ m

Lamination Scheme: 1st layer: graphite-epoxy, 2nd layer: triclinic, 3rd layer: triclinic-soft, 4th layer: graphite-epoxy

Computational Grid: CGL distribution with $I_N = I_M = 31$

When the Murakami's zigzag function [2] is adopted in Eq. (4) within an ESL framework, more accurate results are obtained. However, if LW simulations are performed, a perfect alignment between the 3D FEM-based predictions is outlined. Fig. 1 shows the first eight mode shapes of the structure, calculated by means of the LD4 theory, showing the three-dimensional capability of the proposed formulation.

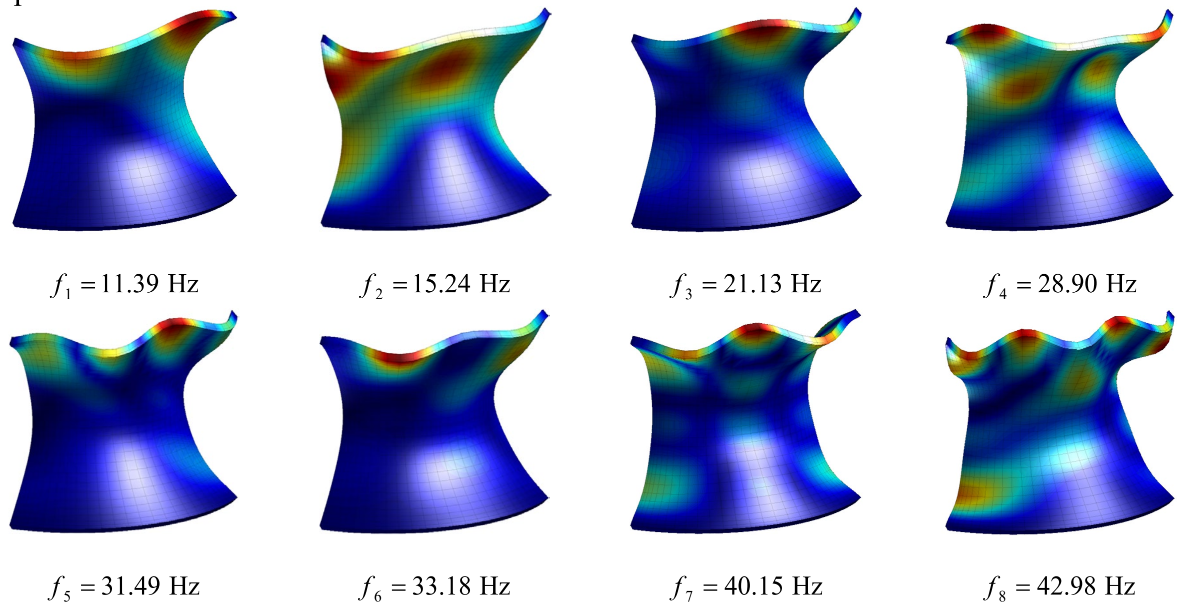

Figure 1. *First eight mode shapes of a laminated anisotropic revolution hyperbolic hyperboloid under general boundary conditions. The modal eigenvectors have been calculated employing the LD4 theory.*

The same structure has been investigated under a static load. In particular, a load $q_3^{(+)} = -2000$ N is applied on the structure, according to Ref. [5]. Taking into account a bivariate super elliptic distribution [5], the external load is distributed only in a limited area within the physical domain, setting $\alpha_{1m} = \alpha_{2m} = 0$ and $\delta_1 = \delta_2 = 0.53$ and $n = 1000$. The three-dimensional through-the-thickness stress distribution is depicted in Fig. 2, referring to the point of the physical domain located at $\left(0.25\left(\alpha_1^1 - \alpha_1^0\right), 0.75\left(\alpha_2^1 - \alpha_2^0\right)\right)$.

Classical approaches like FSDT and TSDT are not capable of predicting the three-dimensional finite element outcomes, as well as higher order ESL theories. The static response of the entire lamination scheme can be properly evaluated only with higher order LW theories for both in-plane and out-of-plane stress components. As can be seen from the three-dimensional solution, the abrupt change of stiffnesses between two adjacent layers leads to very complicated stress distributions, which requires a higher order LW approach among two-dimensional theories.

Conclusions

In the present work a generalized higher order two-dimensional theory has been presented for the static and modal analysis of shell structures made of generally anisotropic laminates. Following the LW approach, the fundamental equations are derived within each layer of the structure. As particular case, a unified ESL theory accounting for zigzag functions has been derived. The equations of motion have been discretized in a strong form via the GDQ method, together with the associated boundary conditions. The proposed methodology has been applied to a doubly-curved shell structure with a generally anisotropic lamination scheme and soft layers, showing the accuracy of the formulation, as well as its computational efficiency.

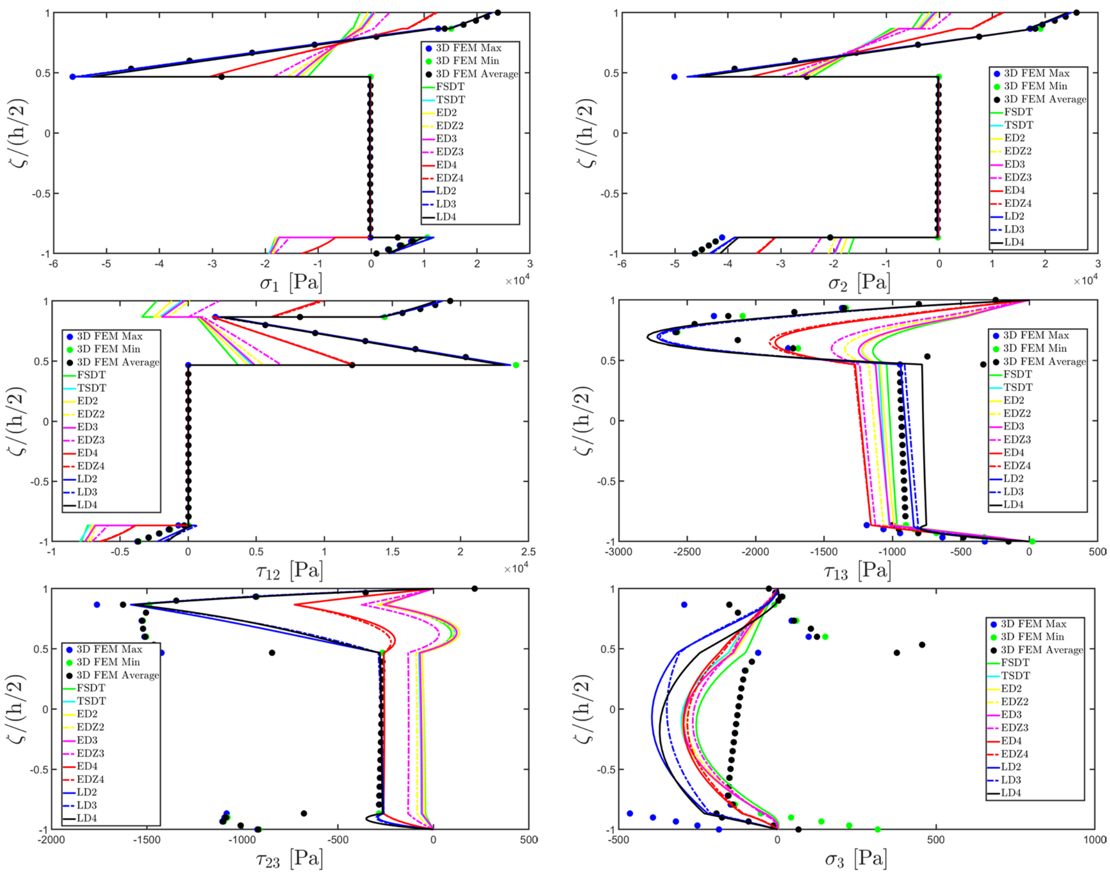

***Figure 2.** Through-the-thickness distributions of the three-dimensional stress components calculated by means of various higher order ESL theories of a fully-clamped ellipsoid subjected to a uniform surface load applied at the top surface.*

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